Coordinate systems in Banach lattices via descriptive set theory

Christian Rosendal
University of Maryland

joint work with Antonio Avilés, Mitchell Taylor and Pedro Tradacete

February 26, 2026

Background on Banach lattices and convergence types

A Banach lattice is a Banach space X equipped with an order relation $\leqslant$ satisfying:

1. $\forall x, y, z \in X \quad x \leqslant y \;\Rightarrow\; x + z \leqslant y + z$.
2. $\forall x, y \in X \; \forall \lambda > 0 \quad x \leqslant y \;\Rightarrow\; \lambda x \leqslant \lambda y$.
3. For any $x \in X$, set $|x| = x \vee (-x)$. Then, if $|x| \leqslant |y|$, also $\|x\| \leqslant \|y\|$.
4. Any two elements $x, y \in X$ have a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$.

The two most familiar examples are the function spaces $C([0,1])$ and $L^p([0,1])$, $1 \leqslant p \leqslant \infty$.

The two most familiar examples are the function spaces $C([0,1])$ and $L^p([0,1])$, $1 \leqslant p \leqslant \infty$.

Here we write $f \leqslant g$ if and only if

$$f(x) \leqslant g(x) \text{ for all, respectively, for almost all } x \in [0,1],$$

whereby

$$\big(f \vee g\big)(x) = \max\{f(x), g(x)\} \quad \& \quad |f|(x) = \big|f(x)\big|.$$

The two most familiar examples are the function spaces $C([0,1])$ and $L^p([0,1])$, $1 \leqslant p \leqslant \infty$.

Here we write $f \leqslant g$ if and only if

$$f(x) \leqslant g(x) \text{ for all, respectively, for almost all } x \in [0,1],$$

whereby

$$\big(f \vee g\big)(x) = \max\{f(x), g(x)\} \quad \& \quad |f|(x) = \big|f(x)\big|.$$

Other examples are sequence spaces such as ℓ_p and c_0, where again

$$x = (x_1, x_2, \ldots) \leqslant y = (y_1, y_2, \ldots) \quad \Leftrightarrow \quad \forall n \, \big(x_n \leqslant y_n\big).$$

The lattice structure on a Banach lattice X gives rise to three classical notions of sequential convergence, not available in a general Banach space.

The lattice structure on a Banach lattice X gives rise to three classical notions of sequential convergence, not available in a general Banach space. Namely,

- a sequence $(x_n)_{n=1}^{\infty}$ converges uniformly to x, denoted $x_n \xrightarrow{\mathrm{u}} x$, if there is some $z \in X_+$ so that

$$\forall m \, \forall^{\infty} n \; |x_n - x| \leqslant \tfrac{z}{m},$$

- a sequence $(x_n)_{n=1}^{\infty}$ σ-order converges to x, denoted $x_n \xrightarrow{\sigma\mathrm{o}} x$, if there is some sequence $z_m \downarrow 0$ in X so that

$$\forall m \, \forall^{\infty} n \; |x_n - x| \leqslant z_m,$$

- a sequence $(x_n)_{n=1}^{\infty}$ order converges to x, denoted $x_n \xrightarrow{\mathrm{o}} x$, if there is some net $z_\mu \downarrow 0$ in X so that

$$\forall \mu \, \forall^{\infty} n \; |x_n - x| \leqslant z_\mu.$$

Explanation of notation:

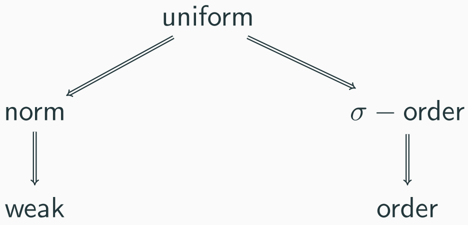
$$\forall^\infty n \text{ means } \exists N\, \forall n \geqslant N,$$

while

$$z_m \downarrow 0 \text{ means } z_1 \geqslant z_2 \geqslant \ldots \geqslant 0 = \inf_n z_n.$$

and similarly $z_\mu \downarrow 0$ means that (z_μ) is decreasing with infimum 0.

Implications between convergence types:



So uniform convergence is the strongest, whereas there is no general relationship between (σ-)order and norm convergence.

None of these new convergence notions stem from topologies on X, but luckily they are still compatible with the algebraic and lattice operations in X.

None of these new convergence notions stem from topologies on X, but luckily they are still compatible with the algebraic and lattice operations in X.

It can be shown that, in all cases above, the limit is unique whenever it exists. Thus, if C is one of the above notions of convergence and $\sum_{n=1}^{\infty} x_n$ is a series in X, we can unambiguously write

$$x =^{C} \sum_{n=1}^{\infty} x_n$$

to express that the sequence

$$\left(\sum_{n=1}^{m} x_n \right)_{m=1}^{\infty}$$

of partial sums C-converges to x.

Coordinate systems in infinite-dimensional spaces

In an infinite-dimensional Banach space, the usual concept of a vector space basis is pretty useless.

In an infinite-dimensional Banach space, the usual concept of a vector space basis is pretty useless.

First of all, such a basis will have to be uncountable,

In an infinite-dimensional Banach space, the usual concept of a vector space basis is pretty useless.

First of all, such a basis will have to be uncountable, secondly, it can only be constructed using the axiom of choice,

In an infinite-dimensional Banach space, the usual concept of a vector space basis is pretty useless.

First of all, such a basis will have to be uncountable, secondly, it can only be constructed using the axiom of choice, and thirdly, it has no good analytical properties.

In an infinite-dimensional Banach space, the usual concept of a vector space basis is pretty useless.

First of all, such a basis will have to be uncountable, secondly, it can only be constructed using the axiom of choice, and thirdly, it has no good analytical properties.

The right concept is somewhat different.

Definition

Let X be a Banach lattice and C one of the following convergence types,

weak, norm, uniform, order, σ-order.

A sequence $(e_n)_{n=1}^{\infty}$ in X is said to be a C-basis for X provided that, for every $x \in X$, there is a unique sequence of scalars $(a_n) \in \mathbb{R}^{\mathbb{N}}$ so that

$$x =^{C} \sum_{n=1}^{\infty} a_n e_n.$$

Definition

Let X be a Banach lattice and C one of the following convergence types,

weak, norm, uniform, order, σ-order.

A sequence $(e_n)_{n=1}^{\infty}$ in X is said to be a C-basis for X provided that, for every $x \in X$, there is a unique sequence of scalars $(a_n) \in \mathbb{R}^{\mathbb{N}}$ so that

$$x =^{C} \sum_{n=1}^{\infty} a_n e_n.$$

In this case, we may define functionals $X \xrightarrow{e_k^\sharp} \mathbb{R}$ by letting

$$e_k^\sharp(x) = a_k,$$

where (a_n) is the uniquely defined sequence referenced above.

Since the sequence constantly equal to e_n will C-converge to e_n, we find that

$$e_k^\sharp(e_n) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

for all k, n, that is, the functionals $e_k^\sharp$ are biorthogonal to the sequence (e_n).

Since the sequence constantly equal to e_n will C-converge to e_n, we find that

$$e_k^\sharp(e_n) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

for all k, n, that is, the functionals $e_k^\sharp$ are biorthogonal to the sequence (e_n).

Observe however that a priori it is not clear that the functionals $e_k^\sharp$ are continuous with respect to the norm on X or with respect to any of the other convergence types for that matter.

Since the sequence constantly equal to e_n will C-converge to e_n, we find that

$$e_k^\sharp(e_n) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

for all k, n, that is, the functionals $e_k^\sharp$ are biorthogonal to the sequence (e_n).

Observe however that a priori it is not clear that the functionals $e_k^\sharp$ are continuous with respect to the norm on X or with respect to any of the other convergence types for that matter.

Without continuity, however, such bases have limited utility.

Theorem (J. Schauder 1927)
The biorthogonal functionals $e_n^\sharp$ *associated with a norm basis* (e_n) *for a Banach space are continuous.*

For this reason, one often denote norm bases as Schauder bases.

Theorem (J. Schauder 1927)
The biorthogonal functionals $e_n^\sharp$ associated with a norm basis (e_n) for a Banach space are continuous.

For this reason, one often denote norm bases as Schauder bases.

Theorem (S. Mazur)
Every weak basis for a Banach space is a norm basis.

On the other hand, the relationship between norm, uniform, σ-order and order bases was pretty much unclear hitherto and subject to detailed studies by A. Gumenchuk, O. Karlova, and M. Popov, and by Taylor and V. Troitsky.

Theorem (Assuming analytic determinacy)

Let (e_n) be a sequence of vectors in a Banach lattice $X = [e_n]$ and $(e_n^\sharp)$ is a sequence of (possibly discontinuous) biorthogonal functionals for (e_n). Consider the following properties:

1. *(e_n) is an order basis for X with functionals $(e_n^\sharp)$,*
2. *(e_n) is a σ-order basis for X with functionals $(e_n^\sharp)$,*
3. *(e_n) is a uniform basis for X with functionals $(e_n^\sharp)$,*
4. *(e_n) is a Schauder basis for X with functionals $(e_n^\sharp)$.*

Theorem (Assuming analytic determinacy)

Let (e_n) be a sequence of vectors in a Banach lattice $X = [e_n]$ and $(e_n^\sharp)$ is a sequence of (possibly discontinuous) biorthogonal functionals for (e_n). Consider the following properties:

1. *(e_n) is an order basis for X with functionals $(e_n^\sharp)$,*
2. *(e_n) is a σ-order basis for X with functionals $(e_n^\sharp)$,*
3. *(e_n) is a uniform basis for X with functionals $(e_n^\sharp)$,*
4. *(e_n) is a Schauder basis for X with functionals $(e_n^\sharp)$.*

Then the $e_n^\sharp$ are continuous in all above cases.

Theorem (Assuming analytic determinacy)

Let (e_n) be a sequence of vectors in a Banach lattice $X = [e_n]$ and $(e_n^\sharp)$ is a sequence of (possibly discontinuous) biorthogonal functionals for (e_n). Consider the following properties:

1. *(e_n) is an order basis for X with functionals $(e_n^\sharp)$,*
2. *(e_n) is a σ-order basis for X with functionals $(e_n^\sharp)$,*
3. *(e_n) is a uniform basis for X with functionals $(e_n^\sharp)$,*
4. *(e_n) is a Schauder basis for X with functionals $(e_n^\sharp)$.*

Then the $e_n^\sharp$ are continuous in all above cases.

Furthermore (1)$\Leftrightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4).

Theorem (Assuming analytic determinacy)

Let (e_n) be a sequence of vectors in a Banach lattice $X = [e_n]$ and $(e_n^\sharp)$ is a sequence of (possibly discontinuous) biorthogonal functionals for (e_n). Consider the following properties:

1. *(e_n) is an order basis for X with functionals $(e_n^\sharp)$,*
2. *(e_n) is a σ-order basis for X with functionals $(e_n^\sharp)$,*
3. *(e_n) is a uniform basis for X with functionals $(e_n^\sharp)$,*
4. *(e_n) is a Schauder basis for X with functionals $(e_n^\sharp)$.*

Then the $e_n^\sharp$ are continuous in all above cases.

Furthermore (1)$\Leftrightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4).

Observe that the implications (1)$\Leftrightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4) do not in any way correspond to the implications between order types.

Let us recall the statement of analytic determinacy.

Let us recall the statement of analytic determinacy.

This says that in every 2-player game with perfect information

$$\begin{array}{llllllllll} \textbf{I} & x_1 & & x_3 & & x_5 & & x_7 & & \dots \\ \textbf{II} & & x_2 & & x_4 & & x_6 & & x_8 & \dots \end{array}$$

in which I and II alternate in choosing natural numbers $x_i \in \mathbb{N}$ and, for every analytic subset $A \subseteq \mathbb{N}^{\mathbb{N}}$, either player I has a strategy to ensure that the outcome

$$x = (x_1, x_2, \dots)$$

lies in A or II has a strategy to ensure that the outcome lies in $\complement A$.

What is really needed for our result is the fact that under analytic determinacy, every $\mathbf{\Sigma}^1_2$-set in a Polish space is Baire measurable, that is, differs from an open set by a meagre set.

What is really needed for our result is the fact that under analytic determinacy, every $\mathbf{\Sigma}^1_2$-set in a Polish space is Baire measurable, that is, differs from an open set by a meagre set.

Furthermore, only the continuity of functionals in cases (1) and (2) plus that every σ-order basis is a uniform basis need this assumption.

Ideas about the proof

The general strategy of the proof is pretty straightforward.

The general strategy of the proof is pretty straightforward.

The main problem lies in showing that if (e_n) is a C-basis with associated biorthogonal functionals $(e_n^\sharp)$, then each $e_n^\sharp$ is continuous.

The general strategy of the proof is pretty straightforward.

The main problem lies in showing that if (e_n) is a C-basis with associated biorthogonal functionals $(e_n^\sharp)$, then each $e_n^\sharp$ is continuous.

The implications between the different types of bases can then be proved using more detailed analytical machinery and the previous work of especially Taylor–Troitsky.

So suppose (e_n) is a C-basis with associated biorthogonal functionals $(e_n^\sharp)$.

So suppose (e_n) is a C-basis with associated biorthogonal functionals $(e_n^\sharp)$.

Then we can write

$$\begin{aligned}
(x,a) \in \mathcal{G}e_k^\sharp \quad &\Leftrightarrow \quad e_k^\sharp(x) = a \\
&\Leftrightarrow \quad \exists (a_n) \in \mathbb{R}^{\mathbb{N}} \left(x =^C \sum_{n=1}^{\infty} a_n e_n \ \ \& \ \ a_k = a \right) \\
&\Leftrightarrow \quad \exists (a_n) \in \mathbb{R}^{\mathbb{N}} \left(x = C - \lim_m \sum_{n=1}^{m} a_n e_n \ \ \& \ \ a_k = a \right).
\end{aligned}$$

So suppose (e_n) is a C-basis with associated biorthogonal functionals $(e_n^\sharp)$.

Then we can write

$$\begin{aligned}
(x,a)\in \mathcal{G}e_k^\sharp \quad &\Leftrightarrow \quad e_k^\sharp(x)=a \\
&\Leftrightarrow \quad \exists (a_n)\in \mathbb{R}^{\mathbb{N}} \left(x =^C \sum_{n=1}^{\infty} a_n e_n \ \ \& \ \ a_k = a\right) \\
&\Leftrightarrow \quad \exists (a_n)\in \mathbb{R}^{\mathbb{N}} \left(x = C-\lim_m \sum_{n=1}^{m} a_n e_n \ \ \& \ \ a_k = a\right).
\end{aligned}$$

Furthermore, if we can show that the graph $\mathcal{G}e_k^\sharp$ of the functional $e_k^\sharp$ is just analytic, i.e., $\mathbf{\Sigma}_1^1$, then it is well-known that this implies continuity.

It follows that, if the set

$$\left\{ \left((x_n)_{n=1}^{\infty}, x\right) \in X^{\mathbb{N}} \times X \;\middle|\; x_n \xrightarrow[n]{\mathsf{C}} x \right\}$$

is $\mathbf{\Sigma}_1^1$, then the graph is also $\mathbf{\Sigma}_1^1$ and the functionals continuous.

It follows that, if the set

$$\left\{ \left((x_n)_{n=1}^{\infty}, x \right) \in X^{\mathbb{N}} \times X \;\middle|\; x_n \xrightarrow[n]{\mathsf{C}} x \right\}$$

is $\mathbf{\Sigma}_1^1$, then the graph is also $\mathbf{\Sigma}_1^1$ and the functionals continuous.

Lemma (Uniform convergence)
In a Banach lattice X, we have

$$x_n \xrightarrow[n]{\mathsf{u}} x \quad \Leftrightarrow \quad \forall \epsilon > 0\; \exists k\; \forall m \geqslant k \quad \left\| \bigvee_{n=k}^{m} |x_n - x| \right\| < \epsilon.$$

In particular, the set

$$\left\{ \left((x_n)_{n=1}^{\infty}, x \right) \in X^{\mathbb{N}} \times X \;\middle|\; x_n \xrightarrow[n]{\mathsf{u}} x \right\}$$

is Borel.

Lemma (Order convergence)
In a Banach lattice X, we have

$$x_n \xrightarrow[n]{\mathrm{o}} x \quad \Leftrightarrow \quad \forall y > 0 \, \exists z \, \big(y \nleqslant z \ \ \& \ \ \forall^\infty n \, |x_n - x| \leqslant z \big).$$

In particular, the set

$$\Big\{ \big((x_n)_{n=1}^\infty, x \big) \in X^{\mathbb{N}} \times X \, \Big| \, x_n \xrightarrow[n]{\mathrm{o}} x \Big\}$$

is $\mathbf{\Pi}^1_2$.

Lemma (σ-order convergence)
In a Banach lattice X, we have

$$x_n \xrightarrow[n]{\sigma o} x \quad \Leftrightarrow \quad \exists (z_m)$$
$$\Big(z_1 \geqslant z_2 \geqslant \ldots \geqslant 0 \;\;\&\;\; \forall y > 0 \, \exists m \; y \nleqslant z_m \;\;\&\;\; \forall m \, \forall^\infty n \, |x_n - x| \leqslant z_m \Big).$$

In particular, the set

$$\Big\{ \big((x_n)_{n=1}^\infty, x \big) \in X^{\mathbb{N}} \times X \;\Big|\; x_n \xrightarrow[n]{\sigma o} x \Big\}$$

is $\mathbf{\Sigma}_2^1$.

Lemma
In a separable Banach lattice X, we have

$$x_n \xrightarrow[n]{\mathrm{o}} x \quad \Leftrightarrow \quad x_n \xrightarrow[n]{\sigma\mathrm{o}} x.$$

In particular, the set

$$\left\{ \left((x_n)_{n=1}^{\infty}, x\right) \in X^{\mathbb{N}} \times X \;\middle|\; x_n \xrightarrow[n]{\sigma\mathrm{o}} x \right\}$$

is $\mathbf{\Delta}^1_2$.

Proposition

The following conditions are equivalent for a separable Banach lattice X.

1. *The set*

$$X_{\downarrow 0} = \left\{ (x_n)_{n=1}^{\infty} \in X^{\mathbb{N}} \;\middle|\; x_1 \geqslant x_2 \geqslant \ldots \geqslant 0 = \inf_n x_n \right\}$$

   is Borel,

2. *the set*

$$\left\{ \left((x_n)_{n=1}^{\infty}, x\right) \in X^{\mathbb{N}} \times X \;\middle|\; x_n \xrightarrow[n]{\sigma\mathrm{o}} x \right\}$$

   is analytic.

Proposition

The following conditions are equivalent for a separable Banach lattice X.

1. *The set*

$$X_{\downarrow 0} = \left\{ (x_n)_{n=1}^{\infty} \in X^{\mathbb{N}} \,\middle|\, x_1 \geqslant x_2 \geqslant \ldots \geqslant 0 = \inf_n x_n \right\}$$

   is Borel,

2. *the set*

$$\left\{ \left((x_n)_{n=1}^{\infty}, x\right) \in X^{\mathbb{N}} \times X \,\middle|\, x_n \xrightarrow[n]{\sigma o} x \right\}$$

   is analytic.

Note that, a priori, the set $X_{\downarrow 0}$ is only coanalytic, i.e., $\mathbf{\Pi}_1^1$.

Problem
When X is a separable Banach lattice, is the set $X_{\downarrow 0}$ Borel?

There are various natural cases in which this holds, for example,

- When X is σ-order continuous, i.e., $(x_n) \in X_{\downarrow 0} \Rightarrow \lim_n \|x_n\| = 0$,
- when X is σ-order complete, i.e., every bounded sequence has a supremum,
- when X has a countable π-basis (b_n), i.e., positive elements $b_n > 0$ minorising every positive element.

A Banach lattice X has the Fatou property, if whenever we have elements $0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x$ with $x = \sup_n x_n$, then $\|x\| = \sup_n \|x_n\|$.

A Banach lattice X has the Fatou property, if whenever we have elements $0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x$ with $x = \sup_n x_n$, then $\|x\| = \sup_n \|x_n\|$.

We define a transfinite hierarchy of α-Fatou properties for $\alpha < \omega_1$ that correspond to a coanalytic rank on $X_{\downarrow 0}$.

A Banach lattice X has the Fatou property, if whenever we have elements $0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x$ with $x = \sup_n x_n$, then $\|x\| = \sup_n \|x_n\|$.

We define a transfinite hierarchy of α-Fatou properties for $\alpha < \omega_1$ that correspond to a coanalytic rank on $X_{\downarrow 0}$.

Thus, for a Banach lattice X,

$$X_{\downarrow 0} \text{ is Borel } \Leftrightarrow X \text{ is } \alpha\text{-Fatou for some } \alpha < \omega_1.$$

A Banach lattice X has the Fatou property, if whenever we have elements $0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x$ with $x = \sup_n x_n$, then $\|x\| = \sup_n \|x_n\|$.

We define a transfinite hierarchy of α-Fatou properties for $\alpha < \omega_1$ that correspond to a coanalytic rank on $X_{\downarrow 0}$.

Thus, for a Banach lattice X,

$$X_{\downarrow 0} \text{ is Borel} \iff X \text{ is } \alpha\text{-Fatou for some } \alpha < \omega_1.$$

Theorem
For every $\alpha < \omega_1$, there is a separable Banach lattice X with a countable π-basis that fails to be α-Fatou.