How computability-theoretic degree structures and topological spaces are related

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The mantra

Continuity is computability relative to some oracle.

The take-home message

Studying substructures of the enumeration degrees is essentially the same thing as studying countably-based topological spaces up to σ-homoemorphism.

A computability theorem with topology proofs

Definition
For $A, B \subseteq \mathbb{N}$, say $A \leq_e B$ if there is a computable procedure transforming enumerations of B into enumerations of A.

Definition
A is *total* iff $A \equiv_e A^c$. A is *almost total*, if $A \otimes B$ is total for all total B with $B \not\leq_e A$.

Theorem
Almost-total but not total enumeration degrees exist.

Proofs.

1. By Miller, using the Kakutani Fixed Point theorem
2. By Day & Miller, using measure theory and randomness
3. By Kihara & P., using topological dimension theory

□

A dimension-theoretic problem with a computability-theoretic solution

Definition

X and **Y** are σ-homeomorphic, if there are partitions $\mathbf{X} = \bigcup_{i \in \mathbb{N}} \mathbf{X}_i$ and $\mathbf{Y} = \bigcup_{i \in \mathbb{N}} \mathbf{Y}_i$ such that $\mathbf{X}_i$ and $\mathbf{Y}_i$ are homeomorphic for all $i \in \mathbb{N}$.

Question (Jayne 1974)

How many σ-homeomorphism types of uncountable Polish spaces are there?

Well, there are at least 3...

Theorem (Kihara & P.)

The space of countable subsets of $\aleph_1$ embeds into the σ-homeomorphism types of Polish spaces.

Defining Turing reducibility

Definition

For $p, q \in \{0,1\}^{\mathbb{N}}$, say that $p \leq_T q$ iff $\exists F :\subseteq \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$, partial computable, such that $F(q) = p$.

Definition (Kihara & P)

For $x \in \mathbf{X}$, $y \in \mathbf{Y}$, say that $x^{\mathbf{X}} \leq_T y^{\mathbf{Y}}$ iff $\exists F :\subseteq \mathbf{Y} \to \mathbf{X}$, partial computable, such that $F(y) = x$.

Represented spaces and computability

Definition

A *represented space* **X** is a pair (X, δ_X) where X is a set and $\delta_X :\subseteq \mathbb{N}^\mathbb{N} \to X$ a surjective partial function.

Definition

$F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is a realizer of $f :\subseteq \mathbf{X} \to \mathbf{Y}$, iff $\delta_Y(F(p)) = f(\delta_X(p))$ for all $p \in \delta_X^{-1}(\mathrm{dom}(f))$. Abbreviate: $F \vdash f$.

$$
\begin{array}{ccc}
\mathbb{N}^\mathbb{N} & \xrightarrow{F} & \mathbb{N}^\mathbb{N} \\
\downarrow \delta_X & & \downarrow \delta_Y \\
\mathbf{X} & \xrightarrow{f} & \mathbf{Y}
\end{array}
$$

Definition

$f :\subseteq \mathbf{X} \to \mathbf{Y}$ is called computable (continuous), iff it has a computable (continuous) realizer.

Where do representations come from?

- For a separable metric space, represent points by fast converging sequences of basic points.
- For a countably-based space, represent points by enumerations of their neighborhood filters.
- This makes continuity as defined for represented space coincide with metric/topological continuity.

Unfolding the definition

Definition (Medvedev reducibility)

For $A, B \subseteq \{0,1\}^{\mathbb{N}}$, let $A \leq_M B$ iff $\exists F : B \to A$, computable.

Observation (Kihara & P.)

$x^{\mathbf{X}} \leq_T y^{\mathbf{Y}}$ iff $\delta_{\mathbf{X}}^{-1}(\{x\}) \leq_M \delta_{\mathbf{Y}}^{-1}(\{y\})$

Observation (Kihara & P.)

For $A, B \subseteq \mathbb{N}$, we have that $A \leq_e B$ iff $A^{\mathcal{O}(\mathbb{N})} \leq_T B^{\mathcal{O}(\mathbb{N})}$.

The spectrum

Definition
Let $\mathrm{Spec}(\mathbf{X})$ be the collection of all $\leq_T$-degrees of points in **X**.

- $\mathrm{Spec}(\mathbb{N}^{\mathbb{N}}) = \mathrm{Spec}(\mathbb{R}) = \mathfrak{T}$ (Turing degrees)
- $\mathrm{Spec}([0,1]^{\mathbb{N}}) =: \mathfrak{C}$ (continuous degrees, MILLER)
- $\mathrm{Spec}(\mathcal{O}(\mathbb{N})) = \mathfrak{E}$ (enumeration degrees)

Relating spectra and σ-homeomorphism

Theorem (Kihara & P.)

The following are equivalent for a represented space $\mathbf{X}$:

1. $\mathit{Spec}(\mathbf{X}) \subseteq \mathit{Spec}(\mathbf{Y})$
2. $\mathbf{X} = \bigcup_{n \in \mathbb{N}} \mathbf{X}_n$ *where there are* $\mathbf{Y}_n \subseteq \mathbf{Y}$ *such* $\mathbf{X}_n$ *and* $\mathbf{Y}_n$ *are computably homeomorphic*

Theorem (Kihara & P.)

The following are equivalent for a represented space $\mathbf{X}$:

1. $\exists t \in \mathfrak{T} \quad t \times \mathit{Spec}(\mathbf{X}) = t \times \mathit{Spec}(\mathbf{Y})$
2. $\mathbb{N} \times \mathbf{X}$ *and* $\mathbb{N} \times \mathbf{Y}$ *are* σ*-homeomorphic*

Defining the continuous degrees

Definition (Miller)

The continuous degrees are the degrees of points in computable metric spaces; i.e. they are $\mathrm{Spec}([0,1]^\omega)$.

Theorem (Miller)

$\mathfrak{T} \subsetneq \mathrm{Spec}([0,1]^\omega) \subsetneq \mathfrak{E}$

Almost-total degrees

Definition

Call $e \in \mathfrak{E}$ *almost-total*, if for all $p \in \mathfrak{T}$, $p \not\leq_e e$ we have that $p \oplus e \in \mathfrak{T}$.

Theorem (Andrews, Igusa, Miller & Soskova)

The continuous degrees are exactly the almost-total degrees.

Small inductive dimension

Definition

Let the (small inductive) dimension of a Polish space be defined inductively via $\dim(\emptyset) = -1$ and:

$$\dim(\mathbf{X}) = \sup_{x \in \mathbf{X}} \sup_{n \in \mathbb{N}} \inf_{U \in \mathcal{O}(\mathbf{X}), x \in U \subseteq B(x, 2^{-n})} \dim(\delta U) + 1$$

- If $\dim(\mathbf{X})$ exists, it is a countable ordinal and we call $\mathbf{X}$ countably dimensional.
- Otherwise $\mathbf{X}$ is infinite-dimensional.

Theorem (Hurewicz & Wallmann)

Uncountable Polish $\mathbf{X}$ *is* σ*-homeomorphic to* $\{0,1\}^{\mathbb{N}}$ *iff* $\mathbf{X}$ *is countably dimensional.*

Corollary

For Polish $\mathbf{X}$ *the following are equivalent:*

1. $\exists p \in \mathfrak{T}\ p \times \mathit{Spec}(\mathbf{X}) \subseteq \mathfrak{T}$
2. $\mathbf{X}$ *has countable dimension.*

Cototal degrees

Definition
Call $A \subseteq \mathbb{N}$ cototal, if $A \leq_e A^C$.

Definition
Call $\mathbf{X}$ an (effective) G_δ-space, if every closed set $A \subseteq \mathbf{X}$ can be expressed as $A = \bigcap_{n \in \mathbb{N}} U_n$ with U_n open (computably open relative to A).

Theorem (Kihara, Ng & P.)
The cototal degrees are exactly the degrees of points in effective G_δ-spaces.

An open question

Question

Is there a universal countably-based G_δ-space?

Theorem (McCarthy)

There is an effectively countably-based G_δ space $\mathbf{A}_{\max}$ such that $\mathrm{Spec}(\mathbf{A}_{\max})$ consists of exactly the cototal enumeration degrees.

Graph-cotototal degrees

Definition

The graph-cotototal degrees are the enumeration degrees of complements of graphs of functions $f : \mathbb{N} \to \mathbb{N}$.

Question (Miller)

Are all almost-total enumeration degrees graph-cotototal?

Definition

Let $\mathbb{N}_{\mathrm{cof}}$ be the natural numbers with the cofinite topology, i.e. $n \in \mathbb{N}_{\mathrm{cof}}$ is represented by enumerating $\mathbb{N} \setminus \{n\}$.

Theorem (Kihara, Ng & P)

$\mathrm{Spec}(\mathbb{N}_{\mathrm{cof}}^{\omega})$ *contains exactly the graph-cotototal enumeration degrees.*

Restating Miller's question

Question (Miller)

Are all almost-total enumeration degrees graph-cototal?

Question (Equivalent)

Does $[0,1]^\omega$ σ*-embed into* $\mathbb{N}^\omega_{\mathrm{cof}}$*?*

The lower reals

Definition
In $\mathbb{R}_<$, real numbers are represented as limits of increasing sequences of rationals.

Definition
$U \in \mathcal{O}(\mathbb{N})$ is called semi-recursive, if there is a computable function $f : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ such that if $n_0 \in U \vee n_1 \in U$, then $n_{f(n_0,n_1)} \in U$. Let $\mathfrak{S} \subseteq \mathfrak{E}$ be all degrees of semi-recursive sets.

Proposition (Ganchev & Soskova)
$Spec(\mathbb{R}_<) = \mathfrak{S}$

An application

Theorem (Kihara & P)

Let $\mathbf{X}$ *be a countably-based* T_1 *space. Then* $\mathbb{R}_<^{n+1}$ *does not piecewise embed into* $\mathbf{X} \times \mathbb{R}_<^n$*.*

Let $\Lambda^n = (\{0,1\}^n, \leq)$ be a partial order on $\{0,1\}^n$ obtained as the n-th product of the ordering $0 < 1$.

Lemma (Kihara & P)

For every countable partition $(P_i)_{i\in\omega}$ *of the n-dimensional hypercube* $[0,1]^n$ *(endowed with the standard product order), there is* $i \in \omega$ *such that* P_i *has a subset which is order isomorphic to the product order* Λ^n*.*

Corollary

For any $n \in \mathbb{N}$ *there exists an enumeration degree which is expressible as the product of* $n+1$ *semirecursive degrees, but not of* n *semirecursive degrees.*

Points don't see everything

Theorem (Kihara,Ng & P.)

$$\mathrm{Spec}(\mathcal{O}(\mathbb{N})) = \bigcup\{\mathrm{Spec}(\mathbf{X}) \mid \mathbf{X} \text{ is effectively cb + } T_{2.5}\}$$

Recall: A space is $T_{2.5}$ iff any two distinct points are separated by open neighborhoods with disjoint closures.

Quasi-minimality

Definition (Kihara, Ng & P.)

Let $\mathfrak{T}$ be a collection of represented spaces. We say that non-computable $x \in \mathbf{X}$ is $\mathfrak{T}$-quasi-minimal, if whenever $y \in \mathbf{Y} \in \mathfrak{T}$ satisfies $y^{\mathbf{Y}} \leq_{\mathrm{T}} x^{\mathbf{X}}$, then y is already computable.

Theorem (Kihara, Ng & P.)

Let $\mathfrak{T}$ be a countable collection of countably-based T_1-spaces. Then there is a $\mathfrak{T}$-quasi-minimal $x \in \mathbb{R}_<$.

A weird consequence

Corollary

There is no countably-based T_1 space **X** *such that every effectively $T_{2.5}$-space* **Y** *computably embeds into* **X***.*

Proof.

1. Assume otherwise.
2. By the second theorem, there is a $\{\mathbf{X}\}$-quasi-minimal $x \in \mathbb{R}_<$.
3. By the first theorem, there is an effectively $T_{2.5}$-space **Y** with some $y \in \mathbf{Y}$ s.t. $x \equiv_{\mathrm{T}} y$.
4. But then the degree of x also appears in **X**, contradiction.

□

Wait, what?

Corollary

*There is no countably-based T_1 space **X** such that every effectively $T_{2.5}$-space **Y** computably embeds into **X**.*

Question

Arno, have you mixed up where to put computable/effective again?

This must work, right?

1. Take an enumeration $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ of the effectively $T_{2.5}$-spaces.
2. Consider $\mathbf{X} = \biguplus_{n\in\mathbb{N}}(\{n\} \times \mathbf{Y}_n)$ – this may not be effective, but it surely is $T_{2.5}$
3. ???

□

The conclusion

Corollary

There are uncountable many effectively $T_{2.5}$ countably-based spaces (for real, not just a metric spaces vs its Cauchy completion thing).

Understanding what is going on

Definition

Let $\mathbb{T} = \{0, 1, \bot\}$ by represented via $\delta_\mathbb{T}(0^{2n}1p) = 0$, $\delta_\mathbb{T}(0^{2n+1}1p) = 1$ and $\delta_\mathbb{T}(0^\omega) = \bot$.

Definition

Call a subspace $\mathbf{X} \subseteq \mathbb{T}^\omega$ separated, if whenever $x, y \in \mathbf{X}$ with $x \neq y$, then there is some $n \in \mathbb{N}$ with $\{x(n), y(n)\} = \{0, 1\}$.

Proposition

The effectively Hausdorff countably-based spaces are exactly those computably isomorphic to separated subspaces of $\mathbb{T}^\omega$.

Understanding what is going on II

Recall: A space is $T_{2.5}$ iff any two distinct points are separated by open neighborhoods with disjoint closures.

Definition

Let $\mathbb{W} = \{\ell, m, r, \bot_\ell, \bot_r\}$ by represented via $\delta_\mathbb{W}(00^{2n}1p) = \ell$, $\delta_\mathbb{W}(00^\omega) = \bot_\ell$, $\delta_\mathbb{W}(b0^{2n+1}1p) = m$, $\delta_\mathbb{W}(10^\omega) = \bot_r$ and $\delta_\mathbb{W}(10^{2n}1p) = r$.

Definition

Call a subspace $\mathbf{X} \subseteq \mathbb{W}^\omega$ well-separated, if whenever $x, y \in \mathbf{X}$ with $x \neq y$, then there is some $n \in \mathbb{N}$ with $\{x(n), y(n)\} = \{\ell, r\}$.

Proposition

The effectively $T_{2.5}$ countably-based spaces are exactly those computably isomorphic to well-separated subspaces of $\mathbb{T}^\omega$.

Some open questions

Question

Is there any (boldface) pointclass Γ such that every separated subspace of $\mathbb{T}^\omega$ is included in a separated Γ-subspace?

Question

What is the cofinality of the Hausdorff countably-based spaces ordered by embeddability? Can we at least relate it to $\mathfrak{c}$?

Question

As above, but for T_1 and $T_{2.5}$.

Spaces of algebraic structures

Observation

For a fixed countable signature $\mathcal{S}$ plus some axioms, there is a represented spaces $\mathfrak{S}$ of $\mathcal{S}$-structures up to isomorphism.

Observation

The spectrum of a model M is the Muchnik degree corresponding to the Medvedev degree $M^{\mathfrak{S}}$.

Question

Are there any interesting connections between spaces $\mathfrak{S}$ of structures and topologically-inspired degree properties?

Russell Miller's recent work explored adding to the signature to obtain nice topologies.

A small observation

Proposition (Shafer & P.)

The successor of a Turing degree in the Medvedev degrees is never the degree of a point in an admissible (ie topological) space.

Theorem (Slaman)

There is a space of the form $\mathfrak{S}$ whose degree is the successor of $\emptyset$.

The articles

T. Kihara & A. Pauly.
Point degree spectra of represented spaces.
arXiv 1405.6866, 2014.

T. Kihara, K.M. Ng & A. Pauly.
Enumeration degrees and non-metrizable topology.
arXiv 1904.04107, 2019.

Plenty of open questions here!