Canonical Models of Determinacy

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Online Logic Seminar

TECHNISCHE UNIVERSITÄT WIEN

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Not all questions in mathematics can be answered in ZFC

Abstractly: Gödel's incompleteness theorems.

→ there are statements that are independent of ZFC

↑ standard axioms for math

Nowadays there are numerous concrete examples:

Is there a set A s.t. $|\mathbb{N}| < |A| < |\mathbb{R}|$?

- Continuum Problem (set theory),
  (Gödel 1938, Cohen 1960's)
- Whitehead Problem (group theory),
  (Shelah 1974)
- Borel Conjecture (measure theory), (Laver, 1976)
- Kaplansky's Conjecture on Banach algebras (analysis),
  (Dales-Esterle, Solovay, 1976)
- Brown-Douglas-Fillmore Problem (operator algebras),
  (Phillips-Weaver, 2006; Farah 2011)
- ....

⇝ We need to find "right axioms" that answer these questions.

The Continuum Problem

Let us focus on the Continuum Problem:

"How many reals are there?"

Question

Is there a set A such that $|\mathbb{N}| < |A| < |\mathbb{R}|$?

What are possible extensions of ZF? How do they decide it?

Determinacy Axioms	Large Cardinals	Forcing Axioms
↓	↓	↓
CH holds, i.e., there is no such A	Don't influence CH.	CH is false, in fact there is exactly one such intermediate size

Actually, the picture is more complicated. –

Determinacy Axioms: Games in set theory

Fix a set $A \subseteq {}^{\mathbb{N}}\mathbb{N}$ ("$= \mathbb{R}$")

I	n_0		n_2		$\dots$
II		n_1		n_3	$\dots$

Player I wins iff $(n_0, n_1, \dots) \in A$.

O/w Player II wins.

A function $\sigma : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ is a winning strategy for I in $G(A)$ iff

I	$\sigma(\emptyset)$		$\sigma(\sigma(\emptyset), n_1)$		$\dots$
II		n_1		n_3	$\dots$

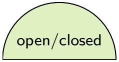
$\in A$.

Def: The set A is determined iff one of the players has a winning strategy.

The Axiom of Determinacy says:

Every set $A \subseteq {}^{\mathbb{N}}\mathbb{N}$ is determined.

Which games are determined?



Gale-Stewart (1953), ZFC

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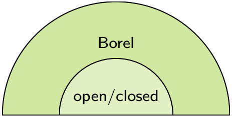

Martin (1975), ZFC

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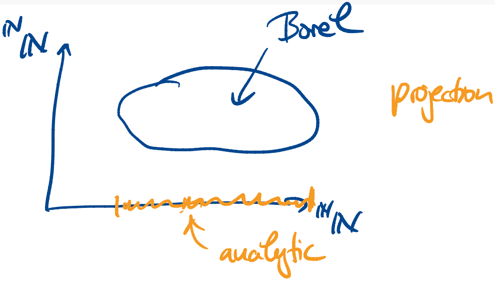

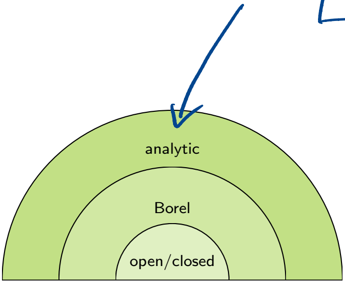

Martin (1970), measurable cardinal

Martin (1975), ZFC

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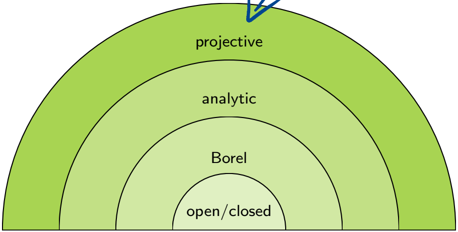
Which games are determined?

Iteratively build complements & projections



Martin-Steel (1985), Woodin cardinals and a measurable cardinal

Martin (1970), measurable cardinal

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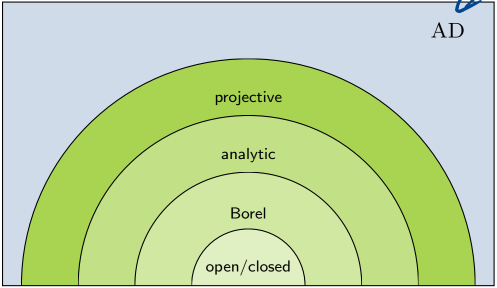
Gale-Stewart (1953), ZFC

Which games are determined?

Axiom of Determinacy



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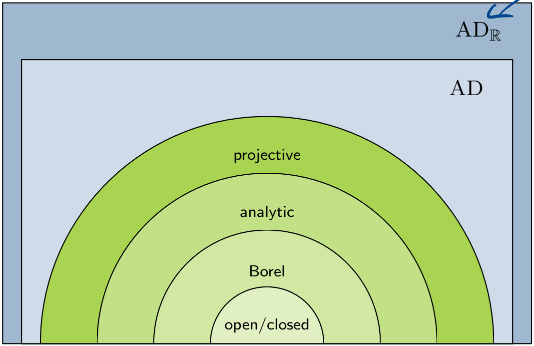
Which games are determined?

Determinacy for games on $\mathbb{R}$

$$\begin{array}{c|cc} \mathrm{I} & x_0 \in \mathbb{R} & \dots \\ \hline \mathrm{II} & & x_1 \in \mathbb{R} \quad \dots \end{array}$$



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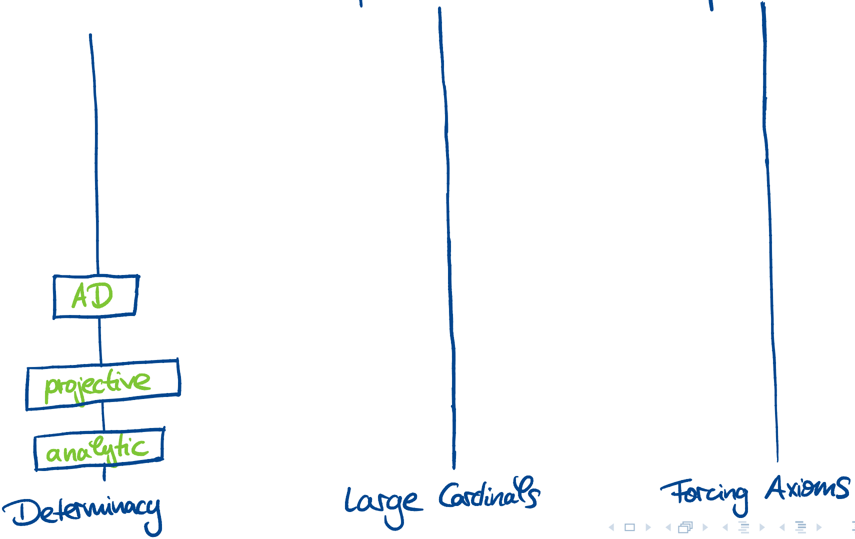
How far are these axioms from ZFC? "Steel's Program"

Consider hierarchies of these axioms and compare their strength.



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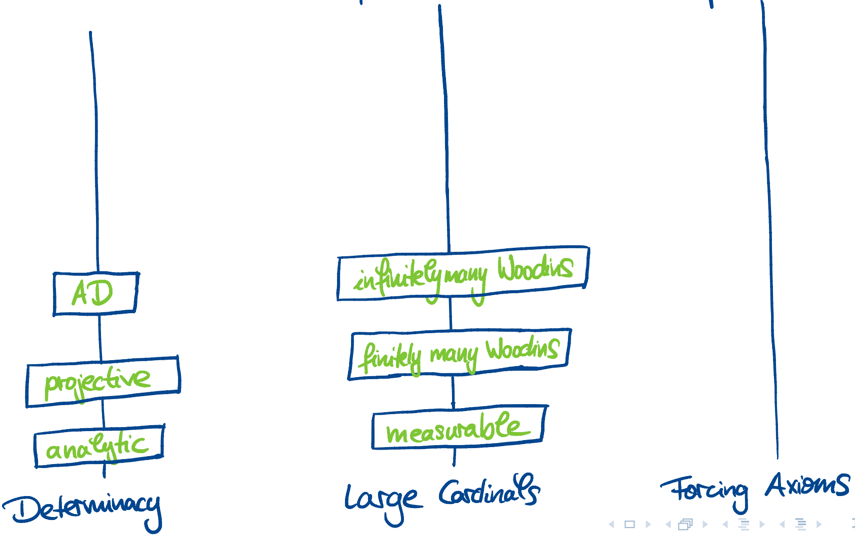

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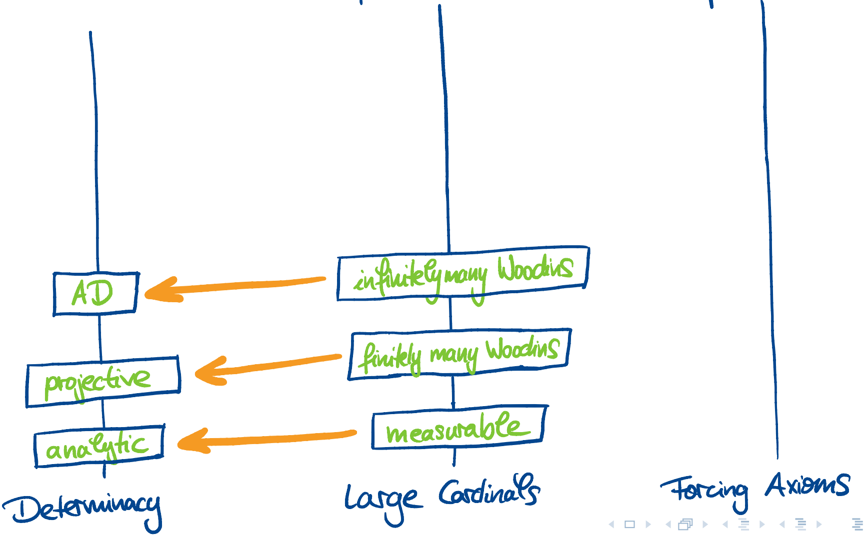
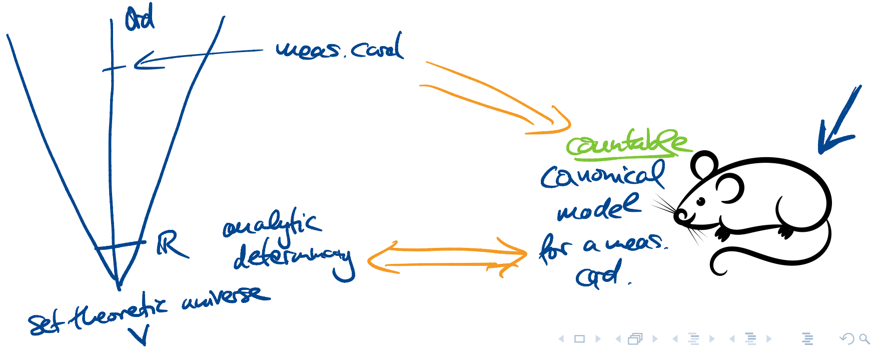


Determinacy and large cardinals

Are large cardinals necessary for the determinacy of these sets of reals?

In some sense...

How can these large cardinals affect what happens with the sets of reals?



Equivalences for analytic and projective determinacy

Theorem (Harrington, Martin)

The following are equivalent.

1. *All analytic sets are determined.*
2. *$x^\#$ exists for all reals x.*

canonical mouse with a meas. card.

Theorem (Neeman, Woodin)

The following are equivalent for all $n \geq 1$.

1. *All Σ^1_{n+1} sets are determined.*
2. *For every real x the ω_1-iterable countable model of set theory with n Woodin cardinals $M_n^\#(x)$ exists.*

levels of projective hierarchy

canonical mouse with n Woodin cards + meas. card

For (1) ⇒ (2) see (M-Schindler-Woodin) "Mice with Finitely many Woodin Cardinals from Optimal Determinacy Hypotheses", JML 2020.

For (2) ⇒ (1) see (Neeman) "Optimal proofs of determinacy II", JML 2002.

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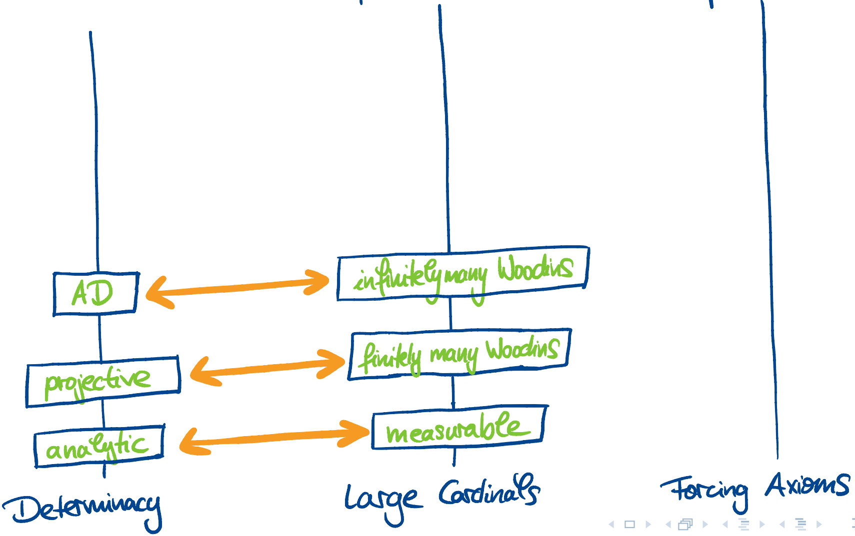

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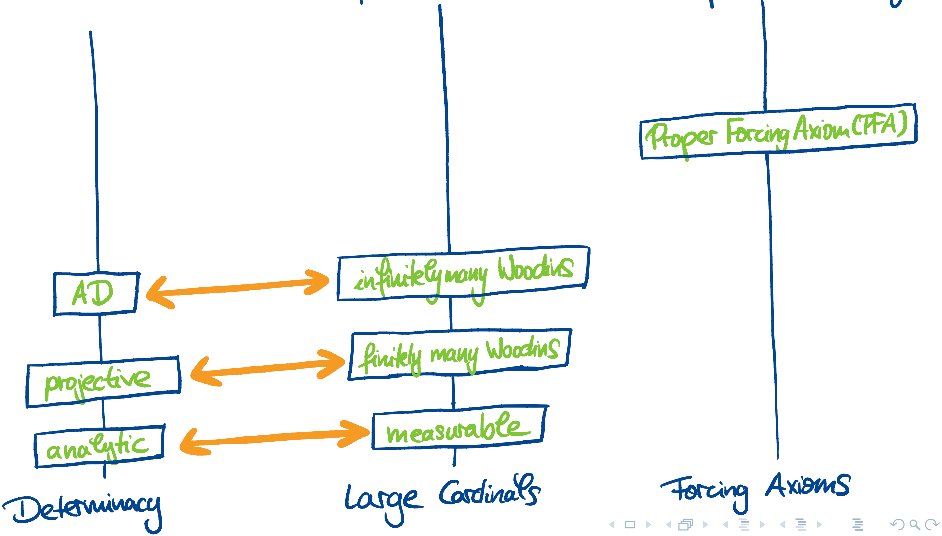

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standard axioms for set theory

Under PFA

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Is there a set A such that $|\mathbb{N}| < |A| < |\mathbb{R}|$? Yes

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  (Gödel 1938, Cohen 1960's)
- Whitehead Problem (group theory), (Shelah, 1974)
  No, there is a non-free Whitehead group
- Borel Conjecture (measure theory), (Laver, 1976)
  is false
- Kaplansky's Conjecture on Banach algebras (analysis),
  is true (Todorčević, 1989) (Dales-Esterle, Solovay, 1976)
- Brown-Douglas-Fillmore Problem (operator algebras),
  (Phillips-Weaver, 2006; Farah, 2011)
  Every automorphism of the Calkin algebra is inner
- ....

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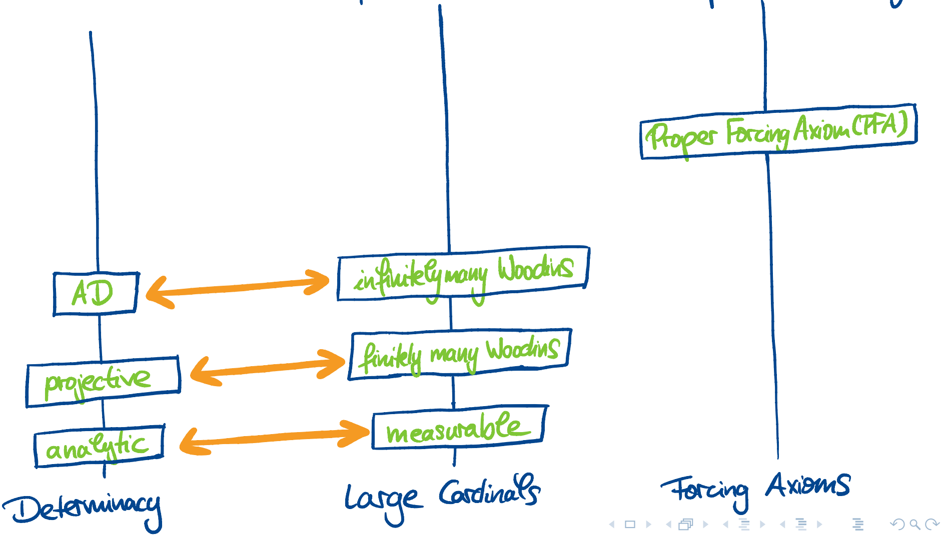

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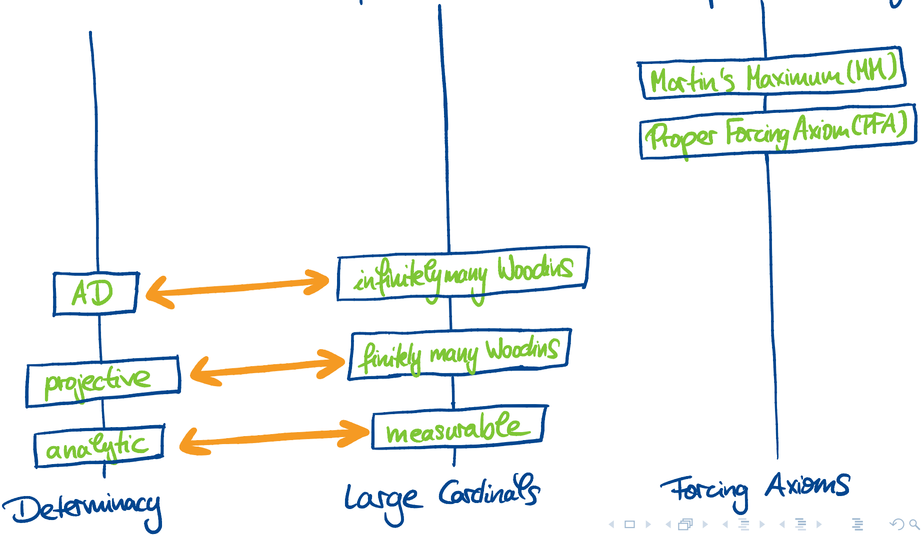

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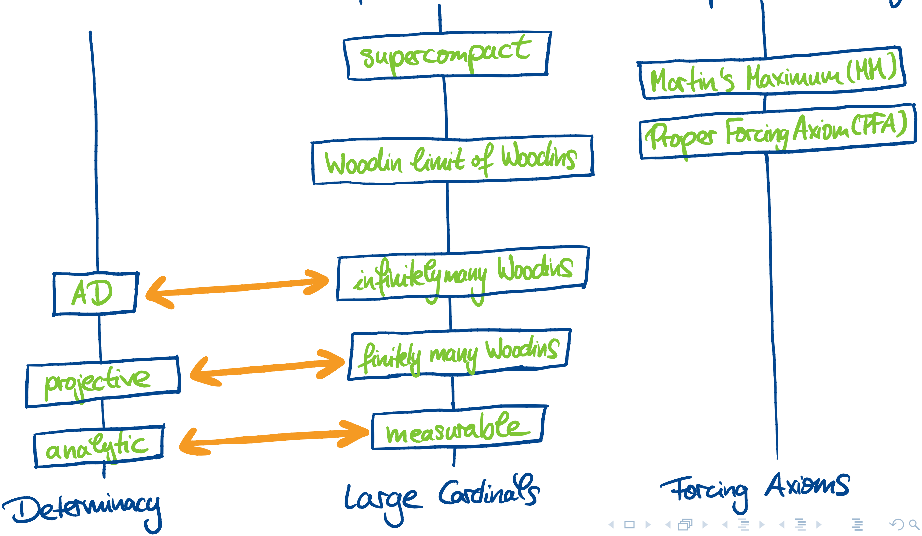

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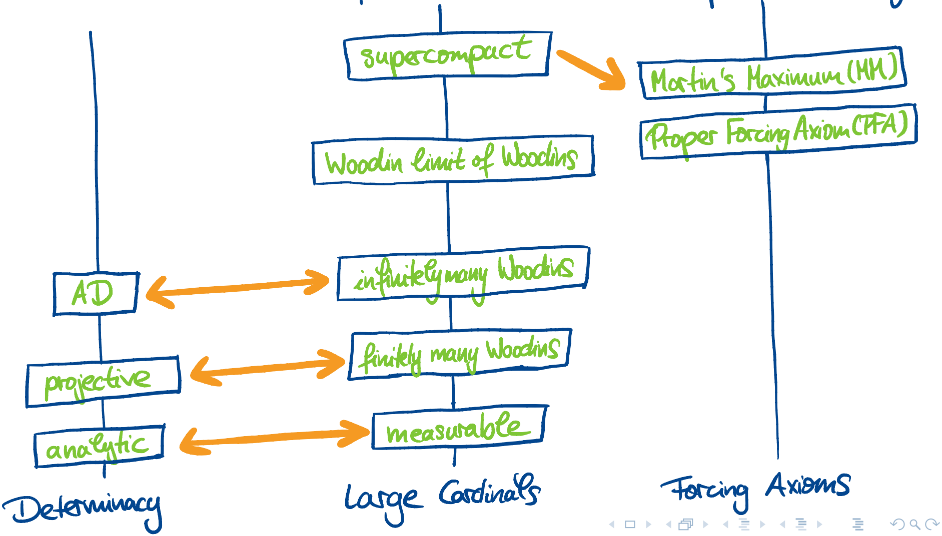

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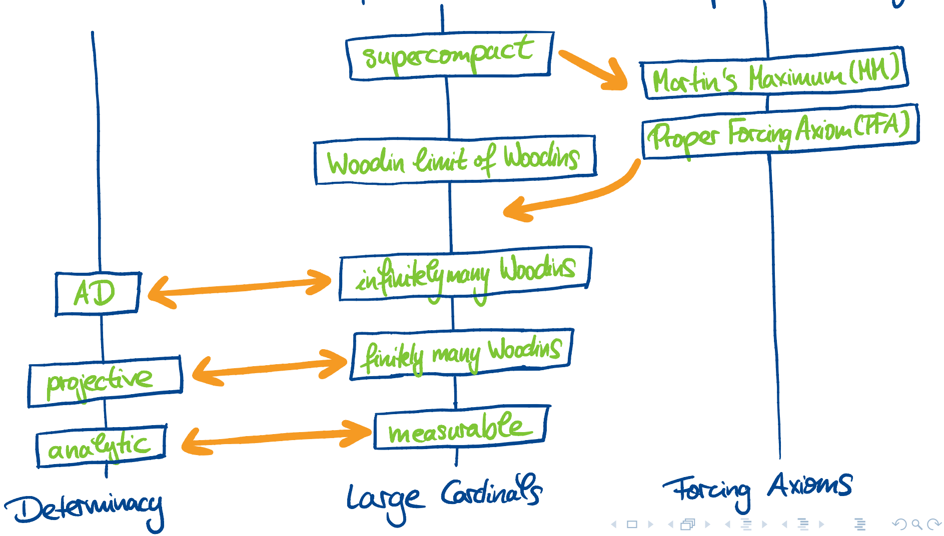

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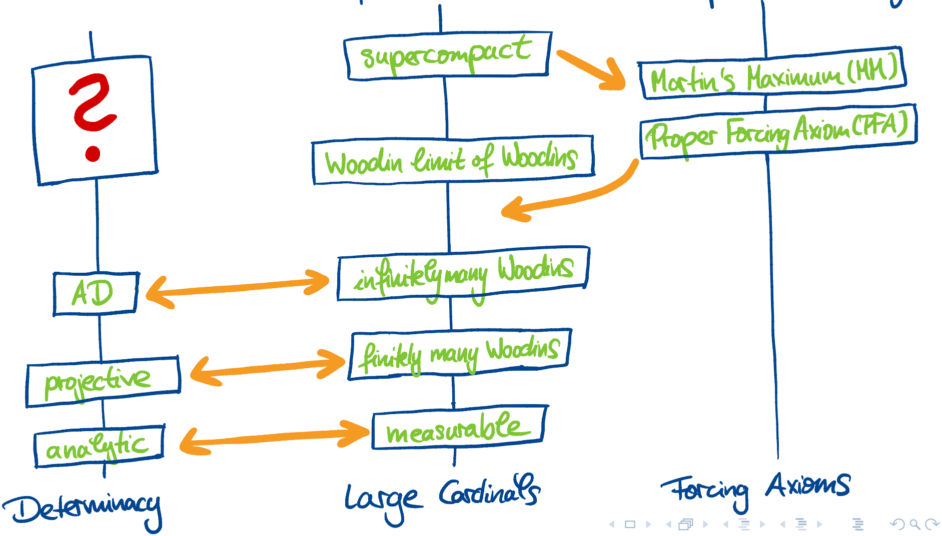

Strong axioms of determinacy

Keep playing games of length ω and impose additional structural properties on the model.

$\mathrm{AD}+$ all sets of reals are Suslin

Being Suslin is a generalization of being analytic. More precisely, a set of reals is *Suslin* if it is the projection of a tree on $\omega \times \kappa$ for some ordinal κ.

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Theorem (Woodin, Derived model construction, 1980's)

Suppose there is a cardinal λ that is

- *a limit of Woodin cardinals, and*
- *a limit of $<\lambda$-strong cardinals.*

Then there is a model of

$$\text{“}\mathrm{AD}+ \textit{all sets of reals are Suslin}\text{”}.$$

Is this optimal?

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Theorem (Steel, 2008)

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A further strengthening: universally Baire sets

Definition (Schilling-Vaught, Feng-Magidor-Woodin)

A subset A of a topological space Y is *universally Baire* if $f^{-1}(A)$ has the property of Baire in any topological space X, where $f\colon X \to Y$ is continuous.

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Definition

Let (S, T) be trees on $\omega \times \kappa$ for some ordinal κ and let Z be any set. We say (S, T) *is Z-absolutely complementing* iff $p[S] = {}^{\omega}\omega \setminus p[T]$ in every $\mathrm{Col}(\omega, Z)$-generic extension of V.

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Definition (Feng-Magidor-Woodin)

A set of reals A is *universally Baire (uB)* if for every Z, there are Z-absolutely complementing trees (S, T) with $p[S] = A$.

Can all sets be universally Baire?

Is there a model of determinacy in which all sets are universally Baire?

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Suppose there is a cardinal λ that is

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Conjecture (Sargsyan)

This is optimal.

Sargsyan's conjecture holds

Theorem (M, 2021)

Suppose there is a proper class model of

$$\text{“}\mathrm{AD} + \textit{all sets of reals are universally Baire}\text{”}.$$

Then there is a transitive model $\mathcal{M}$ of ZFC containing all ordinals such that $\mathcal{M}$ has a cardinal λ that is

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Det | hybrid mice | LC

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The proof is based on a new translation procedure to translate iteration strategies in hybrid mice into large cardinals. This extends work of Steel, Zhu, and Sargsyan.

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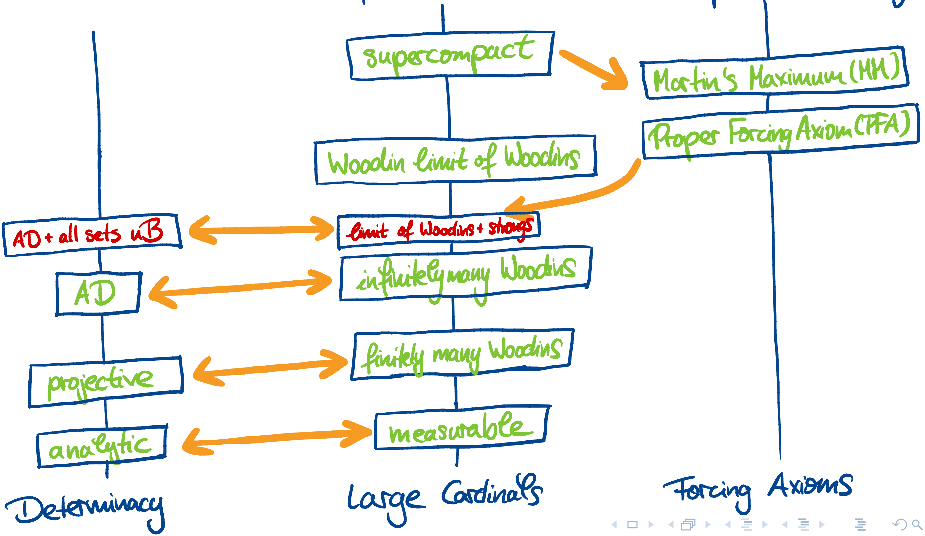

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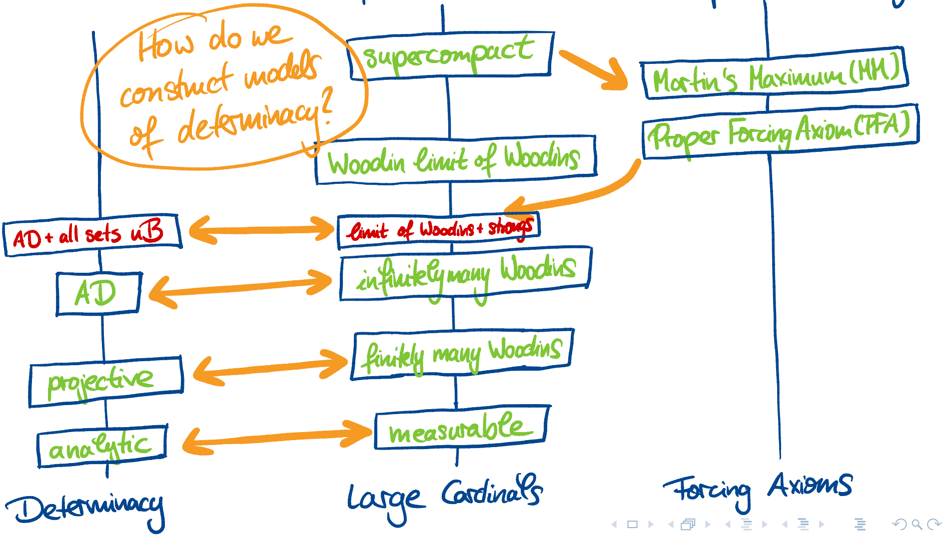

Derived models of determinacy

We say a model M is a *derived model* if it is of the following form:

$$\left(L(\mathbb{R}^*, \mathrm{Hom}^*)\right)^{V[G]}$$

$\bigcup_{\alpha<\lambda} \mathbb{R} \cap V[G\restriction\alpha]$

$\mathrm{Hom}_\gamma = \{A \subseteq {}^\omega\omega \mid A \text{ is hom. Suslin}\}$

$\mathrm{Col}(\omega, <\lambda)$ - generic, λ limit of Woodin cards in V

$\mathrm{Hom}^* = \{A^* \mid \exists \alpha<\lambda \;\; A \in (\mathrm{Hom}_{<\lambda})^{V[G\restriction\alpha]}\}$

In this setting, $L(\mathbb{R}^*, \mathrm{Hom}^*) \models AD^+$

Derived models of determinacy

Woodin showed that all models of AD^+ are elementarily equivalent to a model of AD^+ that can be obtained as a derived model.

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A useful representation for the uB sets.

Theorem (Sargsyan-M, 2022)

Let κ be a supercompact cardinal and suppose there is a proper class of Woodin cardinals. Let $g \subseteq \mathrm{Col}(\omega, 2^\kappa)$ be V-generic, h be $V[g]$-generic and $k \subseteq \mathrm{Col}(\omega, 2^\omega)$ be $V[g*h]$-generic. Then, in $V[g*h*k]$, there is $j : V \to M$ such that $j(\kappa) = \omega_1^{M[g*h]}$ and $L(\Gamma^\infty_{g*h}, \mathbb{R}_{g*h})$ is a derived model of M, i.e., for some M-generic $G \subseteq \mathrm{Col}(\omega, {<}\omega_1^{V[g*h]})$,

$$L(\Gamma^\infty_{g*h}, \mathbb{R}_{g*h}) = (L(\mathrm{Hom}^*, \mathbb{R}^*))^{M[G]}.$$

all uB sets in $V[g*h]$

$\mathbb{R}$ in $V[g*h]$

derived model

Why is this useful?

Woodin showed this in $V[g]$, $g \subseteq Col(\omega, 2^{2^\kappa})$ + proper class of Woodins, κ supercompact card.

Definition (Woodin)

Sealing is the conjunction of the following statements.

1. For every set generic g over V, $L(\Gamma_g^\infty, \mathbb{R}_g) \models \mathrm{AD}^+$ and $\mathcal{P}(\mathbb{R}_g) \cap L(\Gamma_g^\infty, \mathbb{R}_g) = \Gamma_g^\infty$.

   uB sets in $V[g]$

2. For every set generic g over V and set generic h over $V[g]$, there is an elementary embedding

   "freezing the theory of the uB sets"

   $$j: \underbrace{L(\Gamma_g^\infty, \mathbb{R}_g)}_{V[g]} \to \underbrace{L(\Gamma_{g*h}^\infty, \mathbb{R}_{g*h})}_{V[g*h]}$$

   such that for every $A \in \Gamma_g^\infty$, $j(A) = A_h$.

Our DM-representation gives Sealing also for stronger models of determinacy.

Supercompactness in models of determinacy

Conjecture

The following theories are equiconsistent:

1. ZFC + there is a Woodin cardinal that is a limit of Woodin cardinals.
2. $\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular + ω_1 is supercompact.

Supercompactness in models of determinacy

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Theorem (Woodin)

Suppose there is a proper class of Woodin cardinals that are limits of Woodin cardinals. Then there is a model of "$\mathrm{AD}_{\mathbb{R}} + \omega_1$ is supercompact."

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Theorem (Gappo-M-Sargsyan, 2023)

Suppose there is a Woodin cardinal that is a limit of Woodin cardinals. Then there is a model of "$\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular + ω_1 is $< \delta_\infty$-supercompact" for some $\delta_\infty > \Theta$.

Chang-type models

Possibilities for strong models of determinacy:

$L(\mathbb{R})$ $\quad \mathbb{R} = \omega^\omega$

$L(A, \mathbb{R})$ a uB set

$L(\Gamma^\infty, \mathbb{R})$ all uB sets

$L(\Gamma^\infty, \mathbb{R})[\mu]$ measure

$L(\mathrm{Ord}^\omega)$ "Chang model"

$L(\mathrm{Ord}^\omega, \Gamma^\infty, \mathbb{R})$

$L(\mathrm{Ord}^\omega, \Gamma^\infty, \mathbb{R})[\mu_\kappa \mid \kappa \in \mathrm{Ord}]$

$L(\mathcal{P}_{uB}(\Gamma^\infty))$

Conjecture

These models can be respresented as derived models (i.e., canonical models of determinacy).

My vision

The connection between determinacy and inner models should continue throughout the large cardinal hierarchy.

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It should for example ultimately also yield the exact consistency strength of the Proper Forcing Axiom (PFA).

My vision

AD + all sets uB

Det

hybrid mice

LC (mice w/ LC)

Woodins → strongs

The connection between determinacy and inner models should continue throughout the large cardinal hierarchy.

iteration strategies

It should for example ultimately also yield the exact consistency strength of the Proper Forcing Axiom (PFA).

The main barrier we are currently facing is a Woodin limit of Woodin cardinals.

Links to the images:

- https://pixabay.com/illustrations/chess-play-relax-think-chess-board-1019908/
- https://pixabay.com/illustrations/chaos-regulation-chaos-theory-485493/
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