Canonical Models of Determinacy

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Online Logic Seminar

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Not all questions in mathematics can be answered in ZFC

Abstractly: Gödel’s incompleteness theorems.

- There are statements that are independent of ZFC

Nowadays there are numerous concrete examples:

- Continuum Problem (set theory),
  (Gödel 1938, Cohen 1960s)
- Whitehead Problem (group theory),
  (Shelah 1974)
- Borel Conjecture (measure theory),
  (Laver, 1976)
- Kaplansky’s Conjecture on Banach algebras (analysis),
  (Dales-Eskenazi-Solovay, 1976)
- Brown-Douglas-Fillmore Problem (operator algebras),
  (Phillips-Weaver, 2006; Farah 2011)

We need to find “right axioms” that answer these questions.
The Continuum Problem

Let us focus on the Continuum Problem:

Question

Is there a set $A$ such that $|\mathbb{N}| < |A| < |\mathbb{R}|$?

“How many reals are there?”

What are possible extensions of $ZF$? How do they decide it?

Determinacy Axioms

Large Cardinals

Forcing Axioms

$CH$ holds, i.e., there is no such $A$

Don’t influence $CH$.

Actually, the picture is more complicated.

$CH$ is false, in fact, there is exactly one such intermediate size.
Determinacy Axioms: Games in set theory

Fix a set $A \subseteq {}^\omega \omega$ ("= $\mathbb{R}$")

$$
\begin{array}{c|cc}
I & n_0 & n_2 \\
\hline
I & n_1 & n_3 \\
\end{array}
$$

Player I wins iff

$$(n_0, n_1, \ldots) \in A.$$

Or Player II wins.

A function $\sigma : \mathbb{N}^{<\omega} \to \mathbb{N}$ is a winning strategy for I in $G(A)$ iff

$$
\begin{array}{c|cc}
I & \sigma(\emptyset) & \sigma(\sigma(\emptyset), n) \\
\hline
I & n_1 & n_3 \\
\end{array}
$$

$$(n_1, n_3) \in A.
$$

Def: The set $A$ is determined iff one of the players has a winning strategy.

The Axiom of Determinacy says:

Every set $A \subseteq {}^\omega \omega$ is determined.
Which games are determined?

- Open/closed: Gale-Stewart (1953), ZFC
Which games are determined?

- Gale-Stewart (1953), ZFC

- Martin (1975), ZFC

- Gale-Stewart (1953), ZFC
Which games are determined?

- Gale-Stewart (1953), ZFC
- Martin (1970), measurable cardinal
- Martin (1975), ZFC
- Gale-Stewart (1953), ZFC
Determinacy Axioms

Which games are determined?

- Gale-Stewart (1953), ZFC
  - open/closed Borel
- Martin (1975), ZFC
  - open/closed Borel analytic
- Martin (1970), measurable cardinal
- Martin-Steel (1985), Woodin cardinals and a measurable cardinal
- Martin (1975), ZFC
- Gale-Stewart (1953), ZFC

Hereditarily build complements & projections
Which games are determined?

- Martin-Steel (1985), Woodin cardinals and a measurable cardinal
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Determinacy Axioms

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Gale-Stewart (1953), ZFC
How far are these axioms from ZFC?

"Steel's Program"

Consider hierarchies of these axioms and compare their strength.
How far are these axioms from ZFC? "Steel's Program"

Consider hierarchies of these axioms and compare their strength.

Determinacy
  - projective
  - analytic
  - AD

Large Cardinals
  - finitely many Woodins
  - measurable
  - infinitely many Woodins

Forcing Axioms
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Projective

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Infinitely many Woodins

Finitely many Woodins

Measurable

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Forcing Axioms

Determinacy Axioms

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Determinacy and large cardinals

Are large cardinals necessary for the determinacy of these sets of reals?

In some sense...

How can these large cardinals affect what happens with the sets of reals?
Determinacy Axioms

Equivalences for analytic and projective determinacy

Theorem (Harrington, Martin)

The following are equivalent.

1. All analytic sets are determined.
2. $x \#$ exists for all reals $x$.

Theorem (Neeman, Woodin)

The following are equivalent for all $n \geq 1$.

1. All $\Sigma^1_{n+1}$ sets are determined.
2. For every real $x$ the $\omega_1$-iterable countable model of set theory with $n$ Woodin cardinals $M_n^\#(x)$ exists.

For (1) $\Rightarrow$ (2) see (M-Schindler-Woodin) “Mice with Finitely many Woodin Cardinals from Optimal Determinacy Hypotheses”, JML 2020.

For (2) $\Rightarrow$ (1) see (Neeman) “Optimal proofs of determinacy II”, JML 2002.
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- Proper Forcing Axiom (PFA)

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Nowadays there are numerous concrete examples:

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  (Gödel 1938, Cohen 1960’s)
- Whitehead Problem (group theory),
  No, there is a non-free Whitehead group (Shelah, 1974)
- Borel Conjecture (measure theory),
  Is false (Laver, 1976)
- Kaplansky’s Conjecture on Banach algebras (analysis),
  Is true (Todorčević, 1989) (Dales-Esposito, Solovay, 1976)
- Brown-Douglas-Fillmore Problem (operator algebras),
  Every automorphism of the Calkin algebra is inner (Phillips-Weaver, 2006; Farah, 2011)

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Determinacy **→** Forcing Axioms

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- Infinitely many Woodins
- Finitely many Woodins
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- Proper Forcing Axiom (PFA)

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How far are these axioms from ZFC?

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Determinacy
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Forcing Axioms
- Martin's Maximum (MM)
- Proper Forcing Axiom (PFA)

Large Cardinals
- infinitely many Woodins
  - finitely many Woodins
  - measurable

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Ad

Projective

Analytic

Large Cardinals

Infinetly many Woodins

Finitely many Woodins

Measurable

Supercompact

Woodin limit of Woodins

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    - Woodin limit of Woodins
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      - Martin's Maximum (MM)
Strong axioms of determinacy

Keep playing games of length $\omega$ and impose additional structural properties on the model.
AD + all sets of reals are Suslin

Being Suslin is a generalization of being analytic. More precisely, a set of reals is *Suslin* if it is the projection of a tree on $\omega \times \kappa$ for some ordinal $\kappa$. 
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Theorem (Woodin, Derived model construction, 1980’s)

Suppose there is a cardinal $\lambda$ that is

- a limit of Woodin cardinals, and
- a limit of $<\lambda$-strong cardinals.

Then there is a model of

\[ \text{“AD + all sets of reals are Suslin”}. \]

Is this optimal?
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Theorem (Steel, 2008)

This is optimal.
A further strengthening: universally Baire sets

Definition (Schilling-Vaught, Feng-Magidor-Woodin)
A subset $A$ of a topological space $Y$ is universally Baire if $f^{-1}(A)$ has the property of Baire in any topological space $X$, where $f: X \to Y$ is continuous.
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Definition
Let $(S, T)$ be trees on $\omega \times \kappa$ for some ordinal $\kappa$ and let $Z$ be any set. We say $(S, T)$ is $Z$-absolutely complementing iff $p[S] = \omega \omega \setminus p[T]$ in every $\text{Col}(\omega, Z)$-generic extension of $V$. 
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Definition (Feng-Magidor-Woodin)
A set of reals $A$ is universally Baire ($uB$) if for every $Z$, there are $Z$-absolutely complementing trees $(S, T)$ with $p[S] = A$. 
Can all sets be universally Baire?

Is there a model of determinacy in which all sets are universally Baire?
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Is there a model of determinacy in which all sets are universally Baire?

Theorem (Larson-Sargsyan-Wilson, 2014)

Suppose there is a cardinal $\lambda$ that is
- a limit of Woodin cardinals, and
- a limit of (fully) strong cardinals.

Then there is a model of

“$\text{AD} + \text{ all sets of reals are universally Baire}$”.

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Can all sets be universally Baire?

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Conjecture (Sargsyan)

This is optimal.
Sargsyan’s conjecture holds

Theorem (M, 2021)

Suppose there is a proper class model of

“\( \text{AD} \ + \ \text{all sets of reals are universally Baire} \)”.

Then there is a transitive model \( \mathcal{M} \) of ZFC containing all ordinals such that \( \mathcal{M} \) has a cardinal \( \lambda \) that is

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The proof is based on a new translation procedure to translate iteration strategies in hybrid mice into large cardinals. This extends work of Steel, Zhu, and Sargsyan.
How far are these axioms from ZFC?

Consider hierarchies of these axioms and compare their strength.

Determinacy

Analytic → Measurable → Finitely Many Woodins

Projective → Infinitely Many Woodins → Limit of Woodins + Strong

AD → Supercompact → Martin's Maximum (MM)

AD + All Sets nB → Proper Forcing Axiom (PFA)

Large Cardinals

Forcing Axioms

"Steel's Program"
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Determinacy Axioms

Steel's Program

How do we construct models of determinacy?

- AD
- Projective
- Analytic
- AD + all sets nB

Determinacy

- Supercompact
- Woodin limit of Woodins
- Martin's Maximum (MM)
- Proper Forcing Axiom (PFA)

Large Cardinals

- Limit of Woodins + strongs
- Infinitely many Woodins

Forcing Axioms

- Finitely many Woodins
- Measurable
Derived models of determinacy

We say a model $M$ is a derived model if it is of the following form:

\[
\bigcup_{d<\omega} \mathcal{R} \cap \mathcal{V}(\mathcal{G}(\alpha))
\]

\[
\Hom_{\gamma} = \{ A \subseteq \omega \mid A \text{ is non-surj} \}
\]

\[
\Hom^* = \{ A^x \mid x \in 2 \gamma \} \quad A \in (\text{Hom}_{\omega})_{\text{reg}}
\]

In this setting, $L(\mathcal{R}^*, \Hom^*) = AD^+$
Woodin showed that all models of $\mathsf{AD}^+$ are elementarily equivalent to a model of $\mathsf{AD}^+$ that can be obtained as a derived model.
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To useful representation for the $uB$ sets.

Theorem (Sargsyan-M, 2022)

Let $\kappa$ be a supercompact cardinal and suppose there is a proper class of Woodin cardinals. Let $g \subseteq \text{Col}(\omega, 2^\kappa)$ be $V$-generic, $h$ be $V[g]$-generic and $k \subseteq \text{Col}(\omega, 2^\omega)$ be $V[g \ast h]$-generic. Then, in $V[g \ast h \ast k]$, there is $j : V \to M$ such that $j(\kappa) = \omega_1^{M[g \ast h]}$ and $L(\Gamma_{g \ast h}^\infty, R_{g \ast h})$ is a derived model of $M$, i.e., for some $M$-generic $G \subseteq \text{Col}(\omega, <\omega_1^{V[g \ast h]})$,

$$L(\Gamma_{g \ast h}^\infty, R_{g \ast h}) = (L(\text{Hom}^*, R^*))^{M[G]}.$$
Why is this useful?

Definition (Woodin)
Sealing is the conjunction of the following statements.

1. For every set generic $g$ over $V$, $L(\Gamma^\infty_g, \mathbb{R}_g) \models \text{AD}^+$ and $\mathcal{P}(\mathbb{R}_g) \cap L(\Gamma^\infty_g, \mathbb{R}_g) = \Gamma^\infty_g$.

2. For every set generic $g$ over $V$ and set generic $h$ over $V[g]$, there is an elementary embedding $j: L(\Gamma^\infty_g, \mathbb{R}_g) \rightarrow L(\Gamma^\infty_{g*h}, \mathbb{R}_{g*h})$ such that for every $A \in \Gamma^\infty_g$, $j(A) = A_h$.

Woodin showed this in VEG, $g \in \text{Col}(\omega, \mathbb{R})^+$ + supercompact and.

"Sealing the theory of the uB sets" Our DM-representation gives sealing also for stronger models of determinacy.
Supercompactness in models of determinacy

Conjecture

The following theories are equiconsistent:

1. $\text{ZFC} + \text{there is a Woodin cardinal that is a limit of Woodin cardinals.}$
2. $\text{AD}^R + \Theta \text{ is regular} + \omega_1 \text{ is supercompact.}$
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2. $\text{AD}_R + \Theta$ is regular $+ \omega_1$ is supercompact.

Theorem (Woodin)

Suppose there is a proper class of Woodin cardinals that are limits of Woodin cardinals. Then there is a model of “$\text{AD}_R + \omega_1$ is supercompact.”
Conjecture

The following theories are equiconsistent:

1. \( \text{ZFC} + \) there is a Woodin cardinal that is a limit of Woodin cardinals.
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Theorem (Woodin)

Suppose there is a proper class of Woodin cardinals that are limits of Woodin cardinals. Then there is a model of “\( \text{AD}_\mathbb{R} + \omega_1 \text{ is supercompact.} \)”

Theorem (Gappo-M-Sargsyan, 2023)

Suppose there is a Woodin cardinal that is a limit of Woodin cardinals. Then there is a model of “\( \text{AD}_\mathbb{R} + \Theta \text{ is regular } + \omega_1 \text{ is } < \delta_\infty \text{-supercompact} \)” for some \( \delta_\infty > \Theta \).
Chang-type models

Possibilities for strong models of determinacy:

\[
L(C(R))^w, \quad L(A,R) \quad \forall \text{uB set}, \quad L(C(R)[\mu]), \quad L(C(R)\models \text{all uB sets})
\]

\[
L(\text{Ord}^w), \quad L(\text{Ord}^w, \Gamma^w), \quad L(\text{Ord}^w, \Gamma^w, \Gamma^w) \quad \Gamma^w \models \text{Ord}
\]

Conjecture

These models can be represented as derived models (i.e., canonical models of determinacy).
My vision

The connection between determinacy and inner models should continue throughout the large cardinal hierarchy.
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The Inner Model Program

My vision

The connection between determinacy and inner models should continue throughout the large cardinal hierarchy.

It should for example ultimately also yield the exact consistency strength of the Proper Forcing Axiom (PFA).

The main barrier we are currently facing is a Woodin limit of Woodin cardinals.
Links to the images: