On quantum Ramsey theory

José G. Mijares
California State University
Los Angeles

Online Logic Seminar
August 20, 2026

Ramsey's theorem 1929

Notation: $A^{[n]} = \{B \subseteq A : |B| = n\}, \quad n \in \mathbb{N}$ or $n = \infty$.

$M = \{0, 1, \ldots, M-1\},\ M \in \mathbb{N},\ M > 0.$

Infinite version: For every coloring of $c : \mathbb{N}^{[2]} \to 2$ there is $A \in \mathbb{N}^{[\infty]}$ such that $A^{[2]}$ is monochromatic for c.
(i.e., c is constant on $A^{[2]}$).

Finite version: Given $n \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that for every coloring $c : M^{[2]} \to 2$, there is $A \in M^{[n]}$ tal que $A^{[2]}$ is monochromatic for c.

Ramsey's theorem 1929

Notation: $A^{[n]} = \{B \subseteq A : |B| = n\}, \quad n \in \mathbb{N}$ or $n = \infty$.

$M = \{0, 1, \ldots, M-1\},\ M \in \mathbb{N},\ M > 0.$

Generalized infinite version: Given $n \in \mathbb{N}$, for every finite coloring of $\mathbb{N}^{[n]}$ there is $A \in \mathbb{N}^{[\infty]}$ such that $A^{[n]}$ is monochromatic.

Generalized finite version: Given $m, n, r \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that for every coloring $c : M^{[n]} \to r$, there is $A \in M^{[m]}$ tal que $A^{[n]}$ is monochromatic for c.

Classical relations

$R \subseteq A \times A$ (relation)
$\Delta = \{(x,x) : x \in A\}$ (diagonal)
$R^t = \{(x,y) : (y,x) \in R\}$ (transpose)

R es:
reflexive $\Leftrightarrow \Delta \subseteq R$
symmetric $\Leftrightarrow R^t = R$
a **graph**, if it is reflexive and symmetric.

Classical information theory

- A, B finite sets (messages, codes)
  $\mathcal{C} : A \to B$ (channel)
  $P(b/a) =$ probability that $\mathcal{C}(a) = b$
- $a \perp a' \Leftrightarrow P(b/a) \cdot P(b/a') = 0$ for all $b \in B$
  (a and a' are distinguishable)
- $a \not\perp a' \Leftrightarrow P(b/a) \cdot P(b/a') > 0$ for some b
  (a and a' are confusable)

$R = \{(a, a') : a \not\perp a'\} \subseteq A \times A$
$G(\mathcal{C}) = (A, R)$ the **confusability graph** of channel $\mathcal{C}$

Ramsey's theorem $\Rightarrow$ For every $m > 0$, there exists n such that for every channel $\mathcal{C}$ with $|A| \geq n$, $G(\mathcal{C})$ contains an independent set with cardinality m or a complete subgraph K_m.

Ramsey's theorem

Theorem (Ramsey, 1929) For every $m > 0$, there exists n such that every 2-coloring of $E(K_n)$ admits a subgraph K_m such that $E(K_m)$ is monochromatic.

Equivalently: For every $m > 0$, there exists n such that every graph G with $|V(G)| \geq n$ contains an independent set with cardinality m or a complete subgraph K_m.

Quantum relations

$\mathcal{V} \subseteq M_n = M_n(\mathbb{C})$ (quantum relation)
$\mathbb{C}I_n = \{zI_n : z \in \mathbb{C}\}$ (diagonal)
$\mathcal{V}^\dagger = \{A^\dagger : A \in \mathcal{V}\}$ (transpose conjugate)

$\mathcal{V}$ es:
reflexive $\Leftrightarrow$ $\mathbb{C}I_n \subseteq \mathcal{V}$
symmetric $\Leftrightarrow$ $\mathcal{V}^\dagger = \mathcal{V}$
a **quantum graph**, if it is reflexive and symmetric.

Quantum information theory

- (messages, codes) $m, n \in \mathbb{N}$, $M_m = M_m(\mathbb{C})$ and $M_n = M_n(\mathbb{C})$
- (channel) $\Phi : M_n \to M_m$
  Φ es **completely positive**: $\Phi \otimes I_k$ is positive for all integers $k > 0$.
  Φ is **trace-preserving**: $\mathrm{tr}(\Phi(\rho)) = \mathrm{tr}(\rho)$.

Choi's theorem: There exist matrices $K_1, K_2, \ldots, K_N \in M_{m\times n}$, $N \in \mathbb{N}$ (Kraus matrices), such that
(1) $\sum_{i=1}^N K_i^\dagger K_i = I_n$
(2) $\Phi(\rho) = \sum_{i=1}^N K_i \rho K_i^\dagger$

Quantum information theory

$\Phi : M_n \to M_m$, completely positive

Choi's theorem: There exist matrices $K_1, K_2, \ldots, K_N \in M_{m\times n}$, $N \in \mathbb{N}$ (Kraus matrices), such that
(1) $\sum_{i=1}^N K_i^\dagger K_i = I_n$
(2) $\Phi(\rho) = \sum_{i=1}^N K_i \rho K_i^\dagger$

$\mathcal{V}_\Phi = \mathrm{Span}\{K_i^\dagger K_j : 1 \leq i, j \leq N\} \subseteq M_n$.

Pure states (unit vectors) $\alpha = |\alpha\rangle\langle\alpha|$ and $\beta = |\beta\rangle\langle\beta|$ are:

distinguishable $\Leftrightarrow \langle\alpha|A|\beta\rangle = 0, \forall A \in \mathcal{V}_\Phi$
confusable $\Leftrightarrow \langle\alpha|A|\beta\rangle \neq 0$, for some $A \in \mathcal{V}_\Phi$.

Quantum information theory

$\mathcal{V}_\Phi = \mathrm{Span}\{K_i^\dagger K_j : 1 \leq i,j \leq N\}$.

Pure states (unit vectors) $\alpha = |\alpha\rangle\langle\alpha|$ and $\beta = |\beta\rangle\langle\beta|$ are: bounded

distinguishable $\Leftrightarrow \langle\alpha| A |\beta\rangle = 0, \forall A \in \mathcal{V}_\Phi$
confusable $\Leftrightarrow \langle\alpha| A |\beta\rangle \neq 0$, for some $A \in \mathcal{V}_\Phi$.

$\mathbb{C}I_n \subseteq \mathcal{V}_\Phi$ (reflexivity)
$\mathcal{V}_\Phi^\dagger = \mathcal{V}_\Phi$ (symmetry)

$\mathcal{V}_\Phi$ is the **quantum confusability graph** of channel Φ.

Quantum Ramsey theorem?

Quantum graphs

A subspace $\mathcal{V} \subseteq M_n$ is a **quantum graph** (a.k.a., *operator system* or *non commutative graph*) if it satisfies the following:
(1) $\mathbb{C}I_n \subseteq \mathcal{V}$ (equivalently, $I_n \in \mathcal{V}$)
(2) $\mathcal{V}^\dagger = \mathcal{V}$

- The diagonal Δ on a set A with $|A| = n$ (classical independent n-set)
  $\longrightarrow \mathbb{C}I_n$
  (quantum independent n-set)
- K_n (classical complete n-graph)
  $\longrightarrow M_n$ (quantum complete n-graph)

Quantum graphs

If $P \in M_n$ is an orthogonal projection (i.e., $P^2 = P^\dagger = P$) and $\mathcal{V} \subseteq M_n$ is a quantum graph, then
$P\mathcal{V}P = \{PAP : A \in \mathcal{V}\}$ (the compression of $\mathcal{V}$).

A **quantum k-anticlique** of $\mathcal{V}$ is an orthogonal projection $P \in M_n$ with rank k such that $P\mathcal{V}P \cong \mathbb{C}P \Leftrightarrow \dim P\mathcal{V}P = 1$
($\mathcal{V}$ admits a quantum independent k-"set";
Knill-Laflamme error-correction condition.)

A **quantum k-clique** of $\mathcal{V}$ is an orthogonal projection $P \in M_n$ with rank k such that $P\mathcal{V}P = PM_nP \cong M_k \Leftrightarrow \dim P\mathcal{V}P = k^2$
($\mathcal{V}$ admits a quantum complete k-subgraph).

Quantum Ramsey theorem (QRT)

Theorem: (Weaver, 2017) For every $k > 0$ there exists n such that every quantum graph $\mathcal{V} \subseteq M_n$ admits a quantum k-clique or quantum k-anticlique.

Infinite Ramsey theorem

Theorem (Ramsey, 1929) For every 2-coloring of $E(K_{\mathbb{N}})$ there exists an infinite set $A \subseteq \mathbb{N}$ and a subgraph $G < K_{\mathbb{N}}$ with $V(G) = A$ and $G \simeq K_{\mathbb{N}}$ such that $E(G)$ is monochromatic.

Equivalently: Every graph G such that $V(G)$ is countably infinite contains an infinite independent set or a subgraph isomorphic to $K_{\mathbb{N}}$.

Quantum graphs - infinite dimension

Let $\mathcal{H}$ be a Hilbert space.
$B(\mathcal{H})$, the algebra of bounded operator on $\mathcal{H}$. Equip $B(\mathcal{H})$ with the weak topology: a net (T_n) in $B(\mathcal{H})$ converges weakly to $T \in B(\mathcal{H})$ if and only if $\lim_n \langle T_n(u), v\rangle = \langle T(u), v\rangle$, for all $u, v \in \mathcal{H}$.

A subspace $\mathcal{V} \subseteq B(\mathcal{H})$ is a **quantum graph** (a.k.a., *operator system* or *non commutative graph*) if it satisfies the following:
(1) $\mathbb{C}I_{\mathcal{H}} \subseteq \mathcal{V}$ (equivalently, $I_{\mathcal{H}} \in \mathcal{V}$)
(2) $\mathcal{V}^{\dagger} = \mathcal{V}$

Quantum graphs - Example.

Let $G = (V, E)$ be a (classical) graph. Let $\{e_v : v \in V\}$ be an orthonormal basis for the space Hilbert space $\mathcal{H}_G$ with $\dim \mathcal{H}_G = |V|$.

Define $T_{u,v} \in B(\mathcal{H}_G)$ as $T_{u,v}(x) = \langle x, e_v \rangle e_u$.

$$\mathcal{V}_G = \text{ Span}\{I_{\mathcal{H}_G}, T_{u,v} : \{u, v\} \in E\} \text{ is a quantum graph.}$$

Given $A \subseteq V$, let P_A be the projection over Span$\{e_v : v \in A\}$.

A is anticlique $\Leftrightarrow P_A \mathcal{V}_G P_A = \mathbb{C} P_A$

When $|A| = k < \infty$, A is a clique $\Leftrightarrow \dim P_A \mathcal{V}_G P_A = k^2 \Leftrightarrow P_A \mathcal{V}_G P_A \cong M_k$

Quantum graphs - infinite dimension

When $|A| = \infty$, A is a clique $\Leftrightarrow P_A \mathcal{V}_G P_A$ is weakly dense in $P_A B(\mathcal{H}_G) P_A$. (Of course, here $|V| = \infty$, $\dim \mathcal{H}_G = \infty$).

Quantum graphs - infinite dimension

When $|A| = \infty$, A is a clique $\Leftrightarrow P_A \mathcal{V}_G P_A$ is weakly dense in $P_A B(\mathcal{H}_G) P_A$. (Of course, here $|V| = \infty$, $\dim \mathcal{H}_G = \infty$).

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space and $\mathcal{V} \subseteq B(\mathcal{H})$ a quantum graph.

An infinite rank orthogonal projection $P \in B(\mathcal{H})$ is,

an **infinite quantum anticlique** of $\mathcal{V}$, if $P\mathcal{V}P = \mathbb{C}P$.

an **infinite quantum clique** of $\mathcal{V}$, if $P\mathcal{V}P$ is weakly dense in $PB(\mathcal{H})P$.

an **infinite quantum obstruction** of $\mathcal{V}$, if $P\mathcal{V}P$ has finite dimension + **other conditions**.

Quantum graphs - infinite dimension

other conditions = every operator in $P\mathcal{V}P$ can be written as the sum of a scalar multiple of P and a compact operator, and $P\mathcal{V}P$ contains a self-adjoint compact operator with range dense in $P\mathcal{H}$.

Infinite QRT

Theorem: (Kennedy, Kolomatski, Spivak; 2020) Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. Every quantum graph $\mathcal{V} \subseteq B(\mathcal{H})$ admits an infinite quantum clique, an infinite quantum anticlique, or infinite quantum obstruction.

Generalizations of the infinite QRT using selectivity

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, and $B(\mathcal{H})$ denotes the family of bounded operators on $\mathcal{H}$. Fix an orthonormal basis $\mathfrak{B} = \{h_n\}$ for $\mathcal{H}$.

Let $\Pi_\infty(\mathcal{H}) \subset B(\mathcal{H})$ be the family of infinite-rank orthogonal projections.

Given projections $P, Q \in \Pi_\infty(\mathcal{H})$, we write $P \leq Q$ if and only if $P \vee Q = Q$.

Generalizations of the infinite QRT using selectivity

A family of projections $\mathcal{F} \subseteq \Pi_\infty(\mathcal{H})$ is **Ramsey** if for every operator system $\mathcal{V} \subseteq B(\mathcal{H})$ there exists $P \in \mathcal{F}$ satisfying the following trichotomy:

1. Either P is an infinite quantum clique for $\mathcal{V}$.
2. P is infinite quantum anticlique for $\mathcal{V}$, or
3. P is infinite quantum obstruction for $\mathcal{V}$.

Generalizations of the infinite QRT using selectivity

Consider a countably infinite-dimensional Hilbert space $\mathcal{H}$ and an orthonormal basis $\mathfrak{B} = \{h_n\}$ for $\mathcal{H}$.

Let $\mathbb{N}^{[\infty]} = \{A \subseteq \mathbb{N} : |A| = \infty\}$. For $A \in \mathbb{N}^{[\infty]}$, let

$$P_A^{\mathcal{B}} := \text{ the orthogonal projection onto } \overline{\mathrm{span}}\{h_n : n \in A\}.$$

Recall the **diagonal embedding**

$$\mathbb{N}^{[\infty]} \longrightarrow \Pi_\infty(\mathcal{H})$$
$$A \to P_A^{\mathcal{B}}.$$

$P_A^{\mathcal{B}}$ is said to be a **diagonal projection**.

Generalizations of the infinite QRT using selectivity

- A family $\mathcal{C} \subseteq \mathbb{N}^{[\infty]}$ is a **coideal**, if it satisfies the following:

For $A, B \in \mathbb{N}^{[\infty]}$,

1. If $A \in \mathcal{C}$ and $A \subseteq B$, then $B \in \mathcal{C}$.
2. If $A \cup B \in \mathcal{C}$, then $A \in \mathcal{C}$ or $B \in \mathcal{C}$.
3. If $A \in \mathcal{C}$ and $|A \triangle B| < \infty$, then $B \in \mathcal{C}$.

- A coideal $\mathcal{C} \subseteq \mathbb{N}^{[\infty]}$ is **selective**, if for every decreasing sequence $A_1 \supseteq A_2 \supseteq \dots$ in $\mathcal{C}$, there exists $B \in \mathcal{C}$ such that $B/n := \{k \in B : k > n\} \subseteq A_n$, for all $n \in B$.

- Given a coideal $\mathcal{C} \subseteq \mathbb{N}^{[\infty]}$, let

$$\mathcal{F}_{\mathcal{C}} = \{P_A^{\mathcal{B}} : A \in \mathcal{C}\}.$$

Generalizations of the infinite QRT using selectivity

A family $\mathcal{F} \subseteq \Pi_\infty(\mathcal{H})$ of infinite-rank orthogonal projections is **Wide** if there exists a coideal $\mathcal{C} \subseteq \mathbb{N}^{[\infty]}$ such that $\mathcal{F}_\mathcal{C} \subseteq \mathcal{F}$.

Generalizations of the infinite QRT using selectivity

A family $\mathcal{F} \subseteq \Pi_\infty(\mathcal{H})$ of infinite-rank orthogonal projections is **Wide** if there exists a coideal $\mathcal{C} \subseteq \mathbb{N}^{[\infty]}$ such that $\mathcal{F}_\mathcal{C} \subseteq \mathcal{F}$.

Notation. If $v \in \mathcal{H}$ is a vector, define

$$\mathrm{depth}(v) = \min\{n \in \mathbb{N} : v \in \mathrm{span}\{h_1, \ldots, h_n\}\}$$

$P/v :=$ the projection onto $\overline{\mathrm{span}}\{Ph_k \in P\mathcal{H} \ : k > \mathrm{depth}(v)\}$.

For $n \in \mathbb{N}$, $P/n := P/h_n$.

Generalizations of the infinite QRT using selectivity

A family $\mathcal{F} \subseteq \Pi_\infty(\mathcal{H})$ of infinite-rank orthogonal projections is **selective** if it satisfies the following:

1. For every decreasing sequence $\{P_n\}_n$ in $\mathcal{F}$ (i.e., for every n, $P_n \in \mathcal{F}$ and $P_{n+1} \leq P_n$), there exists $P \in \mathcal{F}$ such that $P/n \leq P_n$ for all n.
2. For $P, Q \in \Pi_\infty(\mathcal{H})$, if $P \vee Q \in \mathcal{F}$, then either $P \in \mathcal{F}$ or $Q \in \mathcal{F}$.
3. $\mathcal{F}$ is "closed under finite changes": For $P \in \mathcal{F}$,
    - 3.1 If F is a finite-dimensional subspace of $P\mathcal{H}$ and Q is the orthogonal projection onto $P\mathcal{H} \ominus F$, then $Q \in \mathcal{F}$.
    - 3.2 If F is a finite-dimensional subspace of $(P\mathcal{H})^\perp$ and Q is the orthogonal projection onto $P\mathcal{H} \oplus F$, then $Q \in \mathcal{F}$.

Generalizations of the infinite QRT using selectivity

Lemma 1. Let $\mathcal{F} \subseteq \Pi_\infty(\mathcal{H})$ be a wide family of projections and let $\mathcal{V} \subseteq B(\mathcal{H})$ be an operator system. Let $T \in B(\mathcal{H}, \ell^2(\mathbb{N}))$ be an operator such that $T\mathcal{V}T^*$ is weakly dense in $B(\ell^2(\mathbb{N}))$. Then $\mathcal{F}$ contains an infinite quantum clique for $\mathcal{V}$.

Lemma 2. Let $\mathcal{V} \subseteq B\left(\mathcal{H}\right)$ be an operator system and let $\mathcal{F} \subseteq \Pi_\infty(\mathcal{H})$ be a wide, selective family. Then there exists $P \in \mathcal{F}$ satisfying one of the following conditions:

1. For every n, $P/n\mathcal{H} \ominus P\mathcal{V}Ph_n \notin [\mathcal{F}]$, or
2. The compression, $P\mathcal{V}P$ is diagonalizable.

Theorem. [M - 2026] (A generalization of the infinite quantum Ramsey theorem). Let $\mathcal{F} \subseteq \Pi_\infty(\mathcal{H})$ be a family of infinite-rank orthogonal projections. If $\mathcal{F}$ is wide and selective, then it is Ramsey.

T H A N K S

References:

M. Kennedy, T. Kolomatski, and D. Spivak, *An infinite quantum Ramsey theorem*. J. Operator Theory 84 (2020), no.1, 49-65.

J. G. Mijares, *A generalized infinite quantum Ramsey theorem for operator systems*, (2026). Submitted `https://doi.org/10.48550/arXiv.2604.26894`

N. Weaver, *A "quantum" Ramsey theorem for operator systems*. Proc. Amer. Math. Soc. 145 (2017), 4595-4605.