January 22 and February 19: A. R. D. Mathias (Freiburg im Breisgau, Réunion and Cambridge)

Title: Iteration problems in symbolic dynamics

Abstract: For $\mathcal{X}$ a Polish space, $f : \mathcal{X} \longrightarrow \mathcal{X}$ continuous and $k \in \omega$, write f^k for the k^{th} iterate of f, so that for each $x \in \mathcal{X}$, $f^0(x) = x$ and $f^{k+1}(x) = f(f^k(x))$. Write $\omega_f(x)$ for the set of accumulation points of the forward orbit of x under f, including the periodic points.

Such ω-***limit sets*** are closed in $\mathcal{X}$ and under f; so

$$(*)\ y \in \omega_f(x) \Longrightarrow \omega_f(y) \subseteq \omega_f(x).$$

Define Γ_f on subsets of $\mathcal{X}$ by

$$\Gamma_f(X) = \bigcup\{\omega_f(x) \mid x \in X\}.$$

For $a \in \mathcal{X}$, define $A^0(a, f) = \omega_f(a)$; $A^{\beta+1}(a, f) = \Gamma_f(A^\beta(a, f))$.

By (*) $A^0(a, f) \supseteq A^1(a, f) \supseteq A^2(a, f) \ldots$ so set $A^\lambda(a, f) = \bigcap_{\nu<\lambda} A^\nu(a, f)$ for limit $\lambda > 0$; then for all ordinals $\alpha < \beta \Longrightarrow A^\alpha(a, f) \supseteq A^\beta(a, f)$.

Definition $\theta(a, f) =_{\text{df}}$ the least θ with $A^\theta(a, f) = A^{\theta+1}(a, f)$.

The question raised by the Dynamics Group in Barcelona in 1993 was : ***what are the possible values of the function*** $\theta(a, f)$ ***?***

I showed in [**2a**] that for f the shift function defined in [**1**], all countable ordinals are possible values of $\theta(a, f)$, and whilst in Barcelona I repeatedly attempted to prove that no uncountable ordinal is.

But a note from David Fremlin refuting a related conjecture of mine showed me that I had been barking up the wrong tree, and starting from a refinement of Fremlin's argument, I was led in Réunion in 2001 to develop an extremely difficult new iteration method which led to my proof that, again for f the shift function, there are points a with $\theta(a, f) = \omega_1$; indeed there are recursive such a.

In today's talk I shall recall ideas from [**2a**] and prepare the iteration method of [**2c**]; next week I shall complete the construction of recursive a with $\theta(a, f) = \omega_1$, and then, to encourage young researchers to seek new applications of my iteration method, I shall review related open questions.

Of the papers fully listed in the handout, central to these two talks are

[**1**] Ll. Alsedà, M. Chas, J. Smítal, The structure of the ω-limit sets

[**2a**] A. R. D. Mathias, Delays, recurrence and ordinals

[**2c**] A. R. D. Mathias, Analytic sets under attack

[**6a, b, c**] C. Delhommé, unpublished papers on the shift's attacks.

[**7**] Andreas Blass, *Ultrafilters: where topological dynamics = algebra = combinatorics.* Topology Proc. **18** (1993), 33–56.

In 1993, I found it paid to define this binary relation, where d is the metric of the space $\mathcal{X}$:

$$x \curvearrowright_f y \iff_{\mathrm{df}} \forall m > 0\, \exists \ell \geqslant m\; d(f^\ell(x), y) < \tfrac{1}{m}.$$

Then $\omega_f(x) = \{y \mid x \curvearrowright_f y\}$.

When f is fixed in a discussion, we write $x \curvearrowright y$ for $x \curvearrowright_f y$, and we sometimes write $y \curvearrowleft x$ for $x \curvearrowright y$. We read $x \curvearrowright y$ as "x ***attacks*** y".

PROPOSITION *If $x \curvearrowright y$ and $y \curvearrowright z$ then $x \curvearrowright z$.*

DEFINITION The ***escape set*** or ***boundary*** is the union over all ordinals β of the set of those points in $\omega_f(a)$ eliminated at stage β of the iteration:

$$E(a, f) \;=_{\mathrm{df}} \bigcup_\beta \big(A^\beta(a, f) \smallsetminus A^{\beta+1}(a, f)\big).$$

Here $X \smallsetminus Y$ is the set-theoretic difference $\{x \mid x \in X \text{ and } x \notin Y\}$.

DEFINITION For $x \in E(a, f)$, we write $\beta(x, a, f)$ for the unique β with $x \in A^\beta(a, f) \smallsetminus A^{\beta+1}(a, f)$.

In [**2a**] I used descriptive set theory to show that $\beta(x, a, f)$ is always countable. It follows that $\theta(a, f)$, the least ordinal θ with $A^\theta(a, f) = A^{\theta+1}(a, f)$, is at most ω_1. We call $\theta(a, f)$ the ***f-score*** of a.

DEFINITION We write $A(a, f)$ for $A^{\theta(a,f)}(a, f)$. We call $A(a, f)$ the ***abode***.

Thus $E(a, f) = \omega_f(a) \smallsetminus A(a, f)$. We say that points in $A(a, f)$ ***abide***, and points in $E(a, f)$ ***escape***.

We write $\mathcal{N}$ for *Baire space*, the space of infinite sequences of natural numbers: for each finite such sequence r we have the basic open set $\{\alpha \mid \alpha \restriction \ell h(r) = r\}$. The (backward) ***shift function*** $\mathfrak{s} : \mathcal{N} \longrightarrow \mathcal{N}$ is given by $\mathfrak{s}(\alpha)(n) = \alpha(n + 1)$. As in section 4 of *Delays* we write $\zeta \vartriangleright \xi$, read ζ ***is near to*** ξ, if $\zeta = \mathfrak{s}^n(\xi)$ for some $n \geqslant 0$.

In [**2a**] I constructed for each $\alpha < \omega_1$ a point in $\mathcal{N}$ of score exactly α. Five years later, in [**2c**], when in Réunion, I constructed a recursive point in $\mathcal{N}$ of score ω_1.

My methods did not adapt well to $\mathfrak{s}$ on the Cantor space and my Réunionnais colleague Christian Delhommé in [**6a, b, c**] found a much better treatment of the compact case.

Later my research student Cédric Machefert started to examine the notion of ***uniform attack*** corresponding to the notion of uniform recurrence, but he died before he could finish. In studying that notion, the paper [**7**] of Andreas Blass is hugely helpful.

My work often requires metrizability of the space but not compactness; Blass often needs compactness but not metrizability.

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for the talks on "Iteration problems in symbolic dynamics"
to be given by Adrian Mathias in early 2026
in the Carbondale Zoominar of Wesley Calvert

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More recent work

[**2c**] A. R. D. Mathias, Analytic sets under attack, *Math. Proc. Cambridge Phil. Soc.* **138** (2005) 465–485.

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[**2e**] A. R. D. Mathias, A scenario for transferring high scores, *Acta Universitatis Carolinae - Math. et Phys.,* **45** (2004) 67–73.

Yet more recent work

[**6a**] C. Delhommé. Transfer of scores to the shift's attacks of Cantor space.

[**6b**] C. Delhommé.Representation in the shift's attacks of Baire space.

[*formerly* On embedding transitive relations in that of shift-attack.]

[**6c**] C. Delhommé. Completeness properties of the relation of attack.

Further reading

[**3**] N. N. Lusin, Sur la classification de M. Baire, *Comptes Rendus Acad. Sci. Paris* **164** (1917) 91–94.

[**4a**] C. Dellacherie, Les dérivations en théorie descriptive des ensembles et le théorème de la borne, in: *Séminaire de Probabilités XI*, Lecture Notes in Mathematics Volume 581, Springer–Verlag, Berlin, Heidelberg and New York, 1977, pp 34–46. Erratum in *Séminaire de Probabilités XII* Springer LNM 649, 1978, p. 523.

[**4b**] C. Dellacherie, Un cours sur les ensembles analytiques, in: *Analytic Sets* by C. A. Rogers *et al.*, Academic Press, 1981, pp 183–316.

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[**8a**] T.K. Subrahmonian Moothathu, *Syndetically proximal pairs*, Journal Math. Anal. Appl. **379** (2011) 656–663

[**8b**] Jian LI, T.K. Subrahmonian Moothathu, Piotr Oprocha *Corrigendum to "Syndetically proximal pairs"*, arXiv:1707.07575v1 21 Jul 2017.

[A counterexample to Theorem 9 of [**8a**]]

[**8c**] T. K. S. Moothathu, P. Oprocha, *Syndetic proximality and scrambled sets*, arXiv:1108.1280v4 4 Apr 2013.

[**8d**] T. K. S. Moothathu, *Topological Dynamics* (72 pages, May 2017).

[**9**] E. Akin and S. Kolyada, Li-Yorke sensitivity, *Nonlinearity* **16** (2003) 1421–1433.