January 22 and February 19: A. R. D. Mathias (Freiburg im Breisgau, Réunion and Cambridge)

Title: Iteration problems in symbolic dynamics

Abstract: For $\mathcal{X}$ a Polish space, $f : \mathcal{X} \longrightarrow \mathcal{X}$ continuous and $k \in \omega$, write f^k for the k^{th} iterate of f, so that for each $x \in \mathcal{X}$, $f^0(x) = x$ and $f^{k+1}(x) = f(f^k(x))$. Write $\omega_f(x)$ for the set of accumulation points of the forward orbit of x under f, including the periodic points.

Such ω-***limit sets*** are closed in $\mathcal{X}$ and under f; so

$$(*)\ y \in \omega_f(x) \Longrightarrow \omega_f(y) \subseteq \omega_f(x).$$

Define Γ_f on subsets of $\mathcal{X}$ by

$$\Gamma_f(X) = \bigcup\{\omega_f(x) \mid x \in X\}.$$

For $a \in \mathcal{X}$, define $A^0(a, f) = \omega_f(a)$; $A^{\beta+1}(a, f) = \Gamma_f(A^\beta(a, f))$.

By (*) $A^0(a, f) \supseteq A^1(a, f) \supseteq A^2(a, f) \ldots$ so set $A^\lambda(a, f) = \bigcap_{\nu<\lambda} A^\nu(a, f)$ for limit $\lambda > 0$; then for all ordinals $\alpha < \beta \Longrightarrow A^\alpha(a, f) \supseteq A^\beta(a, f)$.

Definition $\theta(a, f) =_{\text{df}}$ the least θ with $A^\theta(a, f) = A^{\theta+1}(a, f)$.

The question raised by the Dynamics Group in Barcelona in 1993 was : ***what are the possible values of the function*** $\theta(a, f)$ ***?***

I showed in [**2a**] that for f the shift function defined in [**1**], all countable ordinals are possible values of $\theta(a, f)$, and whilst in Barcelona I repeatedly attempted to prove that no uncountable ordinal is.

But a note from David Fremlin refuting a related conjecture of mine showed me that I had been barking up the wrong tree, and starting from a refinement of Fremlin's argument, I was led in Réunion in 2001 to develop an extremely difficult new iteration method which led to my proof that, again for f the shift function, there are points a with $\theta(a, f) = \omega_1$; indeed there are recursive such a.

In today's talk I shall recall ideas from [**2a**] and prepare the iteration method of [**2c**]; next week I shall complete the construction of recursive a with $\theta(a, f) = \omega_1$, and then, to encourage young researchers to seek new applications of my iteration method, I shall review related open questions.

Of the papers fully listed in the handout, central to these two talks are

[**1**] Ll. Alsedà, M. Chas, J. Smítal, The structure of the ω-limit sets

[**2a**] A. R. D. Mathias, Delays, recurrence and ordinals

[**2c**] A. R. D. Mathias, Analytic sets under attack

[**6a, b, c**] C. Delhommé, unpublished papers on the shift's attacks.

[**7**] Andreas Blass, *Ultrafilters: where topological dynamics = algebra = combinatorics*. Topology Proc. **18** (1993), 33–56.