Recent developments in categorical model theory

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We consider the fundamentals of *categorical model theory*, a field that occupies a surprising middle ground between

1. Abstract model theory, chiefly abstract elementary classes (AECs) and their generalizations.
2. Set theory, in the form of large cardinal (and anti-large cardinal) assumptions.
3. Category theory, chiefly accessible categories.

We sketch the history of this field—and the ideas involved—and look at a particularly interesting outgrowth: the connection between *stable independence* and *cofibrant generation*.

If nothing else, hopefully this talk will leave you with the urge to browse the excellent overview of the field in [V].

It will come as no surprise that many categories of structures and morphisms that arise in mathematics are not first-order axiomatizable, e.g.

1. Torsion R-modules with monomorphisms.
2. Abelian p-groups with monomorphisms.

Not to mention the many categories of structures with underlying complete metric spaces...

The natural first response to this problem is to pass to more expressive languages. Among the most popular choices:

- $L_{\kappa\lambda}$, which allows conjunctions/disjunctions of families of $<\kappa$ formulas, and quantification over $<\lambda$-tuples of variables.
- $L(Q)$, incorporating the counting quantifier Q.

The difficulty is that the model theory of each is, in general, radically different from that of the others.

Instead of working with these logics themselves (individually!), Shelah had the idea of working directly with their categories of models and strong embeddings.

Definition

An abstract elementary class (AEC) consists of a class of structures $\mathcal{K}$ in a finitary signature and a strong substructure relation $\prec_{\mathcal{K}}$ with the following properties:

1. *(Coherence) If $M_1 \subseteq M_2 \prec_{\mathcal{K}} M_3$ and $M_1 \prec_{\mathcal{K}} M_3$, $M_1 \prec_{\mathcal{K}} M_3$.*
2. *(Unions of chains) Given a $\prec_{\mathcal{K}}$-chain $\langle M_\alpha \,|\, \alpha < \gamma \rangle$,*
   - 2.1 *The union $\cup_{\alpha<\gamma} M_\alpha$ is in $\mathcal{K}$, and*
   - 2.2 *If $M_\alpha \prec_{\mathcal{K}} N$ for all $\alpha < \gamma$, $\cup_{\alpha<\gamma} M_\alpha \prec_{\mathcal{K}} N$.*
3. *(Löwenheim-Skolem) There is a cardinal $LS(\mathcal{K})$ such that for any $M \in \mathcal{K}$ and $A \subseteq UM$, there is $N \prec_{\mathcal{K}} M$ with $A \subseteq UN$ and $|UN| \leq |A| + LS(\mathcal{K})$.*

In short, an AEC in signature Σ is, as a subcategory of $\mathbf{Str}(\Sigma)$, coherent, closed under *directed colimits*, and, moreover, its structures can be obtained as directed colimits of small ones.

These have been studied intensively. The particular focus:

Conjecture (Shelah's Categoricity Conjecture)

There is a Hanf number for categoricity in AECs—for any AEC $\mathcal{K}$, if $\mathcal{K}$ is categorical in a sufficiently high cardinal, it is categorical in all sufficiently high cardinals.

This **vast** generalization of Morley's Theorem is as close to proved as it likely will be, cf. recent work of Vasey and Espindola.

It is unfortunate, then, that even AECs don't cover everything we encounter in mathematics...

If we consider, for example,

- Complete metric structures.
- λ-saturated models of a first-order theory.

and so on, we see that the unions of chains axiom—closure under directed colimits—will fail.

The first case can be handled by shifting to *metric AECs* or *CATs* but there is a absolute catch-all:

Definition

A μ-AEC is defined in just the same way as an AEC, but with the following changes:

- The signature may be $< \mu$-ary.
- We assume closure only under μ-directed unions.
- The LS axiom is slightly more delicate.

This notion, on its face, has little to recommend it, but it cements a link to a seemingly very different area of inquiry.

The category-theoretic analysis of classes of models has its origins in the following notion of Gabriel and Ulmer (1971):

Definition

For λ a regular cardinal, we say that a category $\mathcal{K}$ is **locally λ-presentable** *if*

1. *$\mathcal{K}$ has all colimits.*
2. *There is a set of λ-***presentable objects***, and every object of $\mathcal{K}$ is a λ-directed colimit thereof.*

Essential is λ-presentability, a notion of size. Of note:

1. It is purely diagrammatic, hence makes sense in any category.
2. Presentability always means what it should in special cases: presentation size, cardinality, density character, etc.

We complained earlier, though, that assuming directed colimits was too much, and in a locally presentable category we insist on *arbitrary* colimits!

Weakening this assumption leads directly to *accessible categories*, developed in, e.g. [MP] and [AR]:

Definition

*For λ a regular cardinal, we say a category $\mathcal{K}$ is λ-**accessible** if*

1. *$\mathcal{K}$ has all λ-**directed colimits**.*
2. *There is a set of λ-presentable objects, and every object of $\mathcal{K}$ is a λ-directed colimit thereof.*

Hopefully this will look familiar: we are within spitting distance of μ-AECs.

As an aside, we note that accessible categories are everywhere, e.g.

- **Set**, with presentability as cardinality.
- AECs, with presentability as cardinality of underlying sets.
- **Ab**, R-**Mod**, **Alg**(Σ), with presentability as presentation size.
- **Met**, the category of complete metric spaces, with presentability as density character.
- Metric AECs, with presentability as density character of underlying spaces.

Accessible categories generalize AECs and metric AECs: how do they relate to μ-AECs?

Theorem (BGLRV)

Every μ-AEC is an accessible category with monomorphisms. Moreover, every accessible category with monomorphisms is equivalent to a μ-AEC (some μ).

This provides a crucial foothold in navigating between category theory and model theory, e.g.

Theorem (LRV1)

- *The μ-AECs admitting intersections are precisely the locally polypresentable categories (accessible with wide pullbacks).*
- *The universal μ-AECs (i.e. those closed under substructure) are precisely the locally multipresentable categories (accessible with connected limits).*

More broadly, this connection allows us to analyze accessible categories with monos using all the tools of (abstract) model theory: element-by-element constructions, types, EM-blueprints, filtrations, etc.

The effects of set-theoretic hypotheses on the structure of accessible categories also become much clearer through this connection.

Example

The existence spectra of accessible categories—the presentability ranks where objects are known to exist—turn out to be highly sensitive to, e.g. SCH.

Going in the other direction, this seems like an excellent place to falsify Shelah's Categoricity Conjecture...

We now turn to stable (nonforking) independence.

Version 1: Fix a theory T, monster model $\mathfrak{C}$. We say the type of a tuple $\bar{a} \in \mathfrak{C}$ over a model B does not fork over $C \subseteq B$ if the type over C has the same complexity, i.e. Morley rank. Notation:

$$\bar{a} \underset{C}{\overset{(\mathfrak{C})}{\perp}} B$$

We now turn to stable (nonforking) independence.

Version 2: Again, given a theory T and monster model $\mathfrak{C}$, we say

$$A \underset{C}{\overset{(\mathfrak{C})}{\downarrow}} B$$

if the type of any $\bar{a} \in A$ over B does not fork over C. One can think of this as a kind of independence relation: A is independent from B over C.

One can think of $\downarrow$ as an abstract ternary relation, and axiomatize stable (or *simple*) independence directly.

We now turn to stable (nonforking) independence.

Version 3: In (μ-)AECs, we can only work over models, and may not have a monster model. We end up with $\downarrow$ as a quaternary relation

$$M_1 \underset{M_0}{\overset{M_3}{\downarrow}} M_2$$

axiomatized as before, cf. [BGKV]. In a sense, we are picking out a family of diagrams of strong embeddings of the form

$$\begin{array}{ccc} M_1 & \rightarrow & M_3 \\ \uparrow & \downarrow & \uparrow \\ M_0 & \rightarrow & M_2 \end{array}$$

The path to stable independence on an abstract category begins to reveal itself...

Definition (LRV2)

An independence notion $\downarrow$ on a category $\mathcal{K}$ is a family of commutative squares in $\mathcal{K}$ (suitably closed). We say that $\downarrow$ is **weakly stable** *if it satisfies*

1. *Existence: Any span $M_1 \leftarrow M_0 \rightarrow M_2$ can be completed to an independent square.*
2. *Uniqueness: there is only one independent square for each span, up to equivalence.*
3. *Transitivity: horizontal and vertical compositions of independent squares are independent.*

Fact

If $\downarrow$ is weakly stable, these squares satisfy the usual cancellation property of pushouts.

We must impose a locality condition—accessibility now appears.

Consider the category $\mathcal{K}_\downarrow$:

- Objects: Morphisms $f : M \to N$ in $\mathcal{K}$.
- Morphisms: A morphism from $f : M \to N$ to $f' : M' \to N'$ is a $\mathop{\smile}\limits^{|}$-independent square

$$\begin{array}{ccc} M' & \longrightarrow & N' \\ \uparrow & \mathop{\smile}\limits^{|} & \uparrow \\ M & \longrightarrow & N \end{array}$$

Definition

1. *We say that* $\mathop{\smile}\limits^{|}$ *is* λ-**continuous** *if* $\mathcal{K}_\downarrow$ *is closed under* λ*-directed colimits.*
2. *We say that* $\mathop{\smile}\limits^{|}$ *is* λ-**accessible** *if* $\mathcal{K}_\downarrow$ *is* λ*-accessible.*
3. *We say* $\mathop{\smile}\limits^{|}$ *is* λ-**stable** *if it is weakly stable and* λ*-accessible.*

Consider a category, $\mathcal{K}$, and a family of morphisms of interest, $\mathcal{M}$.

Definition

We say a square

$$\begin{array}{ccc} M_1 & \longrightarrow & M_3 \\ \uparrow & & \uparrow \\ M_0 & \longrightarrow & M_2 \end{array}$$

in $\mathcal{K}$ *is* $\mathcal{M}$-**effective** *if*

1. *all morphisms are in* $\mathcal{M}$*,*
2. *the pushout of* $M_1 \leftarrow M_0 \rightarrow M_2$ *exists, and*
3. *the induced map from the pushout to* M_3 *is in* $\mathcal{M}$*.*

If $\mathcal{M} = \{\text{regular monos}\}$, these are the *effective unions* of Barr.

Under certain assumptions on $\mathcal{M}$, the $\mathcal{M}$-effective squares yield a weakly stable independence notion on $\mathcal{K}_{\mathcal{M}}$.

Sidebar

One may well ask: is there a reason—beyond the pathology of abstraction—to consider stable independence in categories?

There certainly is.

When working with particular category, say **Ab**, we often wish to consider only a particular family of morphisms, say the pure monos; that is, we study $\mathbf{Ab}_{pure}$.

In such cases (especially when we restrict to monomorphisms) much of the nice structure of the original category is lost: for example, **Ab** is locally presentable, $\mathbf{Ab}_{pure}$ merely accessible.

In particular, we will lose pushouts—roughly, canonical prime amalgams—which are *essential*. (Stably) independent squares are almost as good!

For the next few slides, unless otherwise indicated, we consider the ways in which stable independence notions manifest themselves in μ-*AECs*.

These are an ideal testing ground for the abstract notion, as they are, morally, just accessible categories with monos, but also

- Concrete: reassuringly, we can speak of elements, tuples of elements, etc.
- Equipped with a natural notion of (Galois) type.

This will allow us to determine the relationship between our notion and more straightforwardly model-theoretic ones, e.g. the order property.

Some care is required when translating back to an element- and type-oriented description: we must take a kind of closure ($\overline{\perp}$).

The essential bridge is provided by the *witness property*:

Definition

Let θ be an infinite cardinal. We say $\perp$ has the **right** **$(<\theta)$-witness property** *if $M_1 \underset{M_0}{\overset{M_3}{\perp}} M_2$ holds whenever*

$M_0 \prec_{\mathcal{K}} M_i \prec_{\mathcal{K}} M_3$, $i = 1, 2$, and $M_1 \underset{M_0}{\overset{M_3}{\overline{\perp}}} A$ for all $A \subseteq UM_2$ with $|A| < \theta$. Similarly, **left $(<\theta)$-witness property**, **$(<\theta)$-witness property**.

Theorem

A reasonable independence notion on a μ-AEC is accessible iff it has the witness property and local character (in the usual sense).

Fact (BG, essentially)

The witness property holds in any fully tame and short μ-AEC.

With this rephrasing, model-theoretic consequences become much clearer. For example:

Theorem

Let $\mathcal{K}$ be a μ-AEC with a stable independence relation. Then:

1. *(Galois-stability) For any α, there is a proper class of cardinals S_α such that for any $\lambda \in S_\alpha$ and $M \in \mathcal{K}_\lambda$, $|ga\text{-}S^{<\alpha}(M)| = \lambda$.*
2. *(Tameness) For any α, there is a cardinal λ such that $< \alpha$-types are λ-tame.*

What of the order property? Symmetry? Canonicity?

We consider the version of the order property introduced by Shelah in the context of AECs:

Definition

A μ-AEC $\mathcal{K}$ has the **order property** *if there exists an ordinal α such that for any ordinal theta θ, there is $M \in \mathcal{K}$ and a sequence $\langle \bar{a}_i \mid i < \theta \rangle$ with*

1. $\bar{a}_i \in {}^{\alpha}UM$ *for all $i < \theta$, and*
2. *for all $i_0 < j_0 < \theta$, ga-tp$(\bar{a}_{i_0}\bar{a}_{j_0}/\emptyset, M) \neq$ ga-tp$(\bar{a}_{j_0}\bar{a}_{i_0}/\emptyset, M)$*

One would expect the existence of a stable independence relation to imply failure of the order property...

Theorem

If $\mathcal{K}$ is a μ-AEC with a stable independence notion, $\mathcal{K}$ does not have the order property.

In fact, the same is true even if we drop the assumption that the independence notion has the witness property, provided that $\mathcal{K}$ has *chain bounds*:

Definition

We say a category $\mathcal{K}$ has **chain bounds** if every chain has an upper bound (but not necessarily a union/colimit).

[An important lesson here, and in the canonicity theorem below, is that this weakening of unions of chains is almost always sufficient.]

There is a partial converse, but it is *very* partial:

Theorem

Assume Vopěnka's Principle (VP). Let $\mathcal{K}$ be a μ-AEC with chain bounds, and let κ be strongly compact. If $\mathcal{K}$ does not have the order property, then the κ-AEC of locally κ-model-homogeneous models of $\mathcal{K}$ has a stable independence relation.

Note

If we assume $\mathcal{K}$ has amalgamation, VP is not necessary.

Corollary

Assume VP. Let $\mathcal{K}$ be a μ-AEC with chain bounds. Then $\mathcal{K}$ does not have the order property iff there is a stable independence notion on a cofinal sub-λ-AEC.

An argument very similar to that in [BGKV] gives canonicity of stable independence in μ-AECs.

More remarkably, it holds in far greater generality:

Theorem (LRV2)

Let $\mathcal{K}$ be a category with chain bounds, and let $\overset{1}{\downarrow}$ and $\overset{2}{\downarrow}$ be independence notions with existence and uniqueness such that:

1. $\overset{1}{\downarrow}$ *is (right) monotonic.*
2. $\overset{2}{\downarrow}$ *is transitive and (right) accessible.*

Then $\overset{1}{\downarrow} = \overset{2}{\downarrow}$. So, in particular, $\mathcal{K}$ has at most one stable independence notion.

Granted: the existence of an accessible independence notion $\overset{2}{\smile\!\!\!\!\!|}$ implies $\mathcal{K}$ is accessible. But we are not back in the realm of μ-AECs: here the morphisms **need not be monomorphisms**!

This argument, in [LRV2], resembles that of [BGKV] but is, of necessity, element-free.

Though confined to an appendix, it deserves a lecture all its own: it shows, among other things, that independent sequences can be developed and put to use in an arbitrary (accessible) category.

One expects this to have interesting consequences in algebra...

We mentioned that stable independence can be a useful substitute for pushouts in cases where they have been lost, especially in restricting a locally presentable $\mathcal{K}$ to $\mathcal{K}_{\mathcal{M}}$, $\mathcal{M}$ a family of morphisms.

Note

Recall that if $\mathcal{M}$ is reasonable, $\mathcal{K}_{\mathcal{M}}$ has a weakly stable independence notion, given by $\mathcal{M}$-effective squares.

When is this notion $\aleph_0$-continuous ($\mathcal{K}_{\mathcal{M},\downarrow}$ closed under directed colimits)? When is it $\aleph_0$-accessible ($\mathcal{K}_{\mathcal{M},\downarrow}$ finitely accessible) and therefore very stable?

While existence and uniqueness seem to be the thornier issues in model theory, it is these properties that are most problematic here.

In this special case, though, there is an easy (and altogether surprising!) answer.

We sketch how stable independence relates to *cofibrant generation*, an essential tractability/smallness condition in homotopy theory.

A little motivation:

Note

CW-complexes are built inductively by gluing cells along their boundaries, $S^{n-1} \to D^n$. The corresponding morphisms are constructed in similar fashion...

That is, the morphisms are obtained by

- Pushing out along some $S^{n-1} \to D^n$ (gluing).
- Transfinite composition of such pushouts.

In this case, we say that the morphisms are *cellularly generated* from the set $\{S^{n-1} \to D^n \mid n \in \mathbb{N}\}$.

Being generated in this way from a **set** of morphisms is an important smallness condition...

Definition

Let X be a family of morphisms in a category $\mathcal{K}$. Then

1. $\mathrm{Po}(X)$ *is the closure of X under pushouts.*
2. $\mathrm{Tc}(X)$ *is the closure under transfinite composition.*
3. $\mathrm{Rt}(X)$ *is the closure under retracts.*
4. $\mathrm{cell}(X) = \mathrm{Tc}(\mathrm{Po}(X))$
5. $\mathrm{cof}(X) = \mathrm{Rt}(\mathrm{cell}(X))$

Under certain circumstances, we can dispense with retracts.

Definition

We say that a set of morphisms $\mathcal{M}$ in $\mathcal{K}$ is **cofibrantly generated** *if $\mathcal{M} = \mathrm{cof}(X)$, X a* **set** *of morphisms.*

Theorem

Let $\mathcal{K}$ be locally presentable, $\mathcal{M}$ reasonable. The following are equivalent:

1. *$\mathcal{K}_{\mathcal{M}}$ has a stable independence notion.*
2. *$\mathcal{M}$-effective squares form a stable independence notion on $\mathcal{K}_{\mathcal{M}}$.*
3. *$\mathcal{M}$ is cofibrantly generated (and accessible).*

The proof, in [LRV3], is relatively straightforward, but needs some heavy category-theoretic machinery in places, including the elimination of retracts, [MRV].

In joint work [LRPV] and [LRV4], this is used to derive cofibrant generation (or stable independence!) in a number of categories arising in algebra, building on, e.g. [MA].

Definition

A **weak factorization system** *(or WFS) in a category $\mathcal{K}$ consists of a pair of classes of morphisms $(\mathcal{M}, \mathcal{N})$ such that:*

1. *Any morphism h of $\mathcal{K}$ can be written as $h = gf$, where $f \in \mathcal{M}$ and $g \in \mathcal{N}$.*
2. *Morphisms in $\mathcal{M}$ and $\mathcal{L}$ satisfy certain (nonunique) lifting properties: $\mathcal{M} = {}^{\square}\mathcal{N}$ and $\mathcal{N} = \mathcal{M}^{\square}$.*

Examples

In **Set**: *(epis, monos), as one would expect. Also, (monos, epis)!*

By Quillen's small object argument, if $\mathcal{K}$ is locally presentable, and $\mathcal{M}$ cofibrantly generated, then $(\mathcal{M}, \mathcal{M}^{\square})$ is a WFS on $\mathcal{K}$!

Model Categories

To carry out homotopy theory in a category $\mathcal{K}$, we need a *model structure*, consisting of:

1. $\mathrm{Cof} \subseteq \mathrm{Mor}(\mathcal{K})$, the cofibrations—nice inclusions, roughly.
2. $\mathrm{Fib} \subseteq \mathrm{Mor}(\mathcal{K})$, the fibrations—nice surjections, roughly.
3. $\mathcal{W} \subseteq \mathrm{Mor}(\mathcal{K})$, the weak equivalences—standing in for homotopy equivalences.

subject to the condition (among others) that:

- $(\mathrm{Cof} \cap \mathcal{W}, \mathrm{Fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap \mathcal{W})$ are WFSs.

Combinatorial model structures, in particular, are those where the class of acyclic cofibrations, $\mathrm{Cof} \cap \mathcal{W}$, is cofibrantly generated.

Fact

If $(\mathcal{M}, \mathcal{N})$ is a coherent WFS—that is, $\mathcal{M}$ is coherent—then $\mathcal{M}$ is reasonable.

Corollary

If $(\mathcal{M}, \mathcal{N})$ is a coherent WFS on locally presentable $\mathcal{K}$, the following are equivalent:

1. *$\mathcal{K}_{\mathcal{M}}$ has stable independence.*
2. *$\mathcal{M}$ is cofibrantly generated (and accessible).*

So, modulo a few important technicalities, subcategories $\mathcal{K}_{\mathcal{M}}$ with stable independence are in bijective correspondence with cofibrantly generated WFSs on $\mathcal{K}$!

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