Hilbert Polynomials for Finitary Matroids

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The Hilbert polynomial

Eventual polynomial growth is a common theme in combinatorics and commutative algebra. The first example is the Hilbert polynomial.

Let K be a field and let $R = K[x_1, \ldots, x_m]$ be the polynomial ring over K. Then R is a graded ring $R = \bigoplus_{t=0}^{\infty} R_t$, where R_t consists of homogeneous polynomials of degree t.

Theorem

Let $M = \bigoplus_{t=0}^{\infty} M_t$ be finitely generated graded R-module. Then there is a polynomial $P \in \mathbb{Q}[Y]$ such that $\dim_K(M_t) = P(t)$ for all $t \gg 0$.

This theorem can be applied when $M = R/I$ for some homogeneous ideal $I \subseteq R$ to compute the degree of the projective variety $V(I)$. This in turn can be used to prove Bézout's theorem on the number of intersections of plane curves.

The Kolchin polynomial

Let F be a field of characteristic zero. A **derivation** on F is a map $\delta\colon F \to F$ satisfying $\delta(a+b) = \delta a + \delta b$ and $\delta(ab) = a\delta b + b\delta a$.

Let $\delta_1, \ldots, \delta_m$ be commuting derivations on F. Given a tuple $\bar{a}$ in a differential field extension of F and $t \in \mathbb{N}$, put

$$F(\bar{a})_{\leqslant t} \ := \ F\big(\{\delta_1^{r_1} \cdots \delta_m^{r_m}(\bar{a}) : r_1 + \cdots + r_m \leqslant t\}\big).$$

Theorem (Kolchin 1964)

There is a polynomial $P \in \mathbb{Q}[Y]$ such that $\operatorname{trdeg}(F(\bar{a})_{\leqslant t}|F) = P(t)$ for all $t \gg 0$.

Johnson showed in 1969 that the Kolchin polynomial can be derived from the Hilbert polynomial of a certain differential module.

Khovanskii's polynomial

Khovanskii later made use of the Hilbert polynomial to prove a very general result on sumsets in abelian semigroups.

Theorem (Khovanskii 1992)

Let S be an abelian semigroup and let A, B be finite subsets of S. Then there is a polynomial $P \in \mathbb{Q}[Y]$ such that $|A + tB| = P(t)$ for all $t \gg 0$, where

$$A + tB \; := \; \{a + b_1 + \cdots + b_t : a \in A \text{ and } b_1, \ldots, b_t \in B\}$$

We provide a general framework from which one can easily derive the above theorems.

What do these examples have in common?

In each case, we start with a finite set A.

We then apply commuting maps $\phi_1, \ldots, \phi_m$ to A (multiplying by x_i, applying the derivation δ_i, adding $b_i \in B$).

After applying these maps t times in total, we calculate some rank of the resulting set (K-linear dimension, transcendence degree over F, cardinality).

Finitary matroids

A **finitary matroid** or *pregeometry* is a set X equipped with a closure operator $\mathrm{cl}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ which satisfies:

1. Monotonicity: if $A \subseteq B \subseteq X$, then $A \subseteq \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$;
2. Idempotence: $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ for $A \subseteq X$;
3. Finite character: if $A \subseteq X$ and $a \in \mathrm{cl}(A)$, then $a \in \mathrm{cl}(A_0)$ for some finite subset $A_0 \subseteq A$;
4. Steinitz exchange: For $a, b \in X$ and $A \subseteq X$, if $a \in \mathrm{cl}(A \cup \{b\}) \setminus \mathrm{cl}(A)$, then $b \in \mathrm{cl}(A \cup \{a\})$.

A set $\{a_1, \ldots, a_n\}$ is **independent** if $a_i \notin \mathrm{cl}(a_1, \ldots, a_{i-1})$ for all i. The **rank** of a finite set $A \subseteq X$, denoted $\mathrm{rk}(A)$, is the maximal size of an independent subset of A.

The Hilbert polynomial

Let (X, cl) be a finitary matroid, and let $\Phi = (\phi_1, \ldots, \phi_m)$ be a finite tuple of commuting maps $X \to X$. For $A \subseteq X$, put

$$\Phi^{(t)}(A) \ := \ \{\phi_1^{r_1} \cdots \phi_m^{r_m}(a) : a \in A \text{ and } r_1 + \cdots + r_m = t\}.$$

The tuple Φ is said to be a **triangular system** if for each i:

$$a \in \mathrm{cl}(C) \implies \phi_i a \in \mathrm{cl}\big(\phi_1(C) \cup \cdots \cup \phi_i(C)\big).$$

Theorem (Fornasiero-K. 2023+)

Suppose that Φ is a triangular system and let $A \subseteq X$ be finite. Then there is a polynomial $P_A^\Phi \in \mathbb{Q}[Y]$ of degree $\leqslant m-1$ such that

$$\mathrm{rk}(\Phi^{(t)}(A)) \ = \ P_A^\Phi(t)$$

*for $t \gg 0$. We call P_A^Φ the **Hilbert polynomial for** A.*

Khovanskii's polynomial, revisited

Theorem (Khovanskii 1992)

Let S be an abelian semigroup and let A, B be finite subsets of S. Then there is a polynomial $P \in \mathbb{Q}[Y]$ such that $|A + tB| = P(t)$ for all $t \gg 0$.

Proof.

Let $X = S$ and let cl be the trivial closure $\mathrm{cl}(C) = C$, so $\mathrm{rk}(C) = |C|$.

Write $B = \{b_1, \dots, b_m\}$, and for each i, put $\phi_i(a) \coloneqq a + b_i$.

Then Φ is triangular, since

$$a \in \mathrm{cl}(C) \implies a \in C \implies \phi_i(a) = a + b_i \in C + b_i = \phi_i(C).$$

Note that $\Phi^{(t)}(A) = A + tB$, so $\mathrm{rk}(\Phi^{(t)}(A)) = |A + tB|$. □

The classical Hilbert polynomial, revisited

Theorem

Let $R = K[x_1, \ldots, x_m]$ *and let* $M = \bigoplus_{t=0}^{\infty} M_t$ *be finitely generated graded* R*-module. Then there is a polynomial* $P \in \mathbb{Q}[Y]$ *such that* $\dim_K(M_t) = P(t)$ *for all* $t \gg 0$*.*

Proof.

View M as a $\mathbb{Z}$-graded module $\bigoplus_{t\in\mathbb{Z}} M_t$. Re-index and adjust generators so that $\bigoplus_{t\in\mathbb{N}} M_t$ is generated by a finite set $A \subseteq M_0$.

Let $X = M$, let cl be K-linear span, and put $\phi_i(a) := x_i \cdot a$.

Again, $a \in \mathrm{cl}(C) \Longrightarrow \phi_i(a) \in \mathrm{cl}(\phi_i(C))$, so Φ is triangular.

If $t \geqslant 0$, then $\mathrm{cl}(\Phi^{(t)}(A)) = M_t$, so $\mathrm{rk}(\Phi^{(t)}(A)) = \dim_K(M_t)$. □

In both this and the last example, each ϕ_i is an *endomorphism*: $a \in \mathrm{cl}(C) \Longrightarrow \phi_i(a) \in \mathrm{cl}(\phi_i(C))$.

The Kolchin polynomial, revisited

Theorem (Kolchin 1964)

Let $\bar{a}$ be a tuple in a differential field extension of F. Then there is a polynomial $P \in \mathbb{Q}[Y]$ such that $\operatorname{trdeg}(F(\bar{a})_{\leqslant t}|F) = P(t)$ for all $t \gg 0$.

Proof.

Let $X = \bigcup_t F(\bar{a})_{\leqslant t}$, and let cl be algebraic closure over F.

Put $\phi_i := \delta_i$. Let $a \in \operatorname{cl}(C)$ and take $\bar{b} \in F(C)$ and $Q \in \mathbb{Q}[X, \bar{Y}]$ with

$$Q(a, \bar{b}) = 0, \qquad \frac{\partial Q}{\partial X}(a, \bar{b}) \neq 0.$$

Then $\delta_i Q(a, \bar{b}) = \nabla Q(a, \bar{b}) \cdot (\delta_i a, \delta_i \bar{b}) = 0$, so $\delta_i a \in \operatorname{cl}(C, \delta_i C)$.

Thus, Φ may not be triangular, but $\Phi_* := (\operatorname{id}, \phi_1, \ldots, \phi_m)$ is.

Apply our theorem to Φ_*, noting that $F(\bar{a})_{\leqslant t} = F(\Phi_*^{(t)}(\bar{a}))$. □

A stronger version

Theorem (Fornasiero-K. 2023+)

Let $(\Phi_1, \ldots, \Phi_k)$ *be a partition of* Φ *and let* $A, C \subseteq X$ *with* A *finite. If each* Φ_i *is triangular, then there is* $P^{\Phi}_{A|C} \in \mathbb{Q}[Y_1, \ldots, Y_k]$ *with*

$$\mathrm{rk}(\Phi^{(\bar{s})}(A)|\Phi^{(\bar{s})}(C)) \;=\; P^{\Phi}_{A|C}(\bar{s})$$

for $\bar{s} = (s_1, \ldots, s_k) \in \mathbb{N}^k$ *with* $\min\{s_1, \ldots, s_k\}$ *sufficiently large.*

Corollary (Nathanson 2000, Fornasiero-K. 2023+)

Let $A, B_1, \ldots, B_k$ *be finite subsets of an abelian semigroup* S *and let* C *be an arbitrary subset of* S*. Then there is* $P \in \mathbb{Q}[Y_1, \ldots, Y_k]$ *such that*

$$\big|(A + s_1B_1 + \cdots + s_kB_k) \setminus (C + s_1B_1 + \cdots + s_kB_k)\big| \;=\; P(s_1, \ldots, s_k)$$

when $\min\{s_1, \ldots, s_k\}$ *is sufficiently large.*

The Φ-rank

Let P_A^Φ be the Hilbert polynomial for $A \subseteq X$. Take $\mathrm{rk}^\Phi(A) \in \mathbb{N}$ with

$$P_A^\Phi(Y) \;=\; \frac{\mathrm{rk}^\Phi(A)}{(m-1)!}Y^{m-1} + \text{ lower degree terms.}$$

Define cl^Φ on X by

$$\mathrm{cl}^\Phi(B) \;\coloneqq\; \{a \in X : \mathrm{rk}^\Phi(B_0 a) = \mathrm{rk}^\Phi(B_0) \text{ for some finite } B_0 \subseteq B\}.$$

Theorem (Fornasiero-K. 2023+)

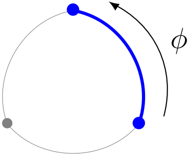
(X, cl^Φ) *is a finitary matroid.*

For the Kolchin polynomial, cl^Φ coincides with differential algebraic closure.

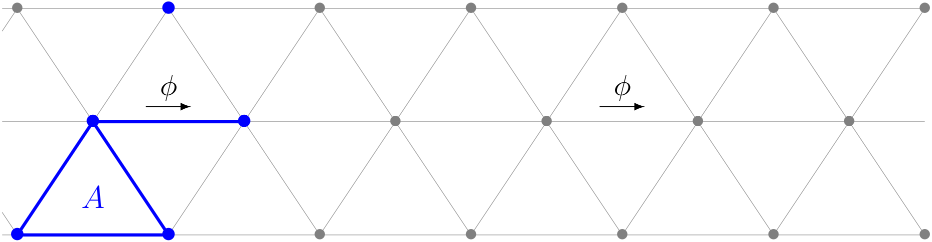
Simplicial maps

Let $\mathcal{K}$ be a simplicial complex, and let $\phi_1, \ldots, \phi_m$ be simplicial maps. Let A be a subcomplex of $\mathcal{K}$. Then for each n, the nth Betti number $b_n(\Phi^{(t)}(A))$ is eventually a polynomial in t.

$$\Phi = (\mathrm{id}, \phi)$$



$$b_0(\Phi^{(0)}(A)) = 3$$
$$b_1(\Phi^{(0)}(A)) = 1$$



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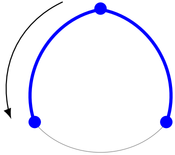
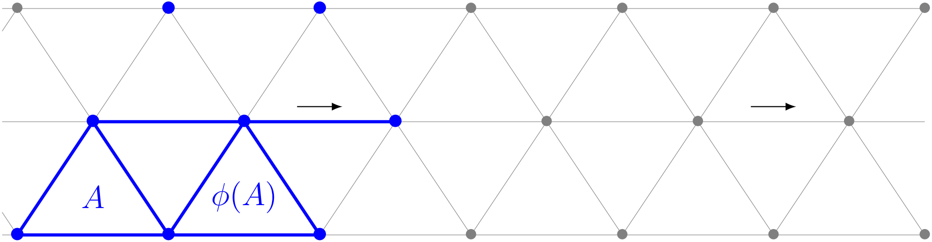


$$b_0(\Phi^{(1)}(A)) = 4$$
$$b_1(\Phi^{(1)}(A)) = 3$$



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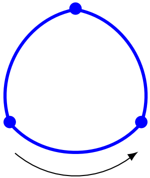
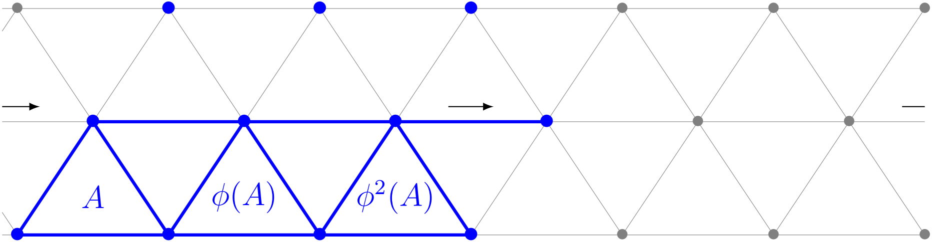


$$b_0(\Phi^{(2)}(A)) = 5$$
$$b_1(\Phi^{(2)}(A)) = 6$$



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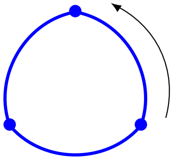
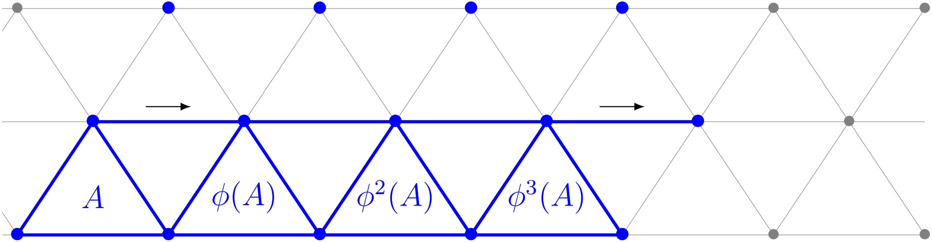


$$b_0(\Phi^{(3)}(A)) = 6$$
$$b_1(\Phi^{(3)}(A)) = 8$$



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Let $\mathcal{K}$ be a simplicial complex, and let $\phi_1, \dots, \phi_m$ be simplicial maps. Let A be a subcomplex of $\mathcal{K}$. Then for each n, the nth Betti number $b_n(\Phi^{(t)}(A))$ is eventually a polynomial in t.

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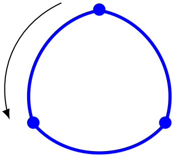
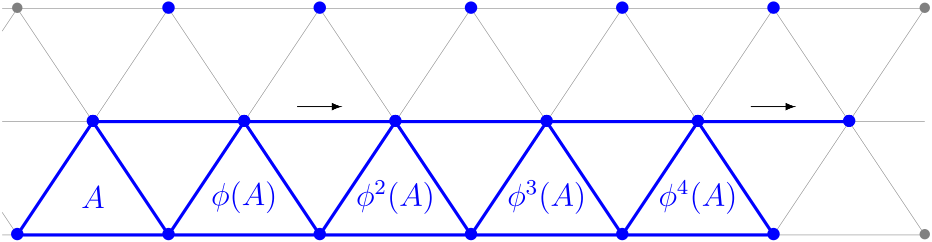


$$b_0(\Phi^{(4)}(A)) = 7$$
$$b_1(\Phi^{(4)}(A)) = 10$$



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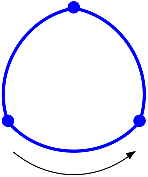
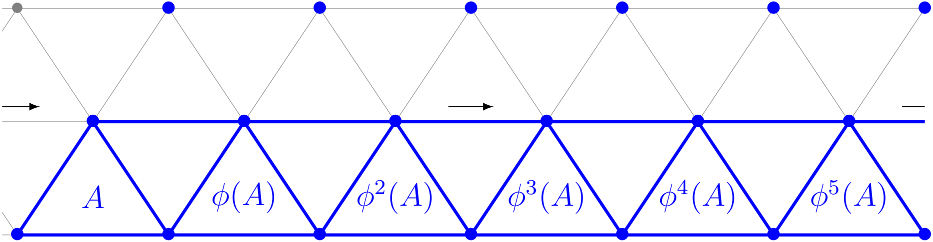


$$b_0(\Phi^{(t)}(A)) = t + 3$$
$$b_1(\Phi^{(t)}(A)) = 2t + 2$$



Simplicial maps

Consider the simplicial chain complex $(C_\bullet, \partial_\bullet)$ associated to $\mathcal{K}$:

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

We assign to a subgroup $B \subseteq C_n$ two ranks: $\mathrm{rk}(B)$ is the rank of the group B, and $\mathrm{rk}^\partial(B)$ is the rank of $\partial_n(B)$.

A simplicial map $\phi\colon \mathcal{K} \to \mathcal{K}$ induces maps $\phi_n : C_n \to C_n$, each of which is an endomorphism of the corresponding closure operators cl and cl^∂.

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\partial_{n+2}} & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots \\ & & \downarrow{\scriptstyle \phi_{n+1}} & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi_{n-1}} & & \\ \cdots & \xrightarrow{\partial_{n+2}} & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots \end{array}$$

It remains to note that for $A \subseteq \mathcal{K}$, we have

$$b_n(A) \;=\; \mathrm{rk}(C_n(A)) - \mathrm{rk}^\partial(C_n(A)) - \mathrm{rk}^\partial(C_{n+1}(A)).$$

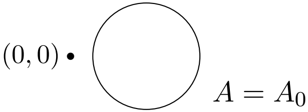
Topological dynamics

The previous result is really about topological dynamics. Let B be a topological space, let $\phi_1, \ldots, \phi_m \colon B \to B$ be commuting continuous maps, and let A be a compact subspace of B.

The system (B, A, Φ) is **triangulable** if there is a triangulation $\tau \colon |\mathcal{K}| \to B$ which is compatible with A and with the maps ϕ_i.

If (B, A, Φ) is triangulable, then $b_n(\Phi^{(t)}(A))$ is eventually polynomial in t for each n. This is not true for arbitrary systems. Which other systems enjoy this phenomenon?

$$
\begin{aligned}
b_1(A_{t+1}) - b_1(A_t) &= b_0(A_t \cap \phi^{t+1}(A)) \\
&\approx t + 1
\end{aligned}
$$



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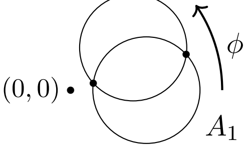

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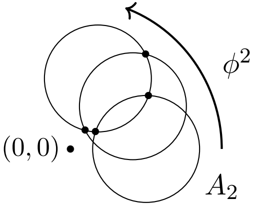

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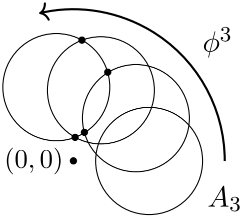
$$b_1(A_{t+1}) - b_1(A_t) = b_0(A_t \cap \phi^{t+1}(A)) \approx t + 1$$



Other applications

- A Hilbert polynomial for homogeneous tropical ideals (originally due to Maclagan and Rincón).
- A Kolchin polynomial for difference-differential fields (various results due to Levin).
- A Kolchin polynomial for difference-differential exponential fields and o-minimal fields with compatible derivations.

Bounding ranks

The Kolchin polynomial for differential fields can be used to bound U-rank in the model completion $\mathrm{DCF}_{0,m}$ (differentially closed fields with m commuting derivations). This is because the Kolchin polynomial can detect whether one type is a forking extension of another.

Explicitly, McGrail showed that for a differential field F and a tuple $\bar{a}$ in a differentially closed extension of F with Kolchin polynomial

$$P_{a|F}(t) = dt^k/k! + \textit{lower degree terms},$$

the type $\mathrm{tp}(\bar{a}/F)$ has U-rank at most $(d+1)\omega^k$.

In previous work, Fornasiero and I showed that for a fixed o-minimal theory T, the theory T^Δ of models of T with finitely many commuting compatible derivations has a model completion.

Our analog of the Kolchin polynomial can be similarly used to bound thorn-rank in this model completion.

A sketch of the proof of the main theorem

For $\bar{r} \in \mathbb{N}^m$, put $\phi^{\bar{r}} := \phi_1^{r_1} \cdots \phi_m^{r_m}$, and define $f\colon \mathbb{N}^m \to \mathbb{N}$ by

$$f(\bar{r}) \;:=\; \operatorname{rk}\left(\phi^{\bar{r}}(A) \middle| \{\phi^{\bar{u}}(A) : |\bar{u}| = |\bar{r}| \text{ and } \bar{u} <_{lex} \bar{r}\}\right).$$

Then f is decreasing in each variable and $\operatorname{rk}(\Phi^{(t)}(A)) = \sum_{|\bar{r}|=t} f(\bar{r})$.

One can show that the generating function

$$\sum_t \operatorname{rk}(\Phi^{(t)}(A))Y^t \;=\; \sum_t \sum_{|\bar{r}|=t} f(\bar{r})Y^t \;=\; \sum_{\bar{r}} f(\bar{r})Y^{|\bar{r}|}$$

is a rational function with denominator $(1-Y)^m$.

It follows that $\operatorname{rk}(\Phi^{(t)}(A))$ is polynomial for t large enough. Exactly how large can be described in terms of the level sets $(f^{-1}(n))_{n\in\mathbb{N}}$.

Thank you!