A criterion for internality of some differential equations

Léo Jimenez, joint with Christine Eagles

September 12, 2024

Parametrizing solutions of differential equations

Consider two complex numbers a and b, and the very basic differential equations:

- $y' = a$
- $y' = by$

the solution sets can be parametrized as:

- $\{y_0 + c, c \in \mathbb{C}\}$
- $\{cy_1, c \in \mathbb{C}\}$

where y_0, y_1 are particular solutions of the equations. Or even:

- $\{at + c, c \in \mathbb{C}\}$
- $\{ce^{bt}, c \in \mathbb{C}\}$

Systems of equations

$$\begin{cases} y_1' = y_1 \\ y_2' = iy_2 \\ z' = 6 \end{cases} \Rightarrow \begin{cases} y_1 \in \{c_1 e^t, c_1 \in \mathbb{C}\} \\ y_2 \in \{c_2 e^{it}, c_2 \in \mathbb{C}\} \\ z \in \{6t + d, d \in \mathbb{C}\} \end{cases}$$

Any three solutions are algebraically independent, so we get three independent parametrizations. Three particular solutions, e^t, e^{it} and t, are needed.

$$\begin{cases} y_1' = 2y_1 \\ y_2' = 4y_2 \\ z_1' = 3 \\ z_2' = 6 \end{cases} \Rightarrow \begin{cases} y_1 \in \{c_1 e^{2t}, c_1 \in \mathbb{C}\} \\ y_2 \in \{c_2 e^{4t}, c_2 \in \mathbb{C}\} \Rightarrow y_2 = \frac{c_2}{c_1^2} y_1^2 \\ z_1 \in \{3t + d_1, d_1 \in \mathbb{C}\} \\ z_2 \in \{6t + d_2, d_2 \in \mathbb{C}\} \Rightarrow z_2 = 2z_1 + d_2 - 2d_1 \end{cases}$$

Only two functions, e^t and t, are needed to parametrize the set of solutions.

More complicated example

$$\begin{cases} y' = \frac{yz}{y+z} \\ z' = -\frac{yz}{y+z} \end{cases} \Rightarrow \begin{cases} y \in \{\frac{ce^x}{e^x - d}, c \in \mathbb{C}, d \in \mathbb{C}^*\} \\ z = c - y \end{cases}$$

So we can still parametrize the solutions as rational functions of e^x, but not in a linear way. This is because this system is in (non-linear) bijection with a linear system:

$$\begin{cases} y' = \frac{yz}{y+z} \\ z' = -\frac{yz}{y+z} \end{cases} \Rightarrow \begin{cases} u = \frac{y}{z} \\ v = y + z \end{cases} \Rightarrow \begin{cases} u' = u \\ v' = 0 \end{cases}$$

Our goal

Consider some system of differential equations of the general form:

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_n(y_1, \cdots, y_n) \end{cases}$$

where $f_i \in \mathbb{C}(x_1, \cdots, x_n)$.

What we saw: it is possible for solutions of such a system to be in rational bijection with solutions of a linear system. In that case, we obtain a rational parametrization by "transferring" the linear one.

What we'll do: the converse is true! If such a system has a parametrization using rational functions, then it must be in rational bijection with a linear system.

How a model theorist thinks about this

We want a convenient structure to work with differential equations.

Bare minimum: differential fields of characteristic zero, i.e. fields equipped with a differential δ that is additive and satisfies Leibniz's rule $\delta(ab) = \delta(a)b + a\delta(b)$. We will often denote $\delta(a) = a'$.

The theory of differential fields of characteristic zero has a model companion, which is the theory DCF_0 of differentially closed fields.

Concretely, this means that if $K \models \mathrm{DCF}_0$ and some finite system of differential (in)equations, defined over some parameters $A \subset K$, has a solution in some differential field extension $K < L$, then it has a solution in K.

DCF_0 has quantifier elimination: any formula is equivalent to a boolean combination of differential equation and inequations.

Types in DCF_0

It will be more convenient to work with types, instead of definable sets. Fix some $M \models \mathrm{DCF}_0$.

Definition

Given $b \in M$ and $A \subset M$, the type of b over A is the set of all formulas, with parameters in A, that are satisfied by b. We denote it $\mathrm{tp}(b/A)$.
In general, a type p over A is a maximal consistent set of formulas with parameters in A. We let $S(A)$ be the set of types over A.

By quantifier elimination, $\mathrm{tp}(b/A)$ is just the set of differential equations and inequations, with parameters in A, satisfied by b.

An important example

We will care about systems of the form:

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_n(y_1, \cdots, y_n) \end{cases}$$

where the f_i are rational functions.
Such a system has a generic type p: this is the type of some $a_1, \cdots, a_n$ satisfying these equations, but no other non-trivial differential equation.

Because the differential equations have order one, this means simply no polynomial equation.

Fixing some model

For the rest of the talk, I will fix some monster model $\mathcal{U} \models \mathrm{DCF}_0$, or equivalently, a model that is homogeneous and saturated in its own cardinality.

Concretely, if $A \subset \mathcal{U}$ and $|A| < |\mathcal{U}|$, then for any $p \in S(A)$:

- the set of realizations of p in $\mathcal{U}$, i.e. elements of $\mathcal{U}$ satisfying all formulas in p, is non-empty. We denote it $p(\mathcal{U})$, and write $a \models p$ for $a \in p(\mathcal{U})$.
- if $b \in p(\mathcal{U})$, then $p(\mathcal{U})$ is the orbit of b under the action of $\mathrm{Aut}(\mathcal{U}/A)$.

To define what I mean by "parametrizing", I need the $\emptyset$-definable set of constants:

$$\mathcal{C} = \{x \in \mathcal{U}, \delta(x) = 0\}$$

Thinking $\mathcal{C} = \mathbb{C}$ is fine.

Rational parametrization = Internality

F will always be an algebraically closed differential field.

Definition

A type $p \in S(F)$ is $\mathcal{C}$-internal if there are

- $a_1, \cdots, a_n$ realizations of p
- an F-definable function $f(x_1, \cdots, x_n, y_1, \cdots, y_m)$

such that for all $a \models p$, there are $c_1, \cdots, c_m \in \mathcal{C}$ with:

$$a = f(a_1, \cdots, a_n, c_1, \cdots, c_m)$$

If we replace f with a one-to-finite correspondence, then we say p is almost $\mathcal{C}$-internal.

What we want

Question

Is there a criterion to determine whether a type $p \in S(F)$ is almost $\mathcal{C}$-internal?

We think of types as representing generic solutions of a differential equations. We will examine systems of the form:

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_n(y_1, \cdots, y_n) \end{cases}$$

where the $f_i \in F(x_1, \cdots, x_n)$ and F is a field *of constants*.

A general system may have some polynomial equations between the y_i. We will not deal with this more general case.

Binding groups

Internal types are structured by the following theorem:

Theorem

If $p \in S(F)$ is $\mathcal{C}$-internal, then the group action of restrictions to $p(\mathcal{U})$ of automorphisms of $\mathcal{U}$ is isomorphic to an F-definable group action.
It is called the binding group of p, and denoted $\mathrm{Aut}_F(p/\mathcal{C})$.

Moreover, the group $\mathrm{Aut}_F(p/\mathcal{C})$ is definably isomorphic to $G(\mathcal{C})$, for some algebraic group G.

Key properties:

- if $\mathrm{Aut}_F(p/\mathcal{C})$ acts transitively, we say p is weakly $\mathcal{C}$-orthogonal.
- if $\mathrm{Aut}_F(p/\mathcal{C})$ acts freely (i.e. without fixed point), we say p is fundamental.

Weakly orthogonal and fundamental

Fact (Kolchin, model-theoretic translation by Jaoui-Moosa)

Let $p \in S(F)$ be a $\mathcal{C}$-internal, weakly $\mathcal{C}$-orthogonal and fundamental type. Then there is an algebraic group G defined over $F \cap \mathcal{C}$ such that p is interdefinable (i.e. in F-definable bijection) with the generic type q of the solution to a full logarithmic differential equation on G over F.

$\mathrm{Aut}_F(p/\mathcal{C})$ must be definably isomorphic to $G(\mathcal{C})$

What we can do:

(A) reduce to weakly $\mathcal{C}$-orthogonal and fundamental types

(B) control what G can appear as a binding group

(C) write concrete equations for the solution to a full logarithmic differential equation

(D) use interdefinability to obtain an explicit condition for internality

(B) Linear binding groups are abelian

We will only need the two most basic algebraic groups:

- $G_a(\mathcal{C}) = (\mathcal{C}, +)$,
- $G_m(\mathcal{C}) = (\mathcal{C} \setminus \{0\}, \cdot)$.

Fact

Let F be a field of constants and $p \in S(F)$ be an internal, weakly $\mathcal{C}$-orthogonal type. If $\mathrm{Aut}_F(p/\mathcal{C})$ is linear, then it is isomorphic to $G_m(\mathcal{C})^k \times G_a(\mathcal{C})^l$, where $k \in \mathbb{N}$ and $l \in \{0, 1\}$.

The action of the binding group is always faithful, and a faithful transitive action of an abelian group is always free, i.e. p must be fundamental!

(B) The binding group is linear

Consider p the generic type of some system:

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_n(y_1, \cdots, y_n) \end{cases}$$

where the $f_i \in F(x_1, \cdots, x_n)$ and F is a field *of constants*.
We see that:

- the action of $\mathrm{Aut}_F(p/\mathcal{C})$ is definably isomorphic to some birational action of an algebraic group $G(\mathcal{C})$ on the affine space $\mathbb{A}^n(\mathcal{C})$
- algebraic geometry $\Rightarrow$ the binding group is linear

By the previous slide, if p is weakly $\mathcal{C}$-orthogonal, then $\mathrm{Aut}_F(p/\mathcal{C})$ is definably isomorphic to $G_m(\mathcal{C})^k \times G_a(\mathcal{C})^l$ for some $k \in \mathbb{N}$ and $l \in \{0, 1\}$.

(C) Logarithmic differential equations

To summarize: let F be a field of constants, and $p \in S(F)$ be a $\mathcal{C}$-internal, weakly $\mathcal{C}$-orthogonal type. Then p is interdefinable with the generic type of a full logarithmic differential equation on $(G_m)^k \times (G_a)^l$, with $k \in \mathbb{N}$ and $l \in \{0, 1\}$.

Such an equation can be expressed by (if $l = 1$):

$$\begin{cases} z_1' = \lambda_1 z_1 \\ \vdots \\ z_k' = \lambda_k z_k \\ z_{k+1}' = 1 \end{cases}$$

and fullness is equivalent to the λ_i being $\mathbb{Q}$-linearly independent.

A dimension argument shows that it's either $(G_m)^{n-1} \times G_a$ or $(G_m)^n$, i.e. $k + 1 = n$

(D) What interdefinability gives

Assume we are in the $(G_m)^{n-1} \times G_a$ case.

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_n(y_1, \cdots, y_n) \end{cases} \xrightarrow{F\text{-definable bijection}} \begin{cases} z_1' = \lambda_1 z_1 \\ \vdots \\ z_{n-1}' = \lambda_{n-1} z_{n-1} \\ z_n' = 1 \end{cases}$$

By quantifier elimination:
the definable bijection is given by rational maps $g_1, \cdots, g_{n-1}, h \in F(x_1, \cdots, x_n)$ such that:

$$\begin{cases} g_1(y_1, \cdots, y_n)' = \lambda_1 g_1(y_1, \cdots, y_n) \\ \vdots \\ g_{n-1}(y_1, \cdots, y_n)' = \lambda_{n-1} g_{n-1}(y_1, \cdots, y_n) \\ h(y_1, \cdots, y_n)' = 1 \end{cases}$$

Main theorem in the weakly orthogonal case

Theorem (Eagles-J.)

Let F be an algebraically closed field of constants, some $f_1, \cdots f_n \in F(x_1, \cdots, x_n)$ and p the generic type of the system:

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_k(y_1, \cdots, y_n) \end{cases}$$

Then p is almost $\mathcal{C}$-internal and weakly $\mathcal{C}$-orthogonal if and only if there are rational functions $g_1, \cdots, g_{n-1}, h \in F(x_1, \cdots, x_n)$, $\mathbb{Q}$-linearly independent $\lambda_1, \cdots, \lambda_{n-1} \in F$ with:

- $\sum_{i=1}^{n} \frac{\partial g_j}{\partial x_i} f_i = \lambda_j g_j$ *for all* $1 \leq j \leq n-1$,
- $\sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i = 1$.

or some similar equations for the $(G_m)^n$ case.

What about the non-weakly $\mathcal{C}$-orthogonal case?

Example

The generic type of:

$$\begin{cases} y' = \frac{yz}{y+z} \\ z' = -\frac{yz}{y+z} \end{cases}$$

is internal, and not weakly $\mathcal{C}$-orthogonal:
$(y+z)' = 0$, so $y + z \in \mathcal{C}$, which must be fixed by the binding group $\Rightarrow$ the binding group does not act transitively.

Non-weak $\mathcal{C}$-orthogonality was witnessed by a definable function $(y, z) \to y + z$ to $\mathcal{C}$. This is true in general:

Lemma

Let F be an algebraically closed field of constants and $p \in S(F)$. Then there is an F-definable map $\pi : p \to \pi(p)$ such that $\pi(p)(\mathcal{U}) \subset \mathcal{C}$ and for any $a \models p$, the type $\mathrm{tp}(a/\pi(a)F)$ is stationary and weakly $\mathcal{C}$-orthogonal.

(A) Reducing to weakly $\mathcal{C}$-orthogonal: proof idea

Let F be an algebraically closed field of constants and some internal type $p \in S(F)$.
Consider the map π from the previous slide, so for any $a \models p$:

- $\pi(a) \in \mathcal{C}$
- $\mathrm{tp}(a/\pi(a)F)$ is stationary and weakly $\mathcal{C}$-orthogonal

Then:

- $\mathrm{Aut}_F(p/\mathcal{C})$ is linear $\Rightarrow$ $\mathrm{Aut}_{\pi(a)F}(\mathrm{tp}(a/\pi(a)F)/\mathcal{C})$ is also linear
- $\mathrm{tp}(a/\pi(a)F)$ is a type over constant parameters. We can (modulo technicalities) apply our previous theorem to get rational maps to logarithmic-differential equations on G_m or G_a
- π is also given by rational maps, we can pick them to be algebraically independent elements of $F(x_1, \cdots, x_n)$

Theorem (Eagles-J.)

The generic type p of the system:

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_k(y_1, \cdots, y_n) \end{cases}$$

is almost $\mathcal{C}$-internal if and only if there are rational functions $g_1, \cdots, g_{k-1}, h, h_1, \cdots h_{n-k} \in F(x_1, \cdots, x_n)$, $\mathbb{Q}$-linearly independent $\lambda_1, \cdots, \lambda_{k-1} \in F$ with:

- $\sum_{i=1}^{n} \frac{\partial g_j}{\partial x_i} f_i = d_j g_j$ for all $1 \leq j \leq k-1$ and $\sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i = 1$,
- the h_j are algebraically independent over F and $\sum_{i=1}^{n} \frac{\partial h_j}{\partial x_i} f_i = 0$ for all $1 \leq j \leq n-k$.

or some similar equations if the binding group is $(G_m)^k$.

Orthogonality to the constants

What if they are no rational functions g_j or h_j? This corresponds to the model-theoretic notion of orthogonality to the constants:

Theorem (Eagles-J.)

The generic type p of the system:

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_k(y_1, \cdots, y_n) \end{cases}$$

is orthogonal to the constants if and only if there are no $g \in F(x_1, \cdots, x_n)$ and $\gamma \in F$ such that $\sum_{i=1}^{n} \frac{\partial g}{\partial x_i} f_i = \gamma g$ or $= \gamma$.

Orthogonality to the constants implies that the generic solutions f_i are not Liouvillian: they cannot be constructed using elementary functions, composition, and integration.

An application: the classic Lotka-Volterra system

The Lotka-Volterra system models predator-prey populations:

- x represents the prey population,
- y represents the predator population,

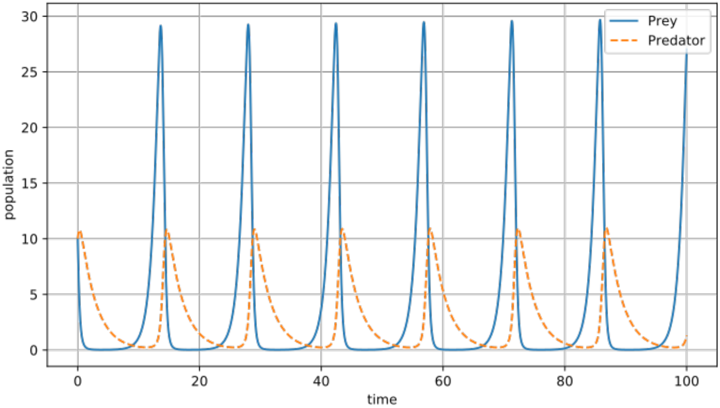
and is given by:

$$\begin{cases} x' = ax - bxy \\ y' = -cy + dxy \end{cases}$$

For realism, one may ask that a, b, c, d are strictly positive real numbers. We instead pick $a, b, c, d \in \mathcal{C} \setminus \{0\}$.

Graphs of (real) solutions

$$\begin{cases} x' = ax - bxy & \text{prey} \\ y' = -cy + dxy & \text{predator} \end{cases}$$



Credit: Ian Alexander (parameters, PNG version) Krishnavedala (vectorisation), from wikipedia.

Mostly not Liouvillian

Theorem (Eagles-J.)

Unless $a = c$, the generic solution of the Lotka-Volterra system:

$$\begin{cases} x' = ax + bxy \\ y' = cy + dxy \end{cases}$$

is not Liouvillian. If $a = c$ it is elementary (proved by Varma [3]).

Using our theorem, it is enough to show that the partial differential equations (where $\mu = \frac{a}{c}$):

$$c\frac{\partial g}{\partial x_0}(\mu x_0 + x_0x_1) + c\frac{\partial g}{\partial x_1}(x_1 + x_0x_1) = \begin{cases} 0 \\ 1 \\ \lambda g \ (\lambda \in \mathbb{Q}(a,b,c,d)^{\mathrm{alg}}) \end{cases}$$

have no rational solutions. We use Laurent series.

Further work

- add polynomial equations. Issue: The binding group need not be linear anymore. But the Chevalley decomposition should help.
- work over non constant parameters.
  - probably not as nice of a result: any algebraic group can appear as a binding group.
  - hope in low dimension. The case $n = 1$ has essentially been solved by Jaoui-Moosa [2]. If $n = 2$, we are essentially interested in connected algebraic groups acting on $\mathbb{P}^2$, which were classified by Enriques [1].
- can model theory say anything about parametrizations by non-rational functions? For example solutions of $y''y - (y')^2 = 0$ are $\{ce^{dx} : c, d \in \mathbb{C}\}$. The generic type is not almost $\mathcal{C}$-internal, essentially because $x \to e^x$ is not definable in DCF_0.

Thank you!

Bibliography

Federigo Enriques.
Sui gruppi continui di trasformazioni cremoniane nel piano.
Tip. della R. Accademia dei Lincei, 1893.

Rémi Jaoui and Rahim Moosa.
Abelian reduction in differential-algebraic and bimeromorphic geometry.
To appear in Annales de l'Institut Fourier, 2022.

VS Varma.
Exact solutions for a special prey-predator or competing species system.
Bulletin of Mathematical Biology, 39:619–622, 1977.