What generic automorphisms of the random poset look like

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What are “generic automorphisms”? 

Definition

For a Polish group $G$, an element $g \in G$ is called generic if it lies in a (necessarily unique) comeagre conjugacy class.

If $K$ is a countable (first-order) structure, its automorphism group $\text{Aut}(K)$ is a Polish group with the pointwise convergence topology. Thus, a generic automorphism of $K$ is a generic element of $\text{Aut}(K)$. 
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What structures do we care about here?

Definition
Fix a relational language $L$, and let $K$ be a countable structure.

- A partial automorphism of $K$ is an isomorphism between substructures of $K$.
- Let $\text{Aut}^{<\omega}(K)$ denote the set of finite partial automorphisms of $K$.
- For each $p \in \text{Aut}^{<\omega}(K)$, let $[p]$ denote the set of $f \in \text{Aut}(K)$ extending $p$.

- $K$ is ultrahomogeneous if every finite partial automorphism of $K$ extends to a (full) automorphism of $K$.
- Equivalently, if $[p]$ is non-empty for all $p \in \text{Aut}^{<\omega}(K)$.

- If $K$ is a class of $L$-structures, $K$ is universal for $K$ if every structure in $K$ embeds into $K$. 
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- If $\mathcal{R}$ is a class of $L$-structures, $K$ is **universal** for $\mathcal{R}$ if every structure in $\mathcal{R}$ embeds into $K$. 

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We are interested in **Fraïssé structures**: ultrahomogeneous countable structures which are universal for a coherent class of finite structures called a **Fraïssé class**.
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Their automorphism groups:

Remark All the examples on the previous slide admit generic automorphisms. (That is, their automorphism groups have comeagre conjugacy classes.)

What might generic automorphisms look like? For example:

Theorem (Truss, 1991)

Let $f \in S_\infty$. Then $f$ is generic if and only if:

• $f$ has no infinite orbits;
• For every $n$, $f$ has infinitely many orbits of length $n$.

Remark Generic automorphisms of $\langle \mathbb{Q}, < \rangle$ and the random graph admit similar kinds of descriptions.
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What about the random poset?

Theorem (Kuske–Truss, 2000)

*The random poset \( P \) admits generic automorphisms.*
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Theorem (Kuske–Truss, 2000)

*The random poset $\mathcal{P}$ admits generic automorphisms.*

Goal

Find an explicit description of generic automorphisms of $\mathcal{P}$. 
Definition
Let $(P, \leq)$ be any poset, and let $f \in \text{Aut}(P)$.

- The **spiral length** of $x$, denoted $\text{sp}(x, f)$, is the least $n \geq 1$ for which $x$ and $f^n(x)$ are comparable, or $\infty$ if no such $n$ exists.

- The **parity** of $x$ is given by:
  
  $$
  \text{par}(x, f) = \begin{cases}
  +1 & \text{if } \text{sp}(x, f) = n < \infty \text{ and } x < f^n(x) \\
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Tools for describing automorphisms’ actions

Example: suppose $\text{sp}(x, f) = 3$ and $\text{par}(x, f) = +1$. 

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Let \((P, <)\) be any poset, and let \(f \in \text{Aut}(P)\).

- Define 
  \(x \sim f y \iff \exists i, j \in \mathbb{Z} \left( f^i(x) \leq y \leq f^j(x) \right) \).

- \(\sim_f\) is an equivalence relation.

Let the quotient map be denoted by 
\(O_f : P \to P/\sim_f\).

- The equivalence classes \(O_f(x)\) are called orbitals.

- The quotient \(O_f[P]\) is called the orbital quotient.
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More tools for describing automorphisms’ actions

Orbitals are the “convex hulls” of orbits.
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Facts

- $f(Z(x)) \subseteq O(f(x))$, and equality holds whenever $\text{par}(x, f) = 0$.
- Parity is orbital-invariant; that is, $x \sim f(y)$ implies $\text{par}(x, f) = \text{par}(y, f)$.
- Spiral length need not be orbital-invariant (unless $\text{par}(x, f) = 0$).
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Example in $\mathbb{Q}$:

Let $f(x) := \begin{cases} 2x + 3 & x \leq -2, \\ \frac{x}{2} & -2 \leq x \leq 2, \\ 2x - 3 & 2 \leq x. \end{cases}$
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Then there are seven orbitals: three of parity 0, two each of parity \(-1\) and \(+1\).
Orbitals inherit order structure from $P$!

**Definition**

Define two orders on $\mathcal{O}_f [P]$ — one strong and one weak:

- $\mathcal{O}_f (x) < s f \mathcal{O}_f (y) \iff x' < y'$ for all $x' \sim_f x$ and $y' \sim_f y$.
- $\mathcal{O}_f (x) \leq w f \mathcal{O}_f (y) \iff x' \leq y'$ for some $x' \sim_f x$ and $y' \sim_f y$.

**Remark**

If $P$ is linearly ordered, these orders agree and are linear orders themselves.
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$$O_f(x) <^s_f O_f(y) \iff x' < y' \text{ for all } x' \sim_f x \text{ and } y' \sim_f y,$$

$$O_f(x) \leq^w_f O_f(y) \iff x' \leq y' \text{ for some } x' \sim_f x \text{ and } y' \sim_f y.$$
Orbitals inherit order structure from $P$!

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**Remark**

If $P$ is linearly ordered, these orders agree and are linear orders themselves.
Theorem (Truss, 1991)

Let $f \in \text{Aut}(Q)$. Then $f$ is generic if and only if $O_f[Q] \sim = Q$, and for each $\sigma \in \{+1, -1, 0\}$, the set of orbitals of parity $\sigma$ is dense in $O_f[Q]$.

Question

Is an analogous statement true for $P$?

Answer

Partially.
Using orbitals to characterize generics

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Using orbitals to characterize generics of Aut (P)?

Theorem (I., 2020)

If $f \in \text{Aut} (\mathbf{P})$ is generic, then

$$(O_f[\mathbf{P}], <) \sim = \mathbf{P}.$$ 

Moreover, the following sets are dense in $(O_f[\mathbf{P}], <)$:

- $\{O_f(x) : \text{par}(x, f) = +1\}$
- $\{O_f(x) : \text{par}(x, f) = -1\}$
- $\{O_f(x) : \text{par}(x, f) = 0, \text{sp}(x, f) = n\}$
for each $1 \leq n \leq \infty$.

Remark This is a partial answer to our goal because we do not know if the converse holds: whether this property implies genericity.
Using orbitals to characterize generics of $\text{Aut}(P)$?

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If $f \in \text{Aut}(P)$ is generic, then $(\mathcal{O}_f[P], \prec_f) \cong P$. Moreover, the following sets are dense in $(\mathcal{O}_f[P], \prec_f)$:

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If $f \in \text{Aut}(\mathbf{P})$ is generic, then $(\mathcal{O}_f[\mathbf{P}], <_f^s) \cong \mathbf{P}$. Moreover, the following sets are dense in $(\mathcal{O}_f[\mathbf{P}], <_f^s)$:

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This is a partial answer to our goal because we do not know if the converse holds: whether this property implies genericity.
Note

Equip $\mathbf{P}$ with a relation symbol for the graph of a unary function, and consider the resulting structures $(\mathbf{P}, f)$ and $(\mathbf{P}, g)$ for $f, g \in \text{Aut}(\mathbf{P})$. Then $f$ and $g$ are conjugate iff $(\mathbf{P}, f) \cong (\mathbf{P}, g)$. 

Question: Can we study this structure to describe the conjugacy relation on $\text{Aut}(\mathbf{P})$?

Answer: Yes and no. Finite substructures in this language don't "remember" enough.
A different tactic

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Can we study this structure to describe the conjugacy relation on $\text{Aut}(\mathbf{P})$?

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Yes and no. Finite substructures in this language don’t “remember” enough.
A first-order language

Definition

• Let $L$ be the language consisting of binary relations $b_i$, for $i \in \mathbb{Z}$.

• For each poset $P$ and each $f \in \text{Aut}(P)$, let $P^f$ be the $L$-structure obtained by letting:
  
  $b^f_i(x, y) \iff x \leq f_i(y)$.

• We identify $b^f_i(x, y)$ with its truth value in $\{0, 1\}$, and we consider the bi-infinite sequence $b^f(x, y) \in 2^\mathbb{Z}$.

Remark: $P^f \sim P^g$ iff $f$ and $g$ are conjugate in $\text{Aut}(P)$.

But also, since $L$ is infinite, finite substructures can encode a lot more information.
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Remark $P_f \sim = P_g$ iff $f$ and $g$ are conjugate in $\text{Aut}(P)$. But also, since $L$ is infinite, finite substructures can encode a lot more information.
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$P_f \cong P_g$ iff $f$ and $g$ are conjugate in $\text{Aut}(P)$. 
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$P_f \cong P_g$ iff $f$ and $g$ are conjugate in $\text{Aut}(P)$. But also, since $L$ is infinite, finite substructures can encode a lot more information.
How to control $b^f$ sequences

Remark

Controlling behavior of $b^f$ sequences with finitary configurations is the name of the game here.
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Example

Suppose the following is a substructure of $P$:

![Diagram](image_url)

Let $p := \{ (y, y'), (z, z') \} \in \text{Aut}^{<\omega}(P)$. Then $\text{sp}(x, f) = \infty$ for every $f \in [p]$, and so $b^f_i(x, x) = 0$ for all $i \neq 0$. 
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```
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   /\    /\  \
  y    x  z'
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How to control $b^f$ sequences

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What generic automorphisms of the random poset look like

Theorem (I., 2020)

Let $f \in \text{Aut}(P)$. Then $f$ is generic iff the following hold:

(A) $f$ has dense conjugacy class;
(B) $Pf$ is ultrahomogeneous (as an $L$-structure);
(C) $b_{f}(x, x)$ is eventually constant on both sides whenever $\text{par}(x, f) \neq 0$;
(D) A technical condition — illustrated on the next slide — that forces $b_{f}(x, y)$ to be eventually periodic on both sides for all $x, y \in P$. 
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Let $x, y \in P$ such that $\text{sp} (x, f) = \infty$ and $y \notin f^\mathbb{Z} (x)$.
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Let \( x, y \in P \) such that \( \text{sp} (x, f) = \infty \) and \( y \notin f^\mathbb{Z} (x) \). (D) asserts there is some chunk of \( f^\mathbb{Z} (x) \) ...
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Let $x, y \in \mathbb{P}$ such that $\text{sp}(x, f) = \infty$ and $y \notin f^\mathbb{Z}(x)$. (D) asserts there is some chunk of $f^\mathbb{Z}(x)$ ... where this configuration exists.
Remark

We hoped model theory might be able to help, but there are some notable complications.

Facts

Suppose $f \in \text{Aut}(P)$ is generic.

- The $L$-theory of $P_f$ is not $\omega$-categorical.
- $P_f$ is not $\omega$-saturated.
- The relation $\text{sp}(x, f) = \infty$ is definable in $P_f$, but not quantifier-freely.

Thus, the $L$-theory of $P_f$ does not have QE.
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Tack så mycket!