Model theory and topological groups

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Prologue - ultrafilters and ultrapowers

A collection $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is a filter if

- $\omega \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$,
- If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$,
- If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$
- $\omega \setminus n \in \mathcal{F}$ for every $n \in \omega$.

A filter $\mathcal{F}$ is an ultrafilter if it is maximal or, equivalently,

- if $\omega = A \cup B$ then $A \in \mathcal{F}$ or $b \in \mathcal{F}$.

(We shall denote ultrafilters by p or q, possibly with indices.)

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Prologue - Fubini products and Frolík sums

Given ultrafilters p and q let

$$p \otimes q = \{A \subseteq \omega \times \omega : \{n : \{m : (n,m) \in A\} \in p\} \in q\}$$

moreover, given a sequence $\{p_n : n \in \omega\}$ of ultrafilters let

$$q\text{-}\sum_{n\in\omega} p_n = \{A \subseteq \omega \times \omega : \{n : \{m : (n,m) \in A\} \in p_n\} \in q\}$$

and, recursively,

$$p^{\alpha+1} = p^\alpha \otimes p \text{ and } p^\alpha = p\text{-}\sum_{n\in\omega} p^{\alpha_n}$$

for some/any $\alpha_n \nearrow \alpha$ for $\alpha < \omega_1$ limit ordinal.

Prologue - Special ultrafilters

Let p be an ultrafilter on ω. Then p is

- selective (Choquet '68)
  $\forall f \in \omega^\omega \; \exists U \in p \; (f \restriction U$ is constant or one-to-one$)$,
- Hausdorff (Daguenet-Teissier '79, R. A. Pitt '71, Choquet '68)
  $\forall f, g \in \omega^\omega \; \exists U \in p \; (f \restriction U = g \restriction U$ or $f[U] \cap g[U] = \emptyset)$.

Facts:

- (Kunen '76) Consistently, selective ultrafilters do not exist.
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Prologue - Ultrapowers of algebraic structures (Hewitt '48, Łoś '55)

Given a group $(\mathbb{G}, \circ)$ and an ultrafilter $p \in \omega^*$

$$ult_p(\mathbb{G}) = (\mathbb{G}^\omega / \equiv_p) \text{ where } f \equiv_p g \text{ iff } \{n : f(n) = g(n)\} \in p.$$

- endowed with the pointwise operation(s),
- having (a copy of) $\mathbb{G}$ as a subgroup (via constant functions), and
- Łoś's Theorem

$$ult_p(\mathbb{G}) \models \varphi([f_0], \ldots, [f_n)]) \text{ iff } \{j : \mathbb{G} \models \varphi(f_0(j), \ldots, f_n(j)) \in p.$$

Prologue - Ultrapowers of topological spaces

Given a topological space (X, τ) with a fixed basis $\mathcal{B}$, and an ultrafilter $p \in \omega^*$ let

$$ult_p(X) = (X^\omega / \equiv_p) \text{ where } f \equiv_p g \text{ iff } \{n : f(n) = g(n)\} \in p$$

endowed it with the topology which has as a basis the collection $\{U^* : U \in \mathcal{B}\}$, where

$$U^* = \{[f] \in ult_p(X) : \{n : f(n) \in U\} \in p\} .$$

The ultrapower with this topology is usually not Hausdorff. In fact,

- p is Hausdorff iff $ult_p(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ is Hausdorff, and
- If $ult_p(X)$ is Hausdorff then p is Hausdorff.
- (Di Nasso-Forti) p^2 is never Hausdorff.

We identify the inseparable classes and denote by $Ult_p(X)$ this quotient.

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Versions of topological compactness

Given an ultrafilter p on ω, a sequence $\{x_n : n \in \omega\}$ and a point x contained in a topological space X we say that

$$x = p\text{-}\lim x_n$$

if $\{n \in \omega : x_n \in U\} \in p$ for every neighbourhood U of x.

Definition

Let X be a topological space and let p be a free ultrafilter p on ω.

- X is compact (Alexandroff-Urysohn '29) if every open cover of X has a finite subcover, eqiv. every ultrafilter on X converges.
- X is p-compact (Bernstein '70) if for every sequence $\{x_n : n \in \omega\} \subseteq X$ there is a point $x \in X$ such that $x = p\text{-}\lim x_n$.
- X is countably compact (Fréchet '28) if every countable open cover of X has a finite subcover, eqiv. every sequence has a p-limit for some $p \in \omega^*$,
- X is pseudo-compact (Hewitt '48) if every continuous function $f : X \to \mathbb{R}$ is bounded.

Versions of compactness and products

- (Tychonoff '30/'35) Any product of compact spaces is compact.
- (Bernstein '70) Any product of p-compact spaces is p-compact.
- (Teresaka '52, Novák '53) There are countably compact spaces whose square is not even pseudo-compact.
- (Comfort-Ross '66) Any product of pseudo-compact topological groups is pseudocompact.

Problem (Comfort '66)

Are there countably compact groups $\mathbb{G}, \mathbb{H}$ such that $\mathbb{G} \times \mathbb{H}$ is not countably compact?

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Comfort's problem - consistent solutions

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- (Hajnal-Juhász '76) (CH) There is a boolean countably compact group without convergent sequences.

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van Douwen's problem - consistent solutions

Problem (van Douwen '80)

Is there countably compact group without convergent sequences?

- (Kuz'minov '58) Every compact topological group contains a convergent sequence.
- (Hajnal-Juhász '76) Yes assuming CH.
- (van Douwen '80) Yes assuming MA.
- (Tomita) Yes assuming MA_{ctble}.
- ...

All of these describe consistent inverse limit constructions of subgroups of $2^{\mathfrak{c}}$.

van Douwen's problem - solution

Theorem (H.–van Mill–Ramos-García–Shelah 2021)

There is a countably compact subgroup of $2^{\mathfrak{c}}$ without convergent sequences in ZFC.

Iterated ultrapowers of topological spaces

X is dense in $Ult_p(X)$ and
every sequence in X has a p-limit in $Ult_p(X)$: $[f] = p\text{-}\lim f(n)$.

The process can, of course be iterated:

Given a space X, ultrafilter $p \in \omega^*$ and $\alpha < \omega_1$ let

$$Ult_p^\alpha(X) = Ult_p(\bigcup_{\beta<\alpha} Ult_p^\beta(X))$$

and finally

$$Ult_p^{\omega_1}(X) = \bigcup_{\beta<\omega_1} Ult_p^\beta(X).$$

- $Ult_p^\alpha(X) = Ult_{p^\alpha}(X)$ for $\alpha < \omega_1$,
- $Ult_p^{\omega_1}(X)$ is p-compact, and
- If X is p-compact then $X = Ult_p(X)$.

  (in particular, iterations beyond ω_1 do not produce new spaces).

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The plan

Theorem

There is a countably compact boolean group without convergent sequences.

Find a suitable topological group $\mathbb{G}$ without convergent sequences and consider $Ult_p^{\omega_1}(\mathbb{G})$.

Then $Ult_p^{\omega_1}(\mathbb{G})$ is a p-compact space and a group.

There are two problems to solve:

- Is $Ult_p^{\omega_1}(\mathbb{G})$ with the ultraproduct topology a topological group?
- Does $Ult_p^{\omega_1}(\mathbb{G})$ have convergent sequences?

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Harald August Bohr (1887-1951) - wikipedia

Harald August Bohr (22 April 1887 – 22 January 1951) was a Danish mathematician and footballer. After receiving his doctorate in 1910, Bohr became an eminent mathematician, founding the field of almost periodic functions on which he published a comprehensive survey in the period from 1954 to 1974 with the help of his son-in-law, mathematician Erling Følner. His brother was the Nobel Prize-winning physicist Niels Bohr. He was a member of the Danish national football team for the 1908 Summer Olympics, where he won a silver medal.

Let

$$Hom([\omega]^{<\omega}) = \{\Phi : \Phi \text{ is a group homomorphism from } [\omega]^{<\omega} \text{ to } 2\}$$

and let τ_{Hom} be the Bohr topology, i.e. the weakest topology making all $\Phi \in Hom([\omega]^{<\omega})$ continuous.

- $([\omega]^{<\omega}, \tau_{Hom})$ is homeomorphic to a countable dense subgroup of $2^{\mathfrak{c}}$ without convergent sequences.

Extensions of homomorphisms to ultrapowers

Claim

$Ult_p^{\omega_1}([\omega]^{<\omega})$ is a p-compact topological group for every p in ω^*.

Every $\Phi \in Hom([\omega]^{<\omega})$ naturally extends to a homomorphism $\overline{\Phi} \in Hom(Ult_p([\omega]^{<\omega}))$ by letting

$$\overline{\Phi}([f]_p) = i \text{ iff } \{k : \Phi(f(k)) = i\} \in p.$$

The ultrapower topology is the topology induced by $\{\bar{\Phi} : \Phi \in Hom([\omega]^{<\omega})\}$ on $Ult_p([\omega]^{<\omega})$.

Similarly, the utrapower topology on $Ult_p^{\omega_1}([\omega]^{<\omega})$ is the topology $\tau_{\overline{Hom}}$ induced by the homomorphisms in $Hom([\omega]^{<\omega})$ extended recursively all the way to $Ult_p^{\omega_1}([\omega]^{<\omega})$ by the same formula as before:

$$\overline{\Phi}([f]) = i \text{ if and only if } \{k : \overline{\Phi}(f(k)) = i\} \in p.$$

The plan works ... for selective ultrafilters

Proposition (H.–van Mill–Ramos-García–Shelah 2021)

p is selective iff for every $\{f_n : n \in \omega\}$ of functions $f_n : \omega \to [\omega]^{<\omega}$ which are not constant or equal on an element of p, there is a sequence $\{U_n : n \in \omega\} \subseteq p$ such that the sequence

$$\{f_n(m) : n \in \omega \text{ and } m \in U_n\}$$

is linearly independent.

Corollary (H.–van Mill–Ramos-García–Shelah 2021)

If p is selective then $Ult_p^{\omega_1}([\omega]^{<\omega})$ is a p-compact topological group without convergent sequences.

Proposition (H.–van Mill–Ramos-García–Shelah 2021)

There is an ultrafilter $p \in \omega^*$ such that $Ult_p^{\omega_1}([\omega]^{<\omega})$ contains a convergent sequence.

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The same ... yet different

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Lemma (H.–van Mill–Ramos-García–Shelah 2019)

There is a familly $\{p_\alpha : \alpha < \mathfrak{c}\} \subseteq \omega^*$ such that for every $D \in [\mathfrak{c}]^\omega$ and every sequence $\{f_\alpha : \alpha \in D\} \subseteq ([\omega]^{<\omega})^\omega$ of one-to-one enumerations of linearly independent sets there are $\{U_\alpha : \alpha \in D\}$ such that

1. $\forall \alpha \in D \;\; U_\alpha \in p_\alpha$, and
2. $\{f_\alpha(n) : \alpha \in D \;\&\; n \in U_\alpha\}$ is a linearly independent subset of $[\omega]^{<\omega}$.

Teorema (Fraïssé 1953)

A countable structure $\mathbb{M}$ in a countable language $\mathcal{L}$ is ultra-homogeneous if and only if the $Age(\mathbb{M})$ - the class of finitely generated structures embeddable in $\mathbb{M}$ satisfies

- JEP - $\forall A, B \in Age(\mathbb{M})\ \exists C \in Age(\mathbb{M})\ A \hookrightarrow C$ and $B \hookrightarrow C$,

  and

- AP - $\forall A, B, C \in Age(\mathbb{M})$, $f : A \hookrightarrow B$ and $g : A \hookrightarrow C$
  $\exists D \in Age(\mathbb{M})$, $f' : B \hookrightarrow D$ and $g' : C \hookrightarrow D$ so that $f' \circ f = g' \circ g$.

Morover, given a class $\mathcal{K}$ of finitely generated structures in the same language which is esentially countable numerable closed under embeddings which satisfies JEP and AP, there is (up to isomorphism) a unique countable ultra-homogeneous structure $\mathbb{K}$ called the Fraïssé limit of $\mathcal{K}$ such that $\mathcal{K} = Age(\mathbb{K})$.

Fraïssé classes

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Automatic continuity

- The grups $(\mathbb{R}, +)$ y $(\mathbb{R}^2, +)$ are isomorphic, but there is no continuous (or Borel, or Lebesgue measurable) such isomorphisms.
- (Banach-Pettis 1950) Every Baire-measurable homomorphism between Polish groups is continuous.
- A Polish group G has the automatic continuity property if for every Polish group H, every homomorphism $\varphi : G \to H$ is continuous.

Automatic continuity of automorphism groups of countable ultrahomogeneous structures

Often, automorphism groups of ultra-homogeneous structures have the automatic continuity property.

- (Hodges-Hodkinson-Lascar-Shelah 1993) S_∞
- (Hrushovski 1992) $Aut(\mathcal{R})$
- (Solecki 2005) $Aut(\mathbb{U}_\mathbb{Q})$
- (Rosendal-Solecki 2007) $Aut(\mathbb{Q}, <)$
- (Kechris-Rosendal 2007) ample generics $\Rightarrow$ AC,

Automatic continuity of groups of homeomorphisms

The following homeomorphism groups have automatic continuity:

- (Rosendal-Solecki 2007) $Homeo(2^{\mathbb{N}})$, $Homeo(\mathbb{R})$
- (Rosendal 2008) $Homeo(\Sigma)$ for every compact surface Σ.
- (Mann 2016) $Homeo(K)$ for every compact manifold.

Mapping class groups

Let Σ be an infinite type surface with E its space of ends .Then:

$$Homeo(\Sigma) \longrightarrow MCG(\Sigma) \longrightarrow Homeo(E)$$

where

$$MCG(\Sigma) = Homeo(\Sigma)/\text{homotopy} \simeq Aut(\text{Curve graph of } \Sigma\ )$$

- (Mann 202?)) $MCG(S^2 \setminus Cantor)$ has the automatic continuity property.
- (Mann 202?) There is a surface Σ such that $MCG(\Sigma)$ does not have the automatic continuity property.
- (Hernández-H.-Morales-Randecker-Sedano-Valdez 2022, Lanier-Vlamis 2022)
  No $MCG(\Sigma)$ has "ample generics ".

Groups of homeomorphisms of countable ordinals

- (Hernández-H.-Rosendal-Valdez 2023) $\mathit{Homeo}(K)$ has the automatic continuity property for every countable compact metric space K (a.k.a countable successor ordinal).
- (Hernández-H.-Rosendal-Valdez 2023) $\mathit{Homeo}(\alpha + 1)$ has ample generics if and only if α is an indecomposable ordinal.

Thank you for your attention!