Distributive ℓ-pregroups: decidability and generation

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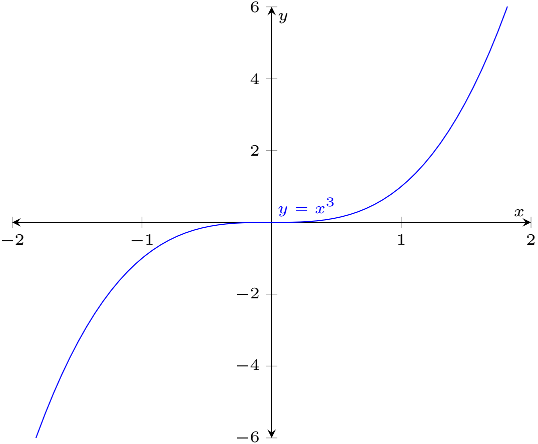
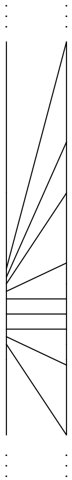
ℓ-groups: $\mathbf{Aut}(\mathbb{Q})$

The collection of the order preserving permutations of $\mathbb{Q}$ i.e. strictly increasing invertible functions from $\mathbb{Q}$ to itself forms an algebra under composition, meet, join and inverse, and we will denote it by $Aut(\mathbb{Q})$. For example:





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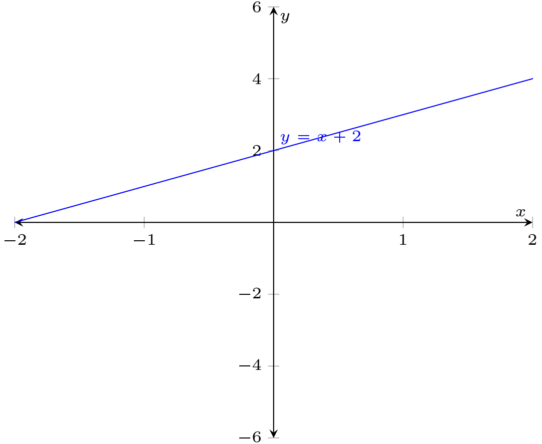

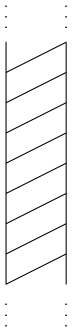

$g(x) = x + 2$

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Every ℓ-group can be embedded $\mathbf{Aut}(\boldsymbol{\Omega})$, for some chain $\boldsymbol{\Omega}$.

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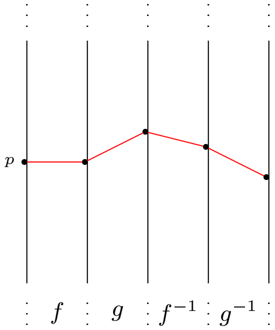

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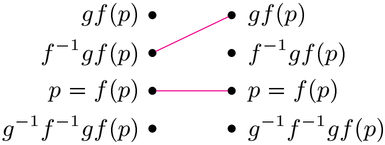

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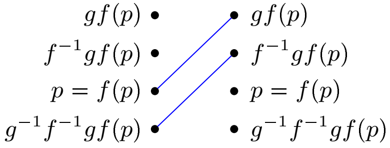

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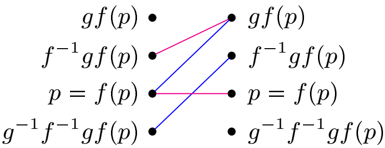

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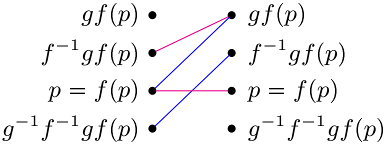

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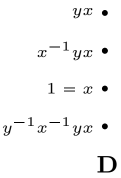
- they are order-preserving partial functions and,
- they are injective.

Building a diagram



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Building a diagram

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$\mathbf{D}$

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Consider

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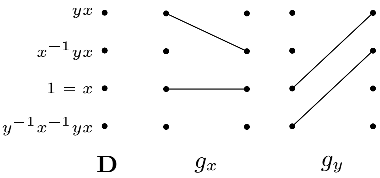

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More formally $|\Delta_\varepsilon|$ with the order on the graphic, $|\Delta_\varepsilon|$ controlled, satisfying that g_x, g_y order preserving, injective, partial functions, satisfies

$$y^{-1}x^{-1}yx < 1$$

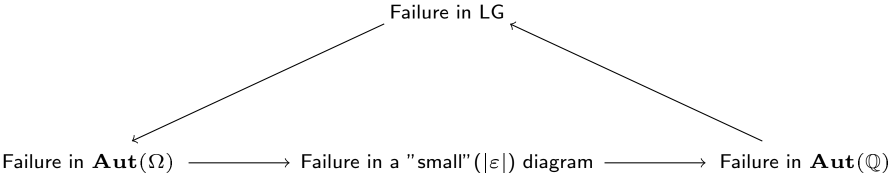
so ε fails.

Theorem (Holland)

If an equation ε fails in an ℓ-group, it fails in a diagram of size at most $|\varepsilon|$.

Theorem (Holland)

If an equation ε fails in a diagram, it fails in $\mathbf{Aut}(\mathbb{Q})$.

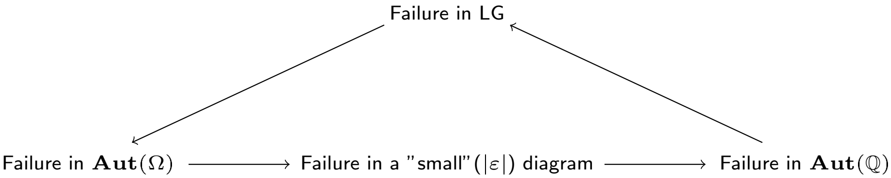


Theorem (Holland - McCleary)

The equational class LG *is decidable*

Theorem (Holland)

The equational class LG *can be generated by* $\mathbf{Aut}(\mathbb{Q})$.



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We will focus for now on distributive ℓ-pregroups, the equational class they form is denoted by DLP.

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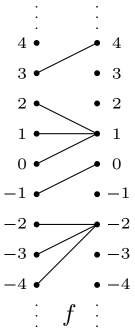

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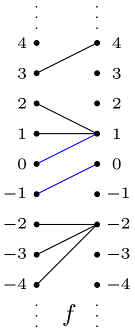

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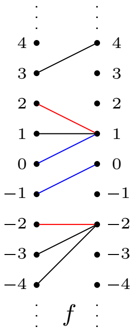

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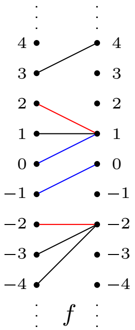

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In general, given a chain $\mathbf{\Omega}$, we denote by $\mathbf{F}(\mathbf{\Omega})$, the collection of all orderpreserving functions f from $\mathbf{\Omega}$ to itself such that $f^{\ell}, f^{\ell\ell}, f^{\ell\ell\ell}, \ldots$ and $f^{r}, f^{rr}, f^{rrr}, \ldots$ exist.

Notation: $f^{\ell} = f^{(1)}, f^{\ell\ell} = f^{(2)}, f^{\ell\ell\ell} = f^{(3)} \ldots$ and $f^{r} = f^{(-1)}, f^{rr} = f^{(-2)}, f^{rrr} = f^{(-3)} \ldots$

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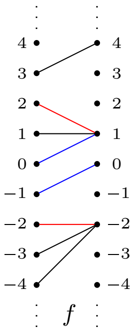

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Theorem (Representation: Galatos-Horcik)

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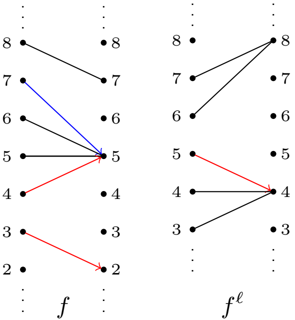
Example of an ℓ-pregroup diagram



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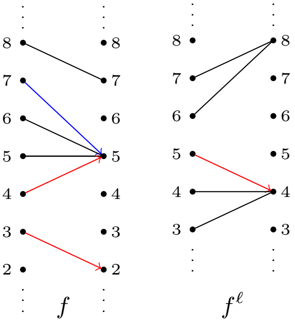

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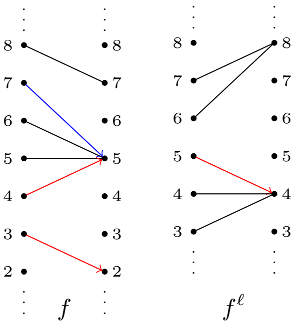

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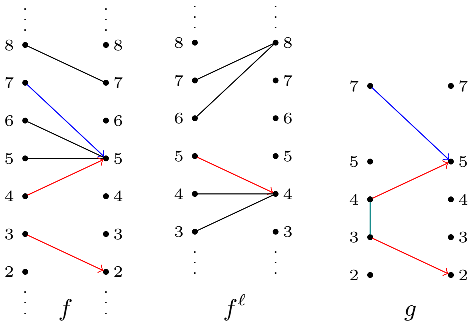

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Also, to make sure that $g^{[\ell]}(5) = 4$ is computed correctly, we need to

- include more elements in the chain
- define g on some of these elements
- mark some covers: $3 \prec 4$.

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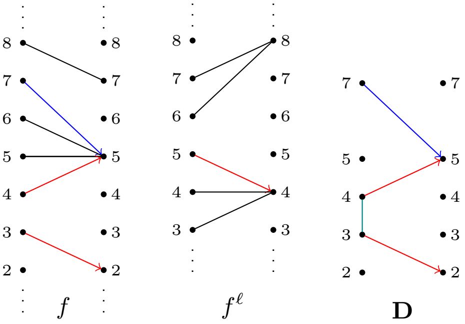

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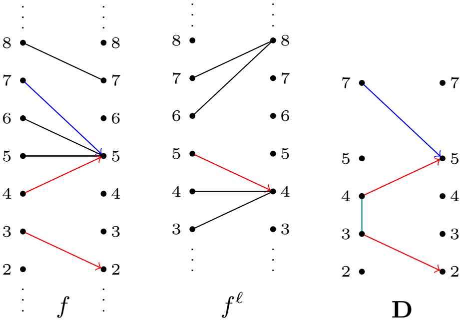

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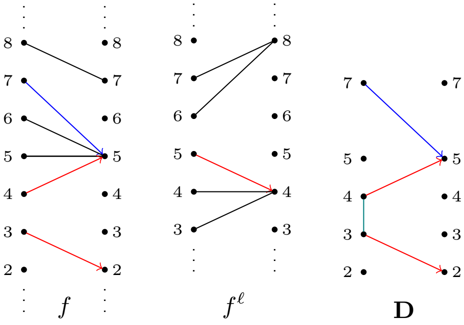

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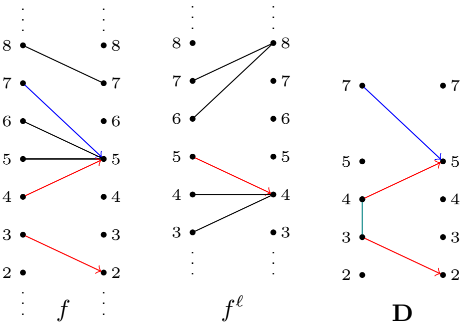

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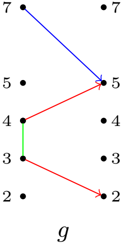

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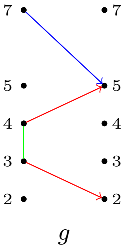

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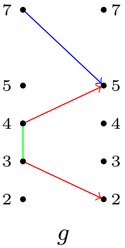

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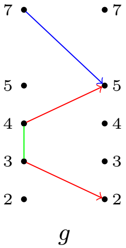

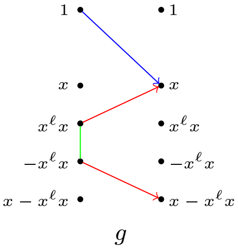

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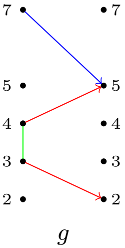

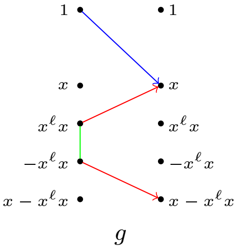

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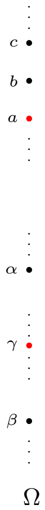
with the ordering on the left, $|\Delta_\varepsilon|$ controlled, satisfying a set of compatibility conditions $x^\ell x < 1$, so the equation fails.

Embedding $\mathbf{F}(\mathbf{\Omega})$ into $\mathbf{F}(\overline{\mathbf{\Omega}})$



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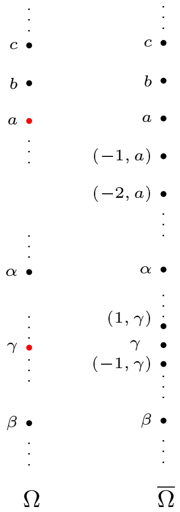

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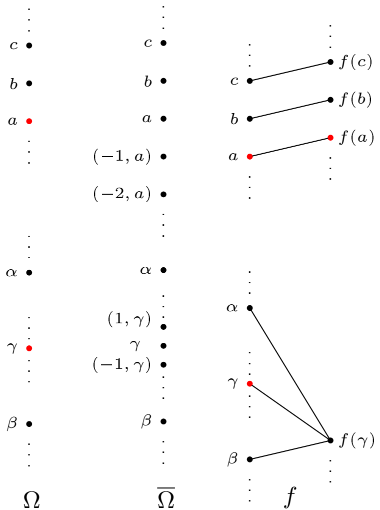

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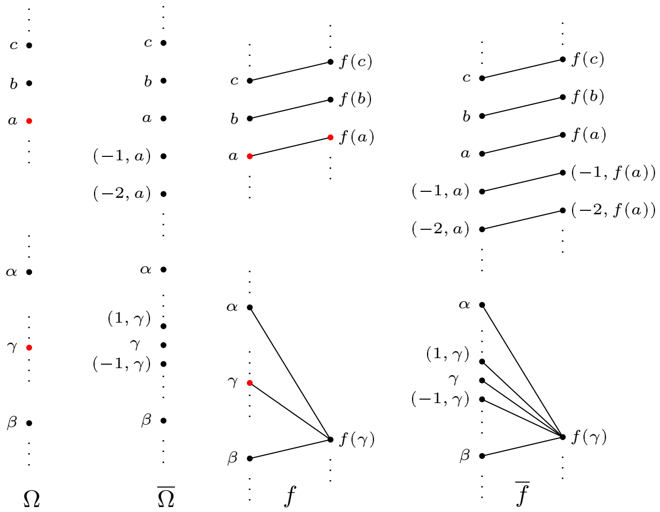

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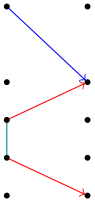
If an equation fails in DLP*, then it fails in $\mathbf{F}(\mathbb{Z})$.*

From a failure to $\mathbf{F}(\mathbb{Z})$



$\mathbf{D}$

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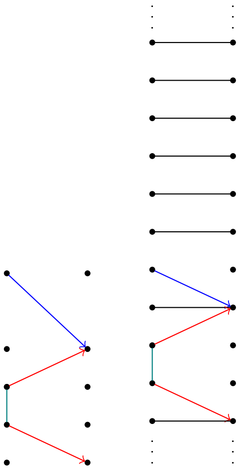

$\mathbf{D}$ $\quad f \in F_{fs}(\mathbb{Z})$

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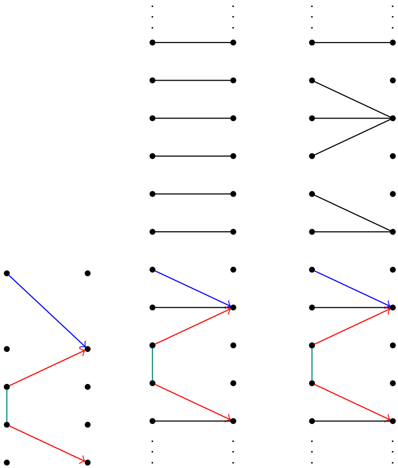

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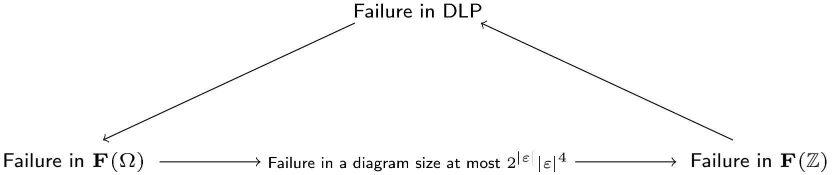

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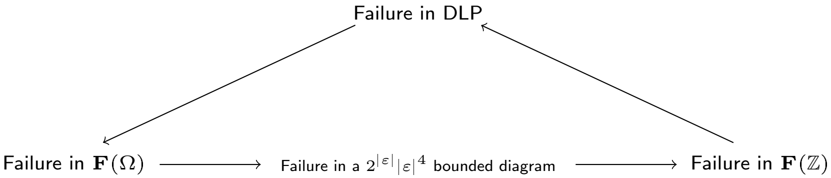

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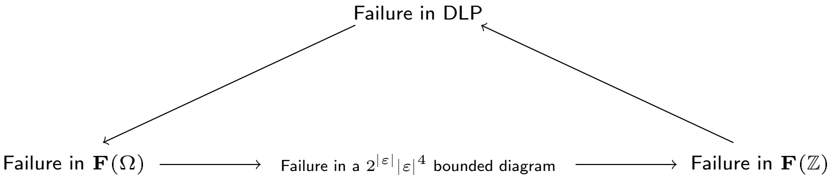
The equational class DLP *can be generated by* $\mathbf{F}(\mathbb{Z})$, $\mathbf{F}_{fs}(\mathbb{Z})$, *or* $\{\mathbf{F}_n(\mathbb{Z}) \mid n \in \mathbb{N}\}$.



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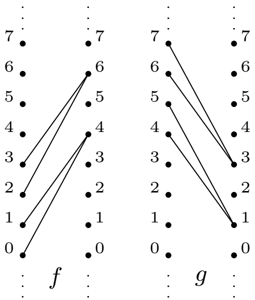

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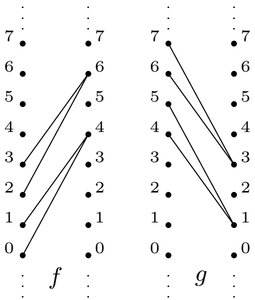

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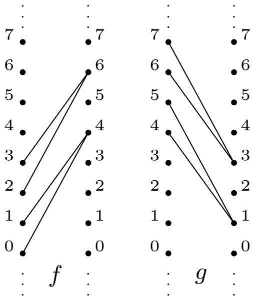

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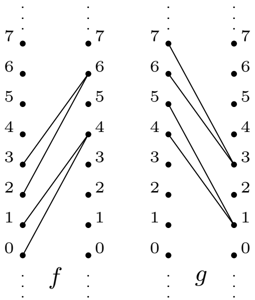

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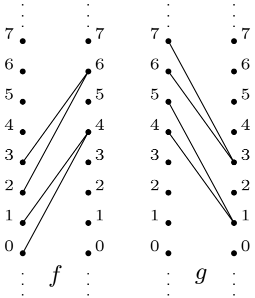

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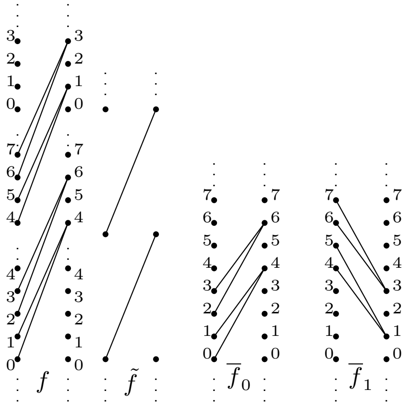
Theorem (Galatos - G.)

For any $n \in \mathbb{Z}$, the equational class LP_n is not generated by $\mathbf{F}_n(\mathbb{Z})$.

Periodic ℓ-pregroups

Theorem (Galatos - G.)

For every chain $\mathbf{J}$ *and* $n \in \mathbb{Z}^+$, $\mathbf{F}_n(\mathbf{J} \overrightarrow{\times} \mathbb{Z}) \cong \mathbf{Aut}(\mathbf{J}) \wr \mathbf{F}_n(\mathbb{Z})$*. Therefore, every* n*-periodic* ℓ*-pregroup can be embedded in a wreath product of an* ℓ*-group and the simple* n*-periodic* ℓ*-pregroup* $\mathbf{F}_n(\mathbb{Z})$*.*

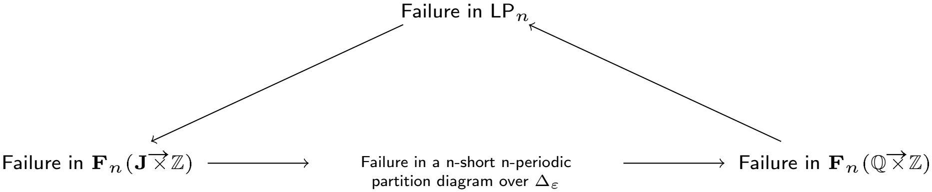


Theorem (Galatos - G.)

If an equation ε fails in an n-periodic ℓ-pregroup, it fails in a n-short n-periodic partition diagram.

Theorem (Galatos - G.)

If an equation ε fails in a n-short n-periodic partition diagram, it fails in $\mathbf{F}_n(\mathbb{Q}\overrightarrow{\times}\mathbb{Z})$.

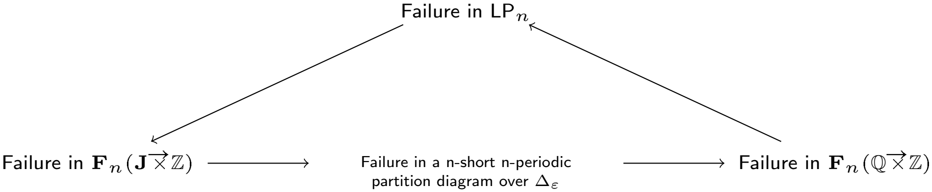


Theorem (Galatos - G.)

The equational class LP_n *is decidable.*

Theorem (Galatos - G.)

The equational class LP_n *is generated by* $\mathbf{F}_n(\mathbb{Q}\overrightarrow{\times}\mathbb{Z})$*.*



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Thank you for your attention!!