Non-Extensional Higher Order Logic with Substitution

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Background: Classical higher-order logic

Type system

We will use the following type system to keep track of the syntactic categories of the language.

Simple relational types

The set of (simple relational) types R is the smallest set that contains e and t, and such that whenever $\sigma \in R$ and $\tau \in R$ and $\tau \neq \sigma$, $\sigma \to \tau \in R$.

- Shorthands: $\rho \to \sigma \to \tau$ for $\rho \to (\sigma \to \tau)$; $\vec{\sigma} \to \tau$ for $\sigma_1 \to \cdots \to \sigma_n \to \tau$.
- In this system e is a 'non-terminal' type. We leave out types like $e \to e$ for convenience (their treatment in the logic raises some not-so-important choice points). Also, some might wish to add more basic types besides e and t.
- A **typed collection** is a set together with a function from the set to R. When C is a typed collection, we write $x :_C \sigma$ for $C(x) = \sigma$, and $C^\sigma for \{x : x :_C \sigma\}$. When C and D are typed collections, D^C is the set of all type-preserving functions from C's carrier set to D's.

Terms

We fix a typed collection Var of variables with infinitely many variables with each type, indicated with superscripts when necessary. Then we define

The language $\mathcal{L}$

$\mathcal{L}$ is the smallest R-typed collection such that:

$$v :_{\mathcal{L}} \sigma \text{ whenever } v :_{\mathsf{Var}} \sigma$$

$$(AB) :_{\mathcal{L}} \tau \text{ whenever } A :_{\mathcal{L}} \sigma \to \tau \text{ and } B :_{\mathcal{L}} \sigma$$

$$(\lambda v.A) :_{\mathcal{L}} \sigma \to \tau \text{ whenever } A :_{\mathcal{L}} \tau \text{ and } v :_{\mathsf{Var}} \sigma \text{ and } \tau \neq e$$

$$\to :_{\mathcal{L}} t \to t \to t$$

$$\forall_{\sigma} :_{\mathcal{L}} (\sigma \to t) \to t \text{ for all } \sigma \in R$$

- A **sentence** is a term P with $P :_{\mathcal{L}} t$.
- We can also add a signature Σ of nonlogical constants to define a language $\mathcal{L}_{\Sigma}$, but my focus will be on the pure language.

Some shorthands

- $A_1 \ldots A_n$ for $(((\cdots(A_1 A_2)\cdots)A_{n-1})A_n)$.
- $\lambda v_1 \ldots v_n.A$ for $(\lambda v_1.(\cdots(\lambda v_n.A)\cdots))$.
- ARB for $((RA)B)$ (when $R : \sigma \to \sigma \to \tau$ is a constant or abbreviation).
- $\forall_\sigma v.P$ for $\forall_\sigma(\lambda v.P)$.
- $\bot$ for $\forall_t q.q$
- $\top$ for $\bot \to \bot$
- $\neg$ for $\lambda p.p \to \bot$
- $\wedge$ for $\lambda pq.\forall_t r.(p \to q \to r) \to r$
- $\vee$ for $\lambda pq.\forall_t r.(p \to r) \to (q \to r) \to r$
- $\exists_\sigma$ for $\lambda X^{\sigma\to t}.\forall_t p.(\forall_\sigma y.Xy \to p) \to p$
- $=_\sigma$ for $\lambda xy.\forall_{\sigma\to t} Z.Zx \to Zy$.

Syntactic definitions

We define in the standard way a syntactic operation of **capture-free substitution** that takes a term A, a variable v, and a term B of the same type as v to a term $A[B/v]$ in which each free occurrences of v is replaced by B, with relettering of bound variables in A when necessary.

In terms of this we define a relationship of **immediate β-equivalence**: A and B are immediately β equivalent iff for some v, C, D, A is $(\lambda v.C)D$ and B is $C[D/v]$, or vice versa.

Also, A and B are **immediately η equivalent** iff for some v, A is $\lambda v.Bv$, or vice versa.

Classical higher-order logic

H-theories

A H-**theory** is a set T of $\mathcal{L}$-sentences closed under the following two rules:

MP $Q \in T$ whenever $P \rightarrow Q \in T$ and $P \in T$

Gen $\forall v.P \in T$ whenever $P \in T$.

and containing all instances of the following schemas

Luk $(P \rightarrow Q \rightarrow R) \rightarrow (R \rightarrow P) \rightarrow S \rightarrow P$ (where $P, Q, R, S : t$)

UI $\forall_\sigma F \rightarrow FA$ (where $F : \sigma \rightarrow t$ and $A : \sigma$)

$\beta\eta$ $\Phi[A] \rightarrow \Phi[B]$ whenever A and B are immediately β- or η-equivalent, and $\Phi[B]$ comes from $\Phi[A]$ by replacing one occurrence of B with one of A.

The logic H

H is the smallest H-theory.

Extensionalism

Since the work of Henkin (1950), the best-known systems of classical higher-order logic (often used for formalizing mathematics) include the following two principles, neither of which is in H:

Fregean Axiom $(p \leftrightarrow q) \rightarrow (p =_t q)$ ('Propositional Extensionality').

Functionality $(\forall_\sigma x.Fx =_\tau Gx) \rightarrow F =_{\sigma\rightarrow\tau} G$ ('Functional Extensionality').

Bearing in mind that all our complex types end in t, the combination of these two is equivalent to

Extensionality $(\forall\vec{x}.(F\vec{x} \leftrightarrow G\vec{x})) \rightarrow F =_{\vec{\sigma}\rightarrow t} G$

Against Extensionalism

The Fregean Axiom is prima facie obviously false!

Counterexample: it's raining iff it's both raining and not raining. But it's not the case that for it to be raining is for it to be both raining and not raining.

Argument: the former but not the latter is *possible*.

- Church (1940) accepts Functionality but not the Fregean Axiom. But in later work, (Church, 1951) he does adopt the Fregean Axiom, along with a version of Frege's (Frege, 1892) ingenious but unconvincing technique for "explaining away" apparent counterexamples like the one above.

Open questions

But without Extensionalism, H is pretty weak. Even in the language without nonlogical constants, it leaves open a wide range of questions, notably including questions about "fineness of grain", such as:

? $$\forall_t p.(p = \neg\neg p)$$

? $$\forall_t p.(p \neq \neg\neg p)$$

? $$\forall_t pq.(p = (p \wedge (q \vee \neg q)))$$

? $$\forall_{e \to t} F.\forall_t p.(F = (\lambda x.Fx \wedge (p \vee \neg p)))$$

Our attitude: these are interesting questions at the intersection of logic and metaphysics. The fact that the answers aren't obvious does not show that the questions aren't intelligible, or that they are to be resolved by stipulation. Rather, we should be exploring different theories that answer them systematically, and assessing the credibility of these theories by the same "abductive" standards used in science.

Classicism

The rule of substitution

A theory extending classical propositional logic is closed under the **rule of substitution** iff logically equivalent formulae can be freely substituted in theorems:

Rule of substitution

T is **closed under substitution** iff $T \vdash \Phi[Q]$ whenever $T \vdash P \leftrightarrow Q$ and $T \vdash \Phi[P]$.

H is not closed under substitution: for example, $\mathbf{H} \vdash p \leftrightarrow \neg\neg p$ and $\mathbf{H} \vdash p =_t p$, but $\mathbf{H} \nvdash p =_t \neg\neg p$.

Let **Classicism**, **C**, be the smallest **H**-theory closed under substitution.

The rule of equivalence

C can also be characterized as the smallest **H**-theory closed under either of the following rules:

Equivalence

If $\vdash P \leftrightarrow Q$ then $\vdash (\lambda\vec{v}.P) = (\lambda\vec{v}.Q)$.

ζ-Equivalence

If $\vdash F\vec{v} \leftrightarrow G\vec{v}$ then $\vdash F = G$ (where none of $\vec{v}$ is free in F or G).

Axiomatizing Classicism

Surprisingly, **C** can also be characterized as the smallest H-theory containing all instances of either of the following axiom-schemas:

Logical Substitution $\Phi[P] \rightarrow \Phi[Q]$, whenever $\mathbf{H} \vdash P \leftrightarrow Q$.

Logical Equivalence $(\lambda \vec{v}.P) = (\lambda \vec{v}.Q)$, whenever $\mathbf{H} \vdash P \leftrightarrow Q$.

Logical ζ-Equivalence $F = G$, whenever $\mathbf{H} \vdash F\vec{v} \leftrightarrow G\vec{v}$ and none of $\vec{v}$ is free in F or G.

Axiomatizing Classicism, more simply

We can also axiomatise **C** using a much smaller collection of identities.

The Boolean identites

Commutativity-$\wedge$	$(\lambda pq.p \wedge q) = (\lambda pq.q \wedge p)$
Commutativity-$\vee$	$(\lambda pq.p \vee q) = (\lambda pq.q \vee p)$
Distribution-$\wedge\vee$	$(\lambda pqr.p \wedge (q \vee r)) = (\lambda pqr.(p \wedge q) \vee (p \wedge r))$
Distribution-$\vee\wedge$	$(\lambda pqr.p \vee (q \wedge r)) = (\lambda pqr.(p \vee q) \wedge (p \vee r))$
Dissolution-$\wedge\vee$	$(\lambda pq.p \wedge (q \vee \neg q)) = (\lambda pq.p)$
Dissolution-$\vee\wedge$	$(\lambda pq.p \vee (q \wedge \neg q)) = (\lambda pq.p)$

The Quantifier Identities

Absorption-$\wedge\forall$	$(\lambda Xy.\forall_\sigma X) = (\lambda Xy.\forall_\sigma X \wedge Xy)$
Distribution-$\vee\forall$	$(\lambda Xp.p \vee \forall_\sigma X) = (\lambda Xp.\forall y.p \vee Xy)$

The modal logic in Classicism

Let $\Box := \lambda p.(p =_t \top)$. Then we have:

- $\mathbf{C} \vdash \Box P$ whenever $\mathbf{C} \vdash P$ (Necessitation)
- $\mathbf{C} \vdash \Box(P \to Q) \to (\Box P \to \Box Q)$ (K axiom)
- $\mathbf{C} \vdash \Box P \to P$ (T axiom)
- $\mathbf{C} \vdash \Box P \to \Box\Box Q$ (4 axiom)

Thus, **C** includes all substitution instances of theorems of modal logic S4. Moreover:

- $\mathbf{C} \vdash \Box(\forall \vec{x}.(R\vec{x} \leftrightarrow S\vec{x})) \to R = S$ (Intensionalism)
- $\mathbf{C} \vdash \forall_\sigma x.\Box(Fx =_\tau Gx) \to F =_{\sigma\to\tau} G$ (Modalized Functionality)
- $\mathbf{C} \vdash \Box(\forall_\sigma x.P) \to \forall_\sigma x.\Box P$ (CBF)
- $\mathbf{C} \vdash N(p \to p) \to \forall q(\Box q \to Nq)$ (Broadness)

C is the smallest H-theory closed under Necessitation and containing K, 4, and Intensionalism.

Between Classicism and Extensionalism

Functionality

Classicism is a lot stronger than H but a lot weaker than Extensionalism. One obvious avenue of exploration is to look at theories intermediate between Classicism and Extensionalism. The most obvious thing to consider is to add one but not the other of the two axioms that together give Extensionlism:

Fregean Axiom $(p \leftrightarrow q) \rightarrow (p =_t q)$

Functionality $(\forall_\sigma x. Fx =_\tau Gx) \rightarrow F =_{\sigma \rightarrow \tau} G$

In fact, the Fregean Axiom just gives Extensionalism again: it implies $P \rightarrow \Box P$ and thus lets us strengthen Modalized Functionality to Functionality.

But adding Functionality or its necessitation gives an interesting intermediate system.

Classicism+Functionality

Functionality is equivalent in Classicism to each of:

BF $\forall_\sigma x.\Box P \to \Box\forall_\sigma x.P$

Tractarianism $(\forall x.P \leq Fx) \to P \leq \forall x.Fx.$

where $P \leq Q := Q = (P \vee Q)$, or equivalently, $\Box(P \to Q)$.

- From Functionality to Tractarianism: suppose $\forall x.P \leq Fx$; then $F = \lambda x.Fx \vee p$ by Functionality, so $(\forall x.Fx) = (\forall x.Fx \vee p)$ (by β) $= (\forall x.Fx) \vee p$ (by Distribution-$\vee\forall$).
- From Tractarianism to BF: plug in $\top$ for p.
- From BF to Functionality: $\forall z(Xz = Yz)$ implies $\forall z\Box(Xz \leftrightarrow Yz)$ by LL, which implies $\Box\forall z(Xz \leftrightarrow Yz)$ by BF, which implies $X = Y$ by Intensionalism.

Another non-theorem

ND $$x \neq_\sigma y \to \Box(x \neq_\sigma y)$$

This is not a theorem of Classicism, unlike

NI $$x =_\sigma y \to \Box(x =_\sigma y)$$

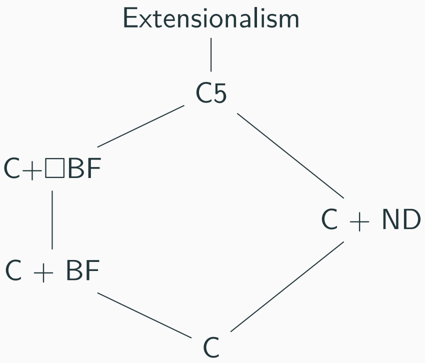
ND is equivalent in Classicism to each of:

5 $$\Diamond p \to \Box\Diamond p$$

B $$p \to \Box\Diamond p$$

We call the result of adding $\Box$ND (or $\Box$5 or $\Box$B) to Classicism '**C5**'.

Logical relationships among these principles



How do we know about non-inclusion facts like those reported here? We use a model-theory I'll explain later.

Three more claims that follow from Extensionalism but not Classicism (or C5)

Atomicity $\forall x(x \neq_\tau \lambda \vec{x}.\bot \to \exists y(\mathrm{Atom}_\tau\, y \wedge y \leq_\tau x))$

where

$$\leq_\tau \;:=\; \lambda X.\lambda Y.(Y =_\tau \lambda \vec{v}.X\vec{v} \vee Y\vec{v})$$

$$\mathrm{Atom}_\tau \;:=\; \lambda y.\forall z((z \leq_\tau y \wedge z \neq y) \leftrightarrow z \leq_\tau \neg_\tau z)$$

Actuality $\exists_t p(p \wedge \forall_t q(q \to p \leq_t q))$

Boolean Completeness $\forall X^{\tau\to t}\exists y^\tau(\mathrm{GLB}_\tau\, yX)$

where

$$\mathrm{LB}_\tau \;:=\; \lambda z^\tau X^{\tau\to t}.\forall_\tau y(Xy \to y \leq_\tau z)$$

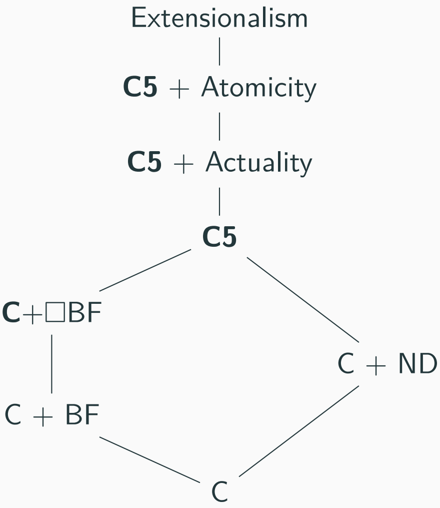
$$\mathrm{GLB}_\tau \;:=\; \lambda y^\tau X^{\tau\to t}.\forall_\tau z(\mathrm{LB}_\tau\, zX \leftrightarrow z \leq_\tau y)$$

Three more claims that follow from Extensionalism but not Classicism (or C5)

Surprisingly, these three are independent in Classicism.

But we do have:

- **C5** $\vdash$ Actuality $\leftrightarrow$ Boolean Completeness
- **C5** $\vdash$ Atomicity $\leftrightarrow$ $\Box$Actuality


Extensionalism
C5 + Atomicity
C5 + Actuality
C5
C+□BF
C + ND
C + BF
C

A better-motivated principle

$$\text{Persistent}_\tau\, Y := Y \leq_\tau (\lambda \vec{z}.\Box Y\vec{z})$$

$$\text{Inextensible}_\tau\, Y := \Box\forall X(\forall \vec{z}(Y\vec{z} \to \Box X\vec{z}) \to Y \leq_\tau X)$$

$$\text{Rigid}_\tau\, Y := \text{Persistent}_\tau\, Y \land \text{Inextensible}_\tau\, Y$$

Any property X with only finitely many instances $x_1 \ldots x_n$ is coextensive with a rigid property $\lambda y.y = x_1 \lor \cdots \lor y = x_n$. It's natural to think that this is true for all properties/relations:

Rigid Comprehension $\qquad \forall_\tau X.\exists_\tau Y(\text{Rigid}_\tau(Y) \land \forall \vec{z}(X\vec{z} \leftrightarrow Y\vec{z}))$

One philosophical motivatation for this principle involves considering the behaviour of *plurals* in natural language. If there is an X thing, then there are *the X things*; the property of being one of *them* is coextensive with X and (arguably) rigid.

Implications involving Rigid Comprehension

In Classicism:

- RC implies Actuality. (Consider $\forall_t p(Rp \rightarrow p)$, where R is a rigid property of propositions coextensive with truth).
- RC implies Boolean Completeness. (Consider $\lambda\vec{x}.\forall_\tau Y(RY \rightarrow Y\vec{x})$, where R is a rigid property coextensive with the X for which we want a GLB.)
- In **C5**, RC is equivalent to both Actuality and Boolean Completeness.
- $\Box$Atomicity, Boolean Completeness, and Functionality imply RC.
- RC and Functionality imply $\Box$Functionality.

A principle that one could easily assume for bad reasons

The system of Church (1940) includes:

Axiom of Descriptions $\quad \text{Functional}_{\vec{\sigma},\tau}\, R \to \exists X^{\vec{\sigma}\to\tau}.\forall\vec{y}.R\vec{y}(X\vec{y})$

where

$$\text{Functional}_{\vec{\sigma},\tau}\, R := \forall\vec{x}.\exists y.R\vec{x}y \wedge \forall z.R\vec{x}z \to y =_{\sigma} z$$

Like all the principles considered in this section, this follows from Extensionalism. If you are used to pronouncing quantification into types of the form $\sigma \to \tau$ using the word 'function' and quantification into types of the form $\sigma \to \tau \to t$ using the word 'relation', and have in the back of your mind the standard set-theoretic equation of functions with functional relations, you might find yourself assuming this by mistake, as I once did. But really it's very strong, and I don't see a strong philosophical motivation for it.

Implications involving the Axiom of Descriptions

In Classicism:

- The Axiom of Descriptions implies both ND and Actuality.
- In C5, Plenitude is equivalent to Actuality/Boolean Completeness/Rigid Comprehension.
- Rigid Comprehension and ND imply Plenitude.

Some combinations we know to be consistent

The following are consistent in Classicism + ¬ND:

- ¬Functionality, ¬RC, ¬Actuality, ¬Atomicity.
- ¬Functionality, ¬RC, ¬Actuality, Atomicity.
- ¬Functionality, RC, Actuality, ¬Atomicity.
- ¬Functionality, RC, Actuality, Atomicity.
- Functionality, ¬RC, ¬Actuality, ¬Atomicity.
- Functionality, ¬RC, ¬Actuality, Atomicity.
- Functionality, ¬RC, Actuality, Atomicity.
- Functionality, ¬RC, Actuality, ¬Atomicity.

Also, in each combination, we can consistently add a $\Box$ to everything.

Fine-grained strengthenings of Classicism

Fine-grained strengthenings

As well as strengthenings that bring us closer to Extensionalism—intuitively, a maximally *coarse-grained* theory of higher-order reality—we can also explore extensions that settle identity questions left open by Classicism *negatively*.

One way to generate such theories is to use the following operation:

Maximalization

For any H-theory T, let Max T be the smallest H-theory that includes T and contains $A \neq B$ whenever $A = B$ is a closed identity that is not in T.

- Note that if T includes Classicism, Max(T) can also be axiomatized the result of adding $\Diamond P$ to T whenever P is closed and $\neg P$ is not in T.
- Except in edge cases, Max(T) is not closed under substitution.
- Except in edge cases, Max(T) is not recursively enumerable.

Which theories can consistently maximalized?

Using our model-theory, we can show that Max(**C**) is consistent. More generally, we can consistently maximalize the result of extending Classicism with any combination of Atomicity, Atomicity, and Functionality, or their necessitations.

Fritz, Lederman and Uzquiano (2021) show that Max(**H**) is consistent.

Max(Extensionalism) is obviously inconsistent, since Extensionalism isn't negation-complete and implies $\Diamond P \to P$ for all P.

Max(**C5**) is also inconsistent, since **C5** is not negation-complete and implies $\Diamond P \to P$ for all **closed** P.

Goodsell [p.c.] shows that Max(Classicism+Rigid Comprehension) and Max(Classicism+$\Box$Rigid Comprehension) are inconsistent.

Philosophical plausibility of Maximalist Classicism

If we had arbitrary non-logical constants floating around (e.g., **bachelor**, **married**, **man**), the maximalization of any of the logics we have considered would be obviously unsound.

But in the language without nonlogical constants, Max(**C**) embodies a philosophically attractive, though controversial, vision—that there is nothing more to the "nature" of the logical constants than is captured by the standard logical rules.

Even if we add nonlogical constants, so long as they only denote distinct "fundamental" properties/relations/individuals, the maximalized theories still have some intuitive plausibility—they answer to a widespread aversion to "brute" or "arbitrary" necessities.

Model theory for Classicism

Henkin premodel

A **Henkin premodel** is a typed collection $\mathbf{H}$ such that $\mathbf{H}^t = \{0, 1\}$, $\mathbf{H}^{\sigma\to\tau} \subseteq (\mathbf{H}^\tau)^{(\mathbf{H}^\sigma)}$, and $\mathbf{H}^\sigma$ is nonempty for all σ.

$\mathbf{H}$ is **full** iff $\mathbf{H}^{\sigma\to\tau} = (\mathbf{H}^\tau)^{(\mathbf{H}^\sigma)}$ for all σ, τ.

Interpretation function of a Henkin premodel

Given a Henkin premodel $\mathbf{H}$, $[\![\cdot]\!]^{\cdot}$ is a function defined on $\mathcal{L} \times \mathbf{H}^{\mathsf{Var}}$, such that:

$$[\![v]\!]^g = g(v)$$

$$[\![AB]\!]^g = [\![A]\!]^g([\![B]\!]^g)$$

$$[\![\lambda v^\sigma.A^\tau]\!]^g = [x \in \mathbf{H}^\sigma \mapsto [\![A]\!]^{g[v\mapsto x]}]$$

$$[\![\to]\!] = [n \in \{0,1\} \mapsto [m \in \{0,1\} \mapsto 1 - m(1-n)]]$$

$$[\![\forall_\sigma]\!] = [f \in \mathbf{H}^{\sigma\to t} \mapsto \max\{fx : x \in \mathbf{H}^\sigma\}]$$

Henkin model

A **Henkin model** is a Henkin premodel such that $[\![A]\!]^g \in \mathbf{H}^\sigma$ for all $A :_{\mathcal{L}} \sigma$ and $g \in H^{\mathsf{Var}}$.

Every full Henkin premodel is a Henkin model, but not vice versa.

A formula P **holds** in $\mathbf{H}$ on g iff $[\![P]\!]^g = 1$.

P is a **consequence** of Γ relative to a class C of Henkin models iff wheever $\mathbf{H} \in C$, $g \in \mathbf{H}^{\mathsf{Var}}$, and every member of Γ holds in $\mathbf{H}$ on g, P holds in $\mathbf{H}$ on g.

Henkin's theorem

Soundness and completeness of Henkin models for Extensionalism

Theorem (Henkin, 1950): P is a consequence of Γ relative to the class of all Henkin models iff there exist $Q_1, \ldots, Q_n \in \Gamma$ such that $Q_1 \rightarrow \cdots \rightarrow Q_n \rightarrow P$ is a theorem of Extensionalism.

From Henkin models to action models: strategy

Choose an arbitrary small category $\mathcal{C}$, the "base category", and an object W_0 of C. Objects of $\mathcal{C}$ will play a role like that of "worlds" in Kripke models; W_0 will be the "actual world".

We'll call a functor from $\mathcal{C}$ to Set an 'action' of $\mathcal{C}$. F is a **subaction** of G iff $F(W) \subseteq G(W)$ for every object W and $F(h)(d) = G(h)(d)$ for all $h : W \to V$ and $d \in F(W)$. Where Henkin models have sets, our more general "action models" will have actions of the base category.

Given two actions F, G of $\mathcal{C}$ there is an "exponential action" G^F:

- For an object W, $G^F(W)$ is the set of all *natural transformations* from $\text{Hom}(W, -) \times F$ to G—i.e., of functions α that take an ordered pair $\{h, d\}$ where h is an arrow from W to some V and $d \in F(V)$ and yield an element of $G(V)$ in such a way that for any $i : V \to U$, $G(i)(\alpha\langle h, d\rangle) = \alpha(i \circ h, F(i)(d))$.
- For any arrow $h : W \to V$, $G^F(h)(\alpha)\langle i, d\rangle = \alpha\langle i \circ h, d\rangle$.

From Henkin models to action models: strategy

For a category $\mathcal{C}$, let the **powerset action** of $\mathcal{C}$ be the functor $P_{\mathcal{C}}$ such that for any object W of $\mathcal{C}$, $P_{\mathcal{C}}(W)$ is the powerset of the set of all $\mathcal{C}$-arrows with source W, and for any $h : W \to V$ and $X \in P_{\mathcal{C}}(W)$, $P_{\mathcal{C}}(h)(X) = \{i : i\mathbf{C}h \in X\}$.

Action models

Action premodel

An **action premodel A** is a tuple $\langle \mathcal{C}, W_0, A^{\cdot} \rangle$ such that $\mathcal{C}$ is any small category, W_0 is any object of $\mathcal{C}$, and for each type σ, A^σ is an action of $\mathcal{C}$, such that:

1. A^t is a subaction of $P_{\mathcal{C}}$.
2. $A^{\sigma \to \tau}$ is a subaction of $(A^\sigma)^{(A^\tau)}$.
3. $A^\sigma(W)$ is nonempty for every object W.

A is **propositionally full** iff $A^t = P_{\mathcal{C}}$, **functionally full** iff $A^{\sigma \to \tau} = (A^\sigma)^{(A^\tau)}$ for all σ, τ, **full** iff both propositionally and functionally full.

Interpretation function of an action model

Given an action model $\mathbf{A}$, $[\![\cdot]\!]^{\cdot}_{\cdot}$ is the function that takes a triple comprising a term A, an arrow $h : W_0 \to V$ for some V, and an assignment function $g \in A(V)^{\mathrm{Var}}$, such that:

$$[\![v]\!]^g_h = g(v)$$

$$[\![AB]\!]^g_h = [\![A]\!]^g_h \langle 1_{\operatorname{trg} h}, [\![B]\!]^g_h \rangle$$

$$[\![\lambda v.A]\!]^g_h = [\langle i, \mathbf{a}\rangle \mapsto [\![A]\!]^{(i \circ g)[v \mapsto \mathbf{a}]}_{i \circ h}]$$

$$[\![\to]\!]^g_h = [\langle i, \mathbf{p}\rangle \mapsto [\langle j, \mathbf{q}\rangle \mapsto (P_{\mathcal{C}}(\operatorname{trg} j) - j^t\mathbf{p}) \cup \mathbf{q}]]$$

$$[\![\forall_\sigma]\!]^g_h = \langle i, \alpha\rangle \mapsto \bigcup_V \{j : \operatorname{trg} i \to V \mid 1_V \in \alpha\langle j, \mathbf{a}\rangle \text{ for every } \mathbf{a} \in V^\sigma\}$$

Action models

Action model

Action premodel $\mathbf{A}$ is an **action model** iff for every term $B : \sigma$, object V, $h : W \to V$, and $g \in A(V)^{\mathsf{Var}}$, $[\![B]\!]^g_h \in A^\sigma(V)$.

Every full action premodel is an action model, but not vice versa.

When g is an assignment function for $A(W_0)$, formula P **holds** in $\mathbf{A}$ iff $1_{W_0} \in [\![P]\!]^g_{1_{W_0}}$.

P is a consequence of Γ relative to a class $\mathcal{C}$ of action models iff whenever $\mathbf{A} \in C$, $g \in A(W_0)^{\mathsf{Var}}$, and every member of Γ holds in $\mathbf{A}$ on g, P holds in $\mathbf{A}$ on g.

Soundness and completeness

Theorem: soundness and completeness of action models for Classicism

P is a consequence of Γ relative to the class of all action models iff there exist $Q_1, \ldots, Q_n \in \Gamma$ such that $Q_1 \to \cdots \to Q_n \to P$ is a theorem of Classicism.

An example

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