Definability problems regarding Campana points and Darmon points

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- If S is a finite subset of Ω_K containing Ω_K^{∞}, denote $\mathcal{O}_{K,S} := \bigcap_{\mathfrak{p} \in \Omega_K \setminus S} (\mathcal{O}_K)_{\mathfrak{p}}$ the **ring of S-integers of K**; i.e., the elements of K which are integral outside S.

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Similarly, **Darmon Points** generalize perfect mth powers.

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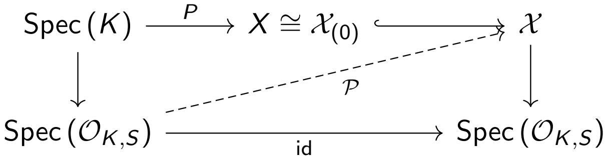

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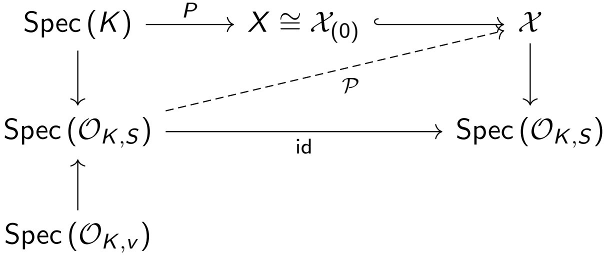

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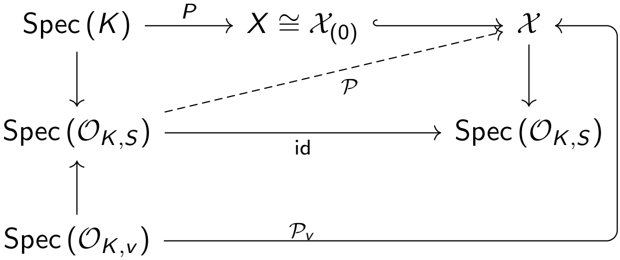

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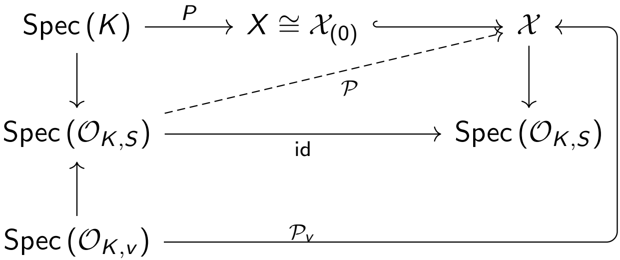

For each $\alpha \in \mathcal{A}$, define *the intersection multiplicity of P and $\mathcal{D}_\alpha$* as

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If $n = \infty$ then in both cases we need $\nu_{\mathfrak{p}}\left(b\left(a,b\right)^{-1}\right) = 0$ for all $\mathfrak{p} \in \Omega_K \setminus S$, hence in both cases we get $\mathcal{O}_{K,S}$, the set of S-integers.

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These are the sets we will try to characterize with first-order language.

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For every $a, b, c, d \in K^\times$, the set $\Omega_{a,b,c,d,K}$ is a finite set of non-archimedean places of K. Conversely, if $S \subseteq \Omega_K^{<\infty}$ is finite, then there exist $a, b, c, d \in K^\times$ such that $S = \Omega_{a,b,c,d,K}$.

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For each $a, b \in K^\times$ define the set

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If $S \subseteq \Omega_K$ is finite, of even cardinality, and does not contain any complex infinite place, there exist $a, b \in K^\times$ such that $S = \Delta_{a,b,K}$ and $a, b \in K^{\mathrm{sf}}_{a,b}$.

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If $S \subseteq \Omega_K^{<\infty}$ is finite and of even cardinality, there exist $a, b \in K^\times$ such that $S = \Delta_{a,b,K} = \Delta^{a,b,K}$ and $a, b \in K_{a,b}^{\mathrm{sf}}$.

If $S \subseteq \Omega_K$ is finite, of even cardinality, and does not contain any complex infinite place, there exist $a, b \in K^\times$ such that $S = \Delta_{a,b,K}$ and $a, b \in K_{a,b}^{\mathrm{sf}}$.

The set $K_{a,b}^{\mathrm{sf}} := \{r \in K : \forall \mathfrak{p} \in \Delta^{a,b,K} \left(|\nu_{\mathfrak{p}}(r)| \leq 1 \right)\}$ is uniformly $\forall\exists$-definable,

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$$\forall a' \forall b' \forall c \forall d \left[\left({}_{\Delta^{a,b,K} \cap \Omega_{a',b',c,d} \neq \emptyset}^{abcda'b' \neq 0} \right) \implies r \notin \left(J_{a',b',c,d,2,K} \cup \left(J_{a',b',c,d,2,K} \setminus \{0\} \right)^{-1} \right) \right].$$

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Corollary

The condition $\Delta_{a,b,K} \cap \Omega_K^\infty = \emptyset$ is uniformly diophantine.

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The formula

$$\varphi(x, a, b) \coloneqq \exists y \exists z \left(z \in T_{a,b,K} \wedge y \in T_{a,b,K} \wedge \gcd_{T_{a,b,K}}(y, z) = 1 \wedge y = xz^n \right)$$

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Theorem (García-Fritz, Pasten, Pheidas; 2022)

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The proof can be adapted to get a slightly more complicated proof of:

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