The Generic Bipartite Graphs of Diameter 3:

Their Ages and Their Almost Sure Theories

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Big Question

- When is arbitrarily large finite different than countably infinite?
- If you have an infinite class of finite structures, and this class has a unique countable structure associated with it, when will the large finite structures look like the countably infinite structure?

Background definitions

- An *amalgamation class* is a class of finite structures which satisfies three properties:
    - Hereditary Property, Joint Embedding Property, Amalgamation Property
- An amalgamation class has a unique countable homogeneous structure which embeds all the finite structures—its *Fraïssé limit*
- This class is the set of all finite structures which embed into the Fraïssé limit—its *age*

0-1 laws

We'll stick with graphs, more or less.

- The class of all finite graphs satisfies a *0-1 law*: for any first order sentence ϕ in the language of graphs,

$$\lim_{n\to\infty} \frac{|G \in \mathcal{G}_n : G \models \phi|}{|\mathcal{G}_n|} \in \{0,1\},$$

  where $\mathcal{G}_n$ is the set of finite graphs on n vertices
- If we are counting up to isomorphism, then this is an *unlabeled* 0-1 law
- If we are counting generally, then this is a *labeled* 0-1 law
- The set of sentences which are asymptotically satisfied form the *almost sure theory*
- The theory of the Fraïssé limit is the *generic theory*

Examples to date

- The class of all graphs satisfies a 0-1 law; its theory matches the theory of the Rado graph
- The class of all triangle-free graphs satisfies a 0-1 law; its theory *diverges* from the theory of the generic triangle-free graph

Why?

Searching for more examples

- Let's stick with graphs
- What other graphs are Fraïssé limits?

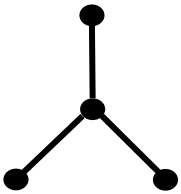
More Examples

- The class of K_n free graphs; almost all K_n graphs are $(n-1)$-partite
- ...That's it. We need to think outside of the box.
- There are other graphs which are Fraïssé limits, but not determined by forbidden graphs

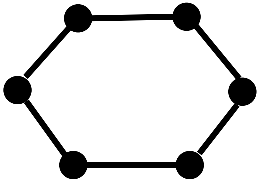
Homogeneity

- A graph is *homogeneous* if every isomorphism between two finite induced subgraphs can be extended to an automorphism of the whole graph

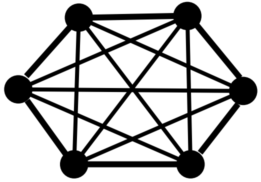
Homogeneous graphs



Not at all homogeneous



Not homogeneous



Homogeneous

Metric homogeneous

- A connected graph is *metrically homogeneous* if, when endowed with the path metric, every finite partial isometry can be extended to a full isometry
- Homogeneous graphs $\subset$ metrically homogeneous graphs

Metrically homogeneous graphs

- Cherlin has a tentative catalog of the metrically homogeneous graphs
- Many of them are Fraïsséllimits
- The configurations they forbid are metric spaces

Graphs as metric spaces

(board work)

Metrically homogeneous graphs: Generic type

All of the known metrically homogeneous graphs of generic type are of the form $\Gamma^{\delta}_{K_1,K_2,C,C',\mathcal{S}}$.

- $(\delta, K_1, K_2, C, C')$ are parameters which restrict the forbidden triangles
- $\mathcal{S}$ is a collection of forbidden $(1, \delta)$-spaces.

The parameters

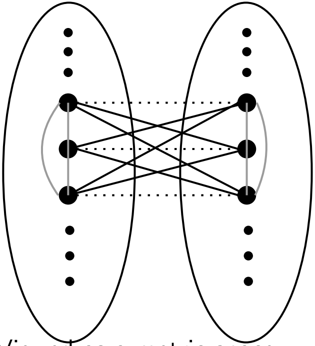
- δ is the diameter;
- K_1, K_2 exclude certain triangles of odd (and not very large) perimeter;
- C_0, C_1 exclude all triangles of sufficiently large even (respectively, odd) perimeter;
- Alternatively — $C = \min(C_0, C_1),\ C' = \max(C_0, C_1)$.

Our graphs

- Cherlin, Amato and MacPherson have found all the metrically homogeneous graphs of diameter 3
- There are exactly 2 metrically homogeneous graphs of diameter 3 which are bipartite and of generic type
- I examined these two graphs

First bipartite graph

- $\mathcal{A}^3_{\infty,0,7,8}$, $\Gamma^3_{\infty,0,7,8}$
- Only keeps (112), (123), (222)
- $\Gamma^3_{\infty,0,7,8}$ can be viewed as a metric space or a graph:

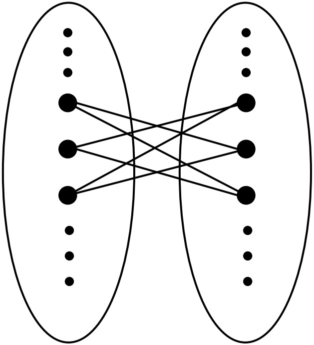


Viewed as a metric space

Distance 1 is written as a solid black line

Distance 2 is written as a solid gray line

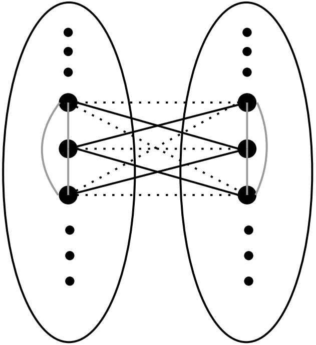
Distance 3 is written as a dotted black line



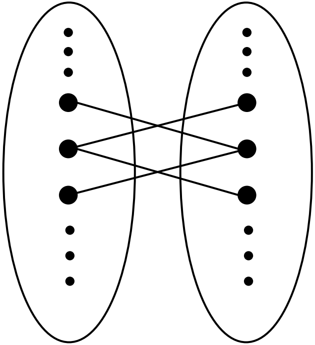
Viewed as a graph

Second bipartite graph

- $\mathcal{A}^3_{\infty,0,7,8}$, $\Gamma^3_{\infty,0,7,8}$
- Only keeps (112), (123), (222), $(2,3,3)$
- $\Gamma^3_{\infty,0,7,8}$ can be viewed as a metric space or a graph:



Viewed as a metric space



Viewed as a graph

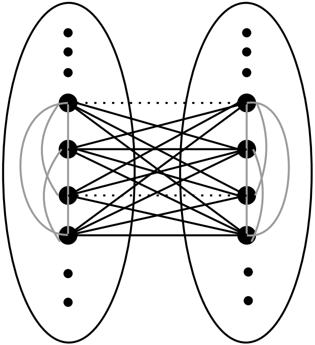
Theorem

Theorem (C.)

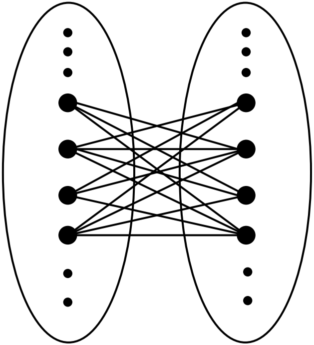
Both $\mathcal{A}^3_{\infty,0,7,8,\emptyset}$ *and* $\mathcal{A}^3_{\infty,0,7,10,\emptyset}$ *have unlabeled and labeled 0-1 laws.*
The almost sure theory of $\mathcal{A}^3_{\infty,0,7,8,\emptyset}$ *diverges from its generic theory.*
The almost sure theory of $\mathcal{A}^3_{\infty,0,7,10,\emptyset}$ *matches its generic theory.*

Why?

Recall that $\mathcal{A}^3_{\infty,0,7,8,\emptyset}$ allows $(1,1,2)$ but not $(3,3,2)$



Viewed as a metric space



Viewed as a graph

Figure: The unique countable model Γ_{as} of the almost sure theory of $\mathcal{A}^3_{\infty,0,7,8,\emptyset}$

But still, why?

It comes down to that triangle

- In $\mathcal{A}^3_{\infty,0,7,8,\emptyset}$, each vertex can have at most one 1-edge
- For homogeneity, every vertex has exactly one 1-edge
- The almost sure limit also has vertices without 1-edges
- 1 and 3 were functionally different
- No such difference in $\mathcal{A}^3_{\infty,0,7,10,\emptyset}$

More musings

- If we put the class of triangle-free graphs in this context, then it has diameter 2, forbids $(1,1,1)$ and allows $(2,1,1)$, $(2,2,1)$, and $(2,2,2)$
- 2 is functionally different than 1

Big picture proof

- First I described the spaces in $\mathcal{A}^3_{\infty,0,7,8,\emptyset}$ and $\mathcal{A}^3_{\infty,0,7,10,\emptyset}$, then I counted them, both up to isomorphism and generally
- I found axioms which describe them
- I showed that these axioms form a complete theory
    - They have no finite models
    - They are $\aleph_0$-categorical
- I showed that these axioms are asymptotically satisfied via a direct count for isomorphism classes for $\mathcal{A}^3_{\infty,0,7,8,\emptyset}$ and $\mathcal{A}^3_{\infty,0,7,10,\emptyset}$, also generally for $\mathcal{A}^3_{\infty,0,7,10,\emptyset}$
- I used a rigidity/asymmetry argument for $\mathcal{A}^3_{\infty,0,7,10,\emptyset}$

Thank you

Thank you for having me!