Zariski Topological Reconstruction Theorems via Model Theory

Ben Castle, joint with Ronan O'Gorman

University of Illinois

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Fundamental Theorem of Affine Geometry, Part 1

Problem: Find all bijections from $\mathbb{R}^2$ to $\mathbb{R}^2$ that send lines to lines in both directions.

Fact

Any such bijection has the form $x \mapsto Ax + b$ where A is an invertible 2x2 matrix and $b \in \mathbb{R}^2$.

- The proof is really a bi-interpretability argument.
- Let $\mathcal{M} = (\mathbb{R}^2, X)$ where $X \subset (\mathbb{R}^2)^3$ is the collinearity relation. Then $\mathcal{M}$ is bi-interpretable with $(\mathbb{R}, +, \cdot)$.
- A similar statement holds for bijections $\phi : \mathbb{R}^n \to \mathbb{R}^n$ ($n \geq 2$) that preserve affine linear subspaces.

Fundamental Theorem of Projective Geometry, Part 1

There is also a projective version of this (which came first):

Fact

Let $P = \mathbb{P}^2(\mathbb{R})$ be the real projective plane. Let $\phi : P \to P$ be a bijection sending lines to lines in both directions. Then ϕ is a projective transformation, i.e. an element of the group $PGL_3(\mathbb{R})$.

Again, there is an n-dimensional version ($n \geq 2$) with 'lines' replaced by 'linear subspaces'.

Other Fields

- What about bijections on $\mathbb{C}^2$ or $\mathbb{P}^2(\mathbb{C}^2)$?
- Negative example: complex conjugation. Send (x, y) to $(\bar{x}, \bar{y})$. Then $y = ax + b$ goes to $y = \bar{a}x + \bar{b}$.
- This works because $z \mapsto \bar{z}$ is a field automorphism.
- In fact, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, σ sends the line $y = ax + b$ to the line $y = \sigma(a)x + \sigma(b)$. But most of the time σ is not a projective transformation.

Fact

Every line preserving bijection of $\mathbb{C}^2$ has the form $x \mapsto A\sigma(x) + b$ where $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Other Fields, Part 2

A fully general version:

Fact

Let K_1, K_2 be fields, and $\phi : K^2 \to K^2$ a line-preserving bijection. Then $\phi(x) = A\sigma(x) + b$ where $\sigma \in \mathrm{Iso}(K_1, K_2)$, $A \in \mathrm{GL}_2(K_2)$, and $b \in K_2^2$.

Fact

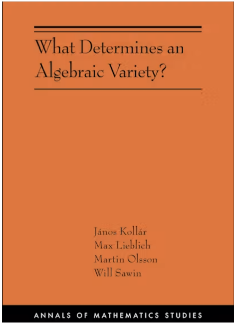
Let K_1, K_2 be fields, and $\phi : \mathbb{P}^2(K_1) \to \mathbb{P}^2(K_2)$ a line-preserving bijection. Then $\phi = f \circ \sigma$ where $\sigma \in \mathrm{Iso}(K_1, K_2)$ and $f \in PGL_3(K_2)$.

Again, the expected n-dimensional versions hold ($n \geq 2$).

Curve Preserving Bijections

Question: What about bijections sending curves to curves?

This turns out to be much harder:



For the rest of the talk, all fields are algebraically closed.

(This is necessary for our result, and it makes KLOS easier to state).

Fact (Kollar, Lieblich, Olsson, Sawin, 2023)

Suppose $\phi : \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ *is a bijection sending curves to curves in both directions. Then* $\phi = f \circ \sigma$ *where* $\sigma \in \mathrm{Aut}(\mathbb{C})$ *and* $f \in \mathrm{PGL}_3(\mathbb{C})$.

Now consider a curve preserving $\phi : \mathbb{C}^2 \to \mathbb{C}^2$.

- Can we write $\phi(x) = A\sigma(x) + b$ as before?
- **No:** take a *polynomial automorphism of the plane*, such as $\phi(x, y) = (x + y^2, y)$.
- **Conjecture:** $\phi(x) = f(\sigma(x))$ for some $\sigma \in \mathrm{Aut}(\mathbb{C})$ and some polynomial automorphism f of the plane.
- the KLOS proof is **only projective**.

Main Theorem, Version 1

Here is a special case of our main theorem:

Theorem (C., O'Gorman)

Let $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ be a curve-preserving bijection. Then $\phi = f \circ \sigma$ for some $\sigma \in \mathrm{Aut}(\mathbb{C})$ and some polynomial automorphism f.

- There is also an n-dimensional version (≥ 2).
- Replace 'curve-preserving' with 'preserving all Zariski closed sets'.
- So this is really about 'Zariski homeomorphisms'.

About the KLOS proof

Let $\phi : \mathbb{P}^2(\mathbb{C}^2) \to \mathbb{P}^2(\mathbb{C}^2)$ be curve preserving.

- KLOS: Show that ϕ sends lines to lines.
- Conclude using the FTPG.
- Their general case is a much more complicated version of this idea, using 'linear systems of divisors'.
- This **cannot work** for maps $\mathbb{C}^2 \to \mathbb{C}^2$ (polynomial automorphisms don't preserve lines).

The Fix

Now consider $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ preserving curves.

Definition

As before, let $\mathcal{M} = (\mathbb{C}^2, X)$ where $X \subset (\mathbb{C}^2)^3$ be the set of collinear triples.

KLOS would want to show $\phi(X) = X$, but this can't be true.

Theorem (C.O'Gorman)

$\phi(X)$ is definable in the structure $(\mathbb{C}, +, \cdot)$.

- This is the punchline of the paper.
- The proof is *very* long and indirect.
- Why does this matter? Knowing something is definable is not great for doing geometry. But it's great for doing model theory.

Model Theory Setup

- Consider now the two structures $\mathcal{M} = (\mathbb{C}^2, X)$ and $\mathcal{N} = (\mathbb{C}^2, \phi(X))$.
- ϕ gives an isomorphism of $\mathcal{M} \to \mathcal{N}$.
- $\mathcal{M}$ (thus also $\mathcal{N}$) is bi-interpretable with $(\mathbb{C}, +, \cdot)$. Now proceed as with FTAG.
- This is more complicated because the two bi-interpretations are different (i.e. $\phi(X)$ is not defined the same way as X). But the fact that $\phi(X)$ is definable at all is the key.
- We actually prove something much more general. For the full theorem, we need the *Zilber trichotomy* in the bi-interpretability step.

KLOS, full version

Fact (KLOS Theorem)

Suppose K_1, K_2 are uncountable algebraically closed fields of characteristic zero, and X_i is an irreducible, normal, projective variety of dimension at least 2 over K_i. Then every Zariski homeomorphism $X_1 \to X_2$ has the form $f \circ \sigma$ where $\sigma \in \mathrm{Iso}(K_1, K_2)$ and f is an isomorphism of K_2-varieties.

- Slogan: a variety as above is determined by the Zariski topology alone.
- Again, they don't need algebraically closed (they really show that every scheme homeomorphism $X_1 \to X_2$ underlies a scheme isomorphism).
- Note that this fails badly if $\dim(X_i) = 1$.

Main Theorem, Version 2

KLOS need a lot of conditions:

- Projective (e.g. can't do $\mathbb{C}^2$).
- Normal (e.g. we can't glue $\mathbb{C}^2$ to itself along a line).
- Irreducible (e.g. we can't take two planes meeting in a line).
- Characteristic zero.

They make several conjectures about what happens without these conditions.

We prove all of their conjectures for all uncountable algebraically closed fields, using model theory.

Uncountable?

It is at least somewhat necessary to assume uncountability:

Fact (Wiegand-Krauter)

For all primes p and q, the projective planes over $\mathbb{F}_p^{alg}$ and $\mathbb{F}_q^{alg}$ are homeomorphic.

- This is *also* a logic fact.
- The proof axiomatizes a common theory in weak second order logic, which is categorical by a back-and-forth argument.
- Ok, the authors didn't know that's what they were doing.
- Uncountability allows 'pigeonhole' arguments: e.g. the values $\deg(\phi(l))$ coincide for uncountably many lines l.

Counterexample 1

How do we formulate a conjecture in general? The 'projective' assumption from KLOS doesn't seem to matter, but the other assumptions do.

- In char p, let $\phi(x, y) = (x, y^p)$ from $K^2 \to K^2$.
- ϕ is a homeomorphism. Can we write $\phi = f \circ \sigma$ where f is a polynomial automorphism? **No.**
- If we try $\sigma(x) = x$, but then $f(x, y) = (x, y^p)$ is not invertible.
- Ok, make $\sigma(x) = x^p$. But then $f(x) = (x^{1/p}, y)$ is not even a polynomial.
- **KLOS Conjecture:** To fix this, allow factorizations $f \circ \sigma$ where f is an 'isomorphism up to pth roots'.

Counterexample 2

A similar problem happens with reducible varieties.

- Let $X = X_1 \cup X_2$ be the union of two disjoint planes in $\mathbb{C}^3$.
- Define $\phi : X \to X$ by $\phi(x) = x$ for $x \in X_1$ and $\phi(x) = \bar{x}$ (complex conjugation) for $x \in X_2$.
- As before, we cannot write $\phi = f \circ \sigma$ with f an isomorphism.
- Even if X_1, X_2 meet in finitely many points, you can still find examples like this.
- **KLOS Conjecture:** Everything should work for varieties X which are *strongly connected* (not decomposable into two closed subsets with finite intersection) and have no one-dimensional components.

Normalizations

Definition

An integral domain is *normal* if it is integrally closed in its field of fractions. An variety is *normal* if it can be covered by affine open sets with normal coordinate rings.

- Non-normal varieties have unavoidable singularities.
- If X is not normal, it has a *normalization* $\widetilde{X} \to X$ (think of a cover of X where all crossings are uncrossed).

Counterexample 3

- Sometimes the normalization can be a homeomorphism that just smooths out a cusp.
- $\phi(x, y) \to (x, y^2, y^3)$ is a homeomorphism from $\mathbb{C}^2$ to its image X (a non-normal surface in $\mathbb{C}^3$).
- This ϕ cannot factor into $f \circ \sigma$ as before. But it does factor as
$$\mathbb{C}^2 \xrightarrow{\sigma=\mathrm{id}} \mathbb{C}^2 \xrightarrow{f} \widetilde{X} \to X.$$
- **KLOS Conjecture:** homeomorphisms between non-normal varieties should lift to homeomorphisms between their normalizations, and everything should work on the normalizations.

Recap: The KLOS Conjectures

We summarize the expected generalizations of the KLOS theorem:

1. **Projective:** Removing 'projective' makes no change (precisely, one should take 'quasi-projective' instead).
2. **Characteristic 0:** In characteristic p, 'isomorphism' should be 'isomorphism up to pth roots'.
3. **Irreducible:** Everything works for strongly connected varieties with no 1-dimensional components.
4. **Normal:** Homeomorphisms should lift to the normalizations, and everything works there.

Universal Homeomorphisms

Definition

A map of varieties $f : X \to Y$ over an algebraically closed field is a *universal homeomorphism* if it is both a homeomorphism and a morphism of varieties.

Universal homeomorphisms act as 'isomorphisms up to pth powers'. In particular:

Fact

A universal homeomorphism of normal varieties in characteristic zero is an isomorphism.

Universal homeomorphisms are unavoidable (you can't classify homeomorphisms without them).

Main Theorem, Version 3

Theorem (C., O'Gorman)

Let K_1, K_2 be uncountable algebraically closed fields. Let X_i be a strongly connected quasi-projective variety over K_i with no 1-dimensional components. Let $\phi : X_1 \to X_2$ be a homeomorphism. Then there is a commutative diagram:

$$\begin{tikzcd}
\widetilde{X_1} \arrow[r, "\widetilde{\sigma}"] \arrow[d] & \widetilde{\sigma(X_1)} \arrow[r, "\widetilde{f}"] \arrow[d] & \widetilde{X_2} \arrow[d] \\
X_1 \arrow[r, "\sigma"] \arrow[rr, bend right, "\phi"'] & \sigma(X_1) \arrow[r, "f"] & X_2
\end{tikzcd}$$

where $\sigma \in \mathrm{Iso}(K_1, K_2)$ and $\widetilde{f}$ is a universal homeomorphism.