An analytic AKE program

with induced structure results on coefficient field and monomial group

The Online Logic Seminar

Neer Bhardwaj
September 15, 2022

Weizmann Institute of Science
AKE-type equivalence for valued fields with analytic structure.

Joint work with Lou van den Dries.
Summary

- AKE-type equivalence for valued fields with *analytic structure*.
- In parallel to the original theory of valued fields, we develop an extension theory in our framework.
» AKE-type equivalence for valued fields with *analytic structure*.

» In parallel to the original theory of valued fields, we develop an extension theory in our framework.

» New is that in addition to AKE-type results for these structures, we obtain induced structure results for the coefficient field and monomial group.
▸ AKE-type equivalence for valued fields with *analytic structure*.

▸ In parallel to the original theory of valued fields, we develop an extension theory in our framework.

▸ New is that in addition to AKE-type results for these structures, we obtain induced structure results for the coefficient field and monomial group.

Joint work with Lou van den Dries.
Outline

1. Classical AKE.

2. Denef – van den Dries’ analytic expansion.

3. Some induced structure by Binyamini – Cluckers – Novikov.

4. Running the AKE program.
All rings are commutative with unity.
The AKE principle

All rings are commutative with unity.

A valuation ring $R$ is an integral domain such that for all $x \neq 0 \in K := \text{Frac}(R)$, $x \in R$ or $x^{-1} \in R$. 

A valued field is a $\mathbb{L} \text{val} := \{0, 1, +, -, \cdot, \leq\}$ structure.

$R$ is local, with maximal ideal $\mathfrak{O}(R)$, $K := \text{Frac}(R)$ is a valued field.

Residue field $k_K := R / \mathfrak{O}(R)$, value group $\Gamma_K := K \times / R \times$.

Residue map $\pi : R \to k_K$, valuation map $v : K \times \to \Gamma_K$.

Theorem (Ax–Kochen–Ersov, 1965) Let $K$ and $L$ be henselian valued fields of equicharacteristic 0. Then $K \equiv L \iff k_K \equiv k_L$ as fields, and $\Gamma_K \equiv \Gamma_L$ as ordered groups.
The AKE principle

All rings are commutative with unity.

A valuation ring $R$ is an integral domain such that for all $x \neq 0 \in K := \text{Frac}(R)$, $x \in R$ or $x^{-1} \in R$.

$R$ is local, with maximal ideal $\mathfrak{o}(R)$, $K := \text{Frac}(R)$ is a valued field.
The AKE principle

All rings are commutative with unity.

A valuation ring $R$ is an integral domain such that for all $x \neq 0 \in K := \text{Frac}(R)$, $x \in R$ or $x^{-1} \in R$.

$R$ is local, with maximal ideal $\mathfrak{o}(R)$, $K := \text{Frac}(R)$ is a valued field. A valued field is a $\mathcal{L}_{\text{val}} := \{0, 1, +, -, \cdot, \leq\}$-structure.
The AKE principle

All rings are commutative with unity.

A valuation ring $R$ is an integral domain such that for all $x \neq 0 \in K := \text{Frac}(R)$, $x \in R$ or $x^{-1} \in R$.

$R$ is local, with maximal ideal $\mathfrak{o}(R)$, $K := \text{Frac}(R)$ is a valued field. A valued field is a $\mathcal{L}_{\text{val}} := \{0, 1, +, -, \cdot, \leq\}$-structure. $a \leq b$ iff $a/b \in R$. 
The AKE principle

All rings are commutative with unity.

A valuation ring $R$ is an integral domain such that for all $x \neq 0 \in K := \text{Frac}(R)$, $x \in R$ or $x^{-1} \in R$.

$R$ is local, with maximal ideal $\mathfrak{o}(R)$, $K := \text{Frac}(R)$ is a valued field. A valued field is a $\mathcal{L}_{\text{val}} := \{0, 1, +, -, \cdot, \leq\}$-structure. $a \leq b$ iff $a/b \in R$.

Residue field $k_K := R/\mathfrak{o}(R)$, value group $\Gamma_K := K^\times/R^\times$. 
The AKE principle

All rings are commutative with unity.

A valuation ring \( R \) is an integral domain such that for all \( x \neq 0 \in K := \text{Frac}(R) \), \( x \in R \) or \( x^{-1} \in R \).

\( R \) is local, with maximal ideal \( \mathfrak{o}(R) \), \( K := \text{Frac}(R) \) is a valued field. A valued field is a \( \mathcal{L}_{\text{val}} := \{0, 1, +, -, \cdot, \leq\} \)-structure. \( a \leq b \) iff \( a/b \in R \).

Residue field \( k_K := R/\mathfrak{o}(R) \), value group \( \Gamma_K := K^\times/R^\times \).
Residue map \( \pi : R \to k_K \), valuation map \( v : K^\times \to \Gamma_K \).
The AKE principle

All rings are commutative with unity.

A valuation ring $R$ is an integral domain such that for all $x \neq 0 \in K := \text{Frac}(R)$, $x \in R$ or $x^{-1} \in R$.

$R$ is local, with maximal ideal $\mathfrak{o}(R)$, $K := \text{Frac}(R)$ is a valued field. A valued field is a $\mathcal{L}_{\text{val}} := \{0, 1, +, -, \cdot, \leq\}$-structure. $a \leq b$ iff $a/b \in R$.

Residue field $k_K := R/\mathfrak{o}(R)$, value group $\Gamma_K := K^x/R^x$. Residue map $\pi : R \to k_K$, valuation map $v : K^x \to \Gamma_K$.

Theorem (Ax–Kochen–Ersov, 1965)

Let $K$ and $L$ be henselian valued fields of equicharacteristic $0$. 
The AKE principle

All rings are commutative with unity.

A valuation ring $R$ is an integral domain such that for all $x \neq 0 \in K := \text{Frac}(R)$, $x \in R$ or $x^{-1} \in R$.

$R$ is local, with maximal ideal $\mathfrak{o}(R)$, $K := \text{Frac}(R)$ is a valued field. A valued field is a $L_{\text{val}} := \{0, 1, +, -, \cdot, \leq\}$-structure. $a \leq b$ iff $a/b \in R$.

Residue field $k_K := R/\mathfrak{o}(R)$, value group $\Gamma_K := K^\times / R^\times$. Residue map $\pi : R \to k_K$, valuation map $v : K^\times \to \Gamma_K$.

Theorem (Ax–Kochen–Ersov, 1965)

Let $K$ and $L$ be henselian valued fields of equicharacteristic 0. Then

\[ K \equiv L \iff k_K \equiv k_L \text{ as fields, and } \Gamma_K \equiv \Gamma_L \text{ as ordered groups.} \]
\( F_p((t)) \) and \( \mathbb{Q}_p \) have the same residue field–\( F_p \) and value group–\( \mathbb{Z} \).
$\mathbb{F}_p((t))$ and \( \mathbb{Q}_p \) have the same residue field-\( \mathbb{F}_p \) and value group-\( \mathbb{Z} \).

**Corollary**

Let \( \sigma \) be any \( \mathcal{L}_{\text{val}} \)-sentence. Then

\[
\mathbb{Q}_p \models \sigma \iff \mathbb{F}_p((t)) \models \sigma
\]

for all but finitely many primes \( p \).
\( \mathbb{F}_p((t)) \) and \( \mathbb{Q}_p \) have the same residue field–\( \mathbb{F}_p \) and value group–\( \mathbb{Z} \).

**Corollary**

*Let \( \sigma \) be any \( \mathcal{L}_{\text{val}} \)-sentence. Then*

\[
\mathbb{Q}_p \models \sigma \iff \mathbb{F}_p((t)) \models \sigma
\]

*for all but finitely many primes \( p \).*

Application: Ax-Kochen theorem.
\[ \mathbb{F}_p((t)) \text{ and } \mathbb{Q}_p \text{ have the same residue field—} \mathbb{F}_p \text{ and value group—}\mathbb{Z}. \]

**Corollary**

Let \( \sigma \) be any \( \mathcal{L}_{\text{val}} \)-sentence. Then

\[ \mathbb{Q}_p \models \sigma \iff \mathbb{F}_p((t)) \models \sigma \]

for all but finitely many primes \( p \).

Application: Ax-Kochen theorem.

The AKE program runs through an extension theory of valued fields.
\( \mathbb{F}_p((t)) \) and \( \mathbb{Q}_p \) have the same residue field–\( \mathbb{F}_p \) and value group–\( \mathbb{Z} \).

**Corollary**

Let \( \sigma \) be any \( \mathcal{L}_{\text{val}} \)-sentence. Then

\[
\mathbb{Q}_p \models \sigma \iff \mathbb{F}_p((t)) \models \sigma
\]

for all but finitely many primes \( p \).

Application: Ax-Kochen theorem.

The AKE program runs through an extension theory of valued fields.

Gives relative elementarity, model completeness, elimination of quantifiers;
\( \mathbb{F}_p((t)) \) and \( \mathbb{Q}_p \) have the same residue field–\( \mathbb{F}_p \) and value group–\( \mathbb{Z} \).

**Corollary**

Let \( \sigma \) be any \( \mathcal{L}_{val} \)-sentence. Then

\[
\mathbb{Q}_p \models \sigma \iff \mathbb{F}_p((t)) \models \sigma
\]

for all but finitely many primes \( p \).

Application: Ax-Kochen theorem.

The AKE program runs through an extension theory of valued fields.

Gives relative elementarity, model completeness, elimination of quantifiers; and *induced structure results for lifts of the residue field and the value group.*
Induced structure in the classical case

Consider an $\mathcal{L}_{val}^{cg}$-structure $(K, C_K, G_K)$, $K$ an $\mathcal{L}_{val}$-structure.
Induced structure in the classical case

Consider an $L_{\text{val}}^{cg}$-structure $(K, C_K, G_K)$, $K$ an $L_{\text{val}}$-structure. $C_K$ and $G_K$ are lifts of the residue field and the value group.
Consider an $\mathcal{L}_{\text{val}}^{\text{cg}}$-structure $(K, C_K, G_K)$, $K$ an $\mathcal{L}_{\text{val}}$-structure. $C_K$ and $G_K$ are lifts of the residue field and the value group.

Example: $(C((t)), C, t^{\mathbb{Z}})$,
Induced structure in the classical case

Consider an $\mathcal{L}_{\text{val}}^{cg}$-structure $(K, C_K, G_K)$, $K$ an $\mathcal{L}_{\text{val}}$-structure. $C_K$ and $G_K$ are lifts of the residue field and the value group.

Example: $(C((t)), C, t^{\mathbb{Z}})$, with $R = C[[t]]$, $C$ a field and $	ext{char } C = 0$. 
Induced structure in the classical case

Consider an $\mathcal{L}_{\text{val}}^{\text{cg}}$-structure $(K, C_K, G_K)$, $K$ an $\mathcal{L}_{\text{val}}$-structure. $C_K$ and $G_K$ are lifts of the residue field and the value group.

Example: $(C((t)), C, t^\mathbb{Z})$, with $R = C[[t]]$, $C$ a field and $\text{char } C = 0$.

**Theorem (folklore)**

*Suppose $K$ and $L$ are henselian of equicharacteristic 0.*
Induced structure in the classical case

Consider an $\mathcal{L}_{\text{val}}^{\text{cg}}$-structure $(K, C_K, G_K)$, $K$ an $\mathcal{L}_{\text{val}}$-structure. $C_K$ and $G_K$ are lifts of the residue field and the value group.

Example: $(C(\langle t \rangle), C, t^\mathbb{Z})$, with $R = C[[t]]$, $C$ a field and $\text{char } C = 0$.

**Theorem (folklore)**

Suppose $K$ and $L$ are henselian of equicharacteristic $0$. Then

$$(K, C_K, G_K) \equiv (L, C_L, G_L) \iff C_K \equiv C_L \text{ and } G_K \equiv G_L.$$
Induced structure in the classical case

Consider an $L_{\text{val}}^{cg}$-structure $(K, C_K, G_K)$, $K$ an $L_{\text{val}}$-structure. $C_K$ and $G_K$ are lifts of the residue field and the value group.

Example: $(C((t)), C, t^{\mathbb{Z}})$, with $R = C[[t]]$, $C$ a field and $\text{char } C = 0$.

**Theorem (folklore)**

Suppose $K$ and $L$ are henselian of equicharacteristic 0. Then

$$(K, C_K, G_K) \equiv (L, C_L, G_L) \iff C_K \equiv C_L \text{ and } G_K \equiv G_L.$$
Induced structure in the classical case

Consider an $\mathcal{L}^{cg}_{val}$-structure $(K, C_K, G_K)$, $K$ an $\mathcal{L}_{val}$-structure. $C_K$ and $G_K$ are lifts of the residue field and the value group.

Example: $(C((t)), C, t\mathbb{Z})$, with $R = C[[t]]$, $C$ a field and $\text{char } C = 0$.

**Theorem (folklore)**

Suppose $K$ and $L$ are henselian of equicharacteristic 0. Then

$$(K, C_K, G_K) \equiv (L, C_L, G_L) \iff C_K \equiv C_L \text{ and } G_K \equiv G_L.$$
Outline

1. Classical AKE.
2. Denef – van den Dries’ analytic expansion.
3. Some induced structure by Binyamini – Cluckers – Novikov.
4. Running the AKE program.
Expansion by $\mathbb{Z}[[t]]$-structure

The valuation rings $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$ are complete local and come with natural analytic structure.
The valuation rings $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$ are complete local and come with natural \textit{analytic} structure.

Moreover, both $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$ are homomorphic images of $\mathbb{Z}[[t]]$:

$$\mathbb{Z}[[t]] \rightarrow \mathbb{Z}_p : a(t) \mapsto a(p)$$
Expansion by $\mathbb{Z}[[t]]$-structure

The valuation rings $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$ are complete local and come with natural analytic structure.

Moreover, both $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$ are homomorphic images of $\mathbb{Z}[[t]]$:

$$\mathbb{Z}[[t]] \rightarrow \mathbb{Z}_p : a(t) \mapsto a(p)$$

$$\mathbb{Z}[[t]] \rightarrow \mathbb{F}_p[[t]] : a(t) \mapsto a(t) \mod p$$
Expansion by $\mathbb{Z}[[t]]$-structure

The valuation rings $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$ are complete local and come with natural *analytic* structure.

Moreover, both $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$ are homomorphic images of $\mathbb{Z}[[t]]$:

\[
\mathbb{Z}[[t]] \to \mathbb{Z}_p : a(t) \mapsto a(p)
\]

\[
\mathbb{Z}[[t]] \to \mathbb{F}_p[[t]] : a(t) \mapsto a(t) \mod p
\]

Can interpret the analytic structure on $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$ through a common language.
For each $n$ we have the ring of *restricted* or *strictly convergent* power series over $\mathbb{Z}[[t]]$: $\mathbb{Z}[[t]]\langle Y_1, \ldots, Y_n \rangle$
Introducing restricted power series

For each $n$ we have the ring of restricted or strictly convergent power series over $\mathbb{Z}[[t]]: \mathbb{Z}[[t]]\langle Y_1, \ldots, Y_n \rangle$ – the $t$-adic completion of $\mathbb{Z}[[t]][Y_1, \ldots, Y_n]$. 
Introducing restricted power series

For each $n$ we have the ring of restricted or strictly convergent power series over $\mathbb{Z}[[t]]$: $\mathbb{Z}[[t]][Y_1, \ldots, Y_n]$ – the $t$-adic completion of $\mathbb{Z}[[t]][Y_1, \ldots, Y_n]$.

$\mathbb{Z}[[t]][Y_1, \ldots, Y_n]$ consists of the formal power series
Introducing restricted power series

For each \( n \) we have the ring of restricted or strictly convergent power series over \( \mathbb{Z}[[t]]: \mathbb{Z}[[t]]\langle Y_1, \ldots, Y_n \rangle \) – the \( t \)-adic completion of \( \mathbb{Z}[[t]][Y_1, \ldots, Y_n] \).

\( \mathbb{Z}[[t]][Y_1, \ldots, Y_n] \) consists of the formal power series

\[
f = f(Y_1, \ldots, Y_n) = \sum_{\nu} a_{\nu} Y_1^{\nu_1} \cdots Y_n^{\nu_n}, \quad \nu = (\nu_1, \ldots, \nu_n) \text{ ranging over } \mathbb{N}^n,
\]

9/22
Introducing restricted power series

For each $n$ we have the ring of \textit{restricted} or \textit{strictly convergent} power series over $\mathbb{Z}[[t]]$: $\mathbb{Z}[[t]]\langle Y_1, \ldots, Y_n \rangle$ – the $t$-adic completion of $\mathbb{Z}[[t]][Y_1, \ldots, Y_n]$.

$\mathbb{Z}[[t]]\langle Y_1, \ldots, Y_n \rangle$ consists of the formal power series

$$f = f(Y_1, \ldots, Y_n) = \sum_{\nu} a_{\nu} Y_1^{\nu_1} \cdots Y_n^{\nu_n}, \quad \nu = (\nu_1, \ldots, \nu_n) \text{ ranging over } \mathbb{N}^n,$$

with all $a_{\nu} \in \mathbb{Z}[[t]]$ such that $a_{\nu} \to 0$, $t$-adically, as $|\nu| = \nu_1 + \cdots + \nu_n \to \infty$. 


Introducing restricted power series

For each $n$ we have the ring of restricted or strictly convergent power series over $\mathbb{Z}[[t]]$: $\mathbb{Z}[[t]][Y_1, \ldots, Y_n]$ – the $t$-adic completion of $\mathbb{Z}[[t]][Y_1, \ldots, Y_n]$.

$\mathbb{Z}[[t]][Y_1, \ldots, Y_n]$ consists of the formal power series

$$f = f(Y_1, \ldots, Y_n) = \sum_\nu a_\nu Y_1^{\nu_1} \cdots Y_n^{\nu_n}, \quad \nu = (\nu_1, \ldots, \nu_n) \text{ ranging over } \mathbb{N}^n,$$

with all $a_\nu \in \mathbb{Z}[[t]]$ such that $a_\nu \to 0$, $t$-adically, as $|\nu| = \nu_1 + \cdots + \nu_n \to \infty$.

Extend the language $\mathcal{L}_{\text{val}}$ to $\mathcal{L}^{\mathbb{Z}[[t]]}_{\text{val}}$ by augmenting an $n$-ary function symbol for each $f \in \mathbb{Z}[[t]][Y_1, \ldots, Y_n]$. 
Introducing restricted power series

For each $n$ we have the ring of restricted or strictly convergent power series over $\mathbb{Z}[[t]]$: $\mathbb{Z}[[t]]\langle Y_1, \ldots, Y_n \rangle$ – the $t$-adic completion of $\mathbb{Z}[[t]][Y_1, \ldots, Y_n]$.

$\mathbb{Z}[[t]]\langle Y_1, \ldots, Y_n \rangle$ consists of the formal power series

$$f = f(Y_1, \ldots, Y_n) = \sum_\nu a_\nu Y_1^{\nu_1} \cdots Y_n^{\nu_n}, \quad \nu = (\nu_1, \ldots, \nu_n) \text{ ranging over } \mathbb{N}^n,$$

with all $a_\nu \in \mathbb{Z}[[t]]$ such that $a_\nu \to 0$, $t$-adically, as $|\nu| = \nu_1 + \cdots + \nu_n \to \infty$.

Extend the language $\mathcal{L}_{\text{val}}$ to $\mathcal{L}^{\mathbb{Z}[[t]]}_{\text{val}}$ by augmenting an $n$-ary function symbol for each $f \in \mathbb{Z}[[t]]\langle Y_1, \ldots, Y_n \rangle$.

Construe $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ as $\mathcal{L}^{\mathbb{Z}[[t]]}_{\text{val}}$-structures.
Introducing restricted power series

For each $n$ we have the ring of restricted or strictly convergent power series over $\mathbb{Z}[[t]]: \mathbb{Z}[[t]]\langle Y_1, \ldots, Y_n \rangle$ – the $t$-adic completion of $\mathbb{Z}[[t]][Y_1, \ldots, Y_n]$.

$\mathbb{Z}[[t]]\langle Y_1, \ldots, Y_n \rangle$ consists of the formal power series

$$f = f(Y_1, \ldots, Y_n) = \sum_{\nu} a_{\nu} Y_1^{\nu_1} \cdots Y_n^{\nu_n}, \quad \nu = (\nu_1, \ldots, \nu_n) \text{ ranging over } \mathbb{N}^n,$$

with all $a_{\nu} \in \mathbb{Z}[[t]]$ such that $a_{\nu} \to 0$, $t$-adically, as $|\nu| = \nu_1 + \cdots + \nu_n \to \infty$.

Extend the language $\mathcal{L}_{\text{val}}$ to $\mathcal{L}^{\mathbb{Z}[[t]]}_{\text{val}}$ by augmenting an $n$-ary function symbol for each $f \in \mathbb{Z}[[t]]\langle Y_1, \ldots, Y_n \rangle$.

Construe $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ as $\mathcal{L}^{\mathbb{Z}[[t]]}_{\text{val}}$-structures.

$f \in \mathbb{Z}[[t]]\langle Y \rangle$ only takes values in $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$. 
An analytic AKE principle

Theorem (van den Dries, 1992)

Let $\sigma$ be any $L_{\text{val}}^{\mathbb{Z}[[t]]}$-sentence. Then

$$\mathbb{Q}_p \models \sigma \iff \mathbb{F}_p((t)) \models \sigma$$

for all but finitely many primes $p$. 

Followed seminal work of Denef and van den Dries.

Strategy: Directly reduce to AKE-theory by Weierstrass division.

Application: solution to a problem posed by Serre.

Gives relative elementarity, model completeness, elimination of quantifiers, but not induced structure results for the coefficient field and monomial group.
Theorem (van den Dries, 1992)

Let $\sigma$ be any $\mathcal{L}^{\mathbb{Z}[[t]]}_{\text{val}}$-sentence. Then

$$\mathbb{Q}_p \models \sigma \iff \mathbb{F}_p((t)) \models \sigma$$

for all but finitely many primes $p$.

Followed seminal work of Denef and van den Dries.
An analytic AKE principle

Theorem (van den Dries, 1992)

Let $\sigma$ be any $\mathcal{L}_{\text{val}}^{\mathbb{Z}[[t]]}$-sentence. Then

$$\mathbb{Q}_p \models \sigma \iff \mathbb{F}_p((t)) \models \sigma$$

for all but finitely many primes $p$.

Followed seminal work of Denef and van den Dries.
Strategy: Directly reduce to AKE-theory by Weierstrass division.
An analytic AKE principle

Theorem (van den Dries, 1992)

Let $\sigma$ be any $L_{\text{val}}^{\mathbb{Z}[[t]]}$-sentence. Then

$$\mathbb{Q}_p \models \sigma \iff \mathbb{F}_p((t)) \models \sigma$$

for all but finitely many primes $p$.

Followed seminal work of Denef and van den Dries. Strategy: Directly reduce to AKE-theory by Weierstrass division.

Application: solution to a problem posed by Serre.
An analytic AKE principle

Theorem (van den Dries, 1992)

Let $\sigma$ be any $\mathcal{L}_{\text{val}}^{\mathbb{Z}[[t]]}$-sentence. Then

$$\mathbb{Q}_p \models \sigma \iff \mathbb{F}_p((t)) \models \sigma$$

for all but finitely many primes $p$.

Followed seminal work of Denef and van den Dries.
Strategy: Directly reduce to AKE-theory by Weierstrass division.

Application: solution to a problem posed by Serre.

Gives relative elementarity, model completeness, elimination of quantifiers,
An analytic AKE principle

Theorem (van den Dries, 1992)

Let \( \sigma \) be any \( L_{val}^{\mathbb{Z}[t]} \)-sentence. Then

\[
\mathbb{Q}_p \models \sigma \iff \mathbb{F}_p((t)) \models \sigma
\]

for all but finitely many primes \( p \).

Followed seminal work of Denef and van den Dries. Strategy: Directly reduce to AKE-theory by Weierstrass division.

Application: solution to a problem posed by Serre.

Gives relative elementarity, model completeness, elimination of quantifiers, but not induced structure results for the coefficient field and monomial group.
1. Classical AKE.

2. Denef – van den Dries’ analytic expansion.

3. Some induced structure by Binyamini – Cluckers – Novikov.

4. Running the AKE program.
In the role of $\mathbb{Z}[[t]]$ we consider a general noetherian ring $A$ with a distinguished ideal $\mathfrak{o}(A) \neq A$, and $A$ is $\mathfrak{o}(A)$-adically complete.
In the role of $\mathbb{Z}[[t]]$ we consider a general noetherian ring $A$ with a distinguished ideal $\mathfrak{o}(A) \neq A$, and $A$ is $\mathfrak{o}(A)$-adically complete.

A ring $R$ has $A$-analytic structure if there is a ring morphism

$$\iota_n : A\langle Y_1, \ldots, Y_n \rangle \to \text{ring of } R\text{-valued functions on } R^n$$
Formal analytic structure

In the role of $\mathbb{Z}[[t]]$ we consider a general noetherian ring $A$ with a distinguished ideal $\mathfrak{o}(A) \neq A$, and $A$ is $\mathfrak{o}(A)$-adically complete.

A ring $R$ has $A$-analytic structure if there is a ring morphism

$$\iota_n : A\langle Y_1, \ldots, Y_n \rangle \to \text{ring of } R\text{-valued functions on } R^n$$

for every $n$, 
Formal analytic structure

In the role of $\mathbb{Z}[[t]]$ we consider a general noetherian ring $A$ with a distinguished ideal $\mathfrak{o}(A) \neq A$, and $A$ is $\mathfrak{o}(A)$-adically complete.

A ring $R$ has $A$-analytic structure if there is a ring morphism

$$\iota_n : A\langle Y_1, \ldots, Y_n \rangle \rightarrow \text{ring of } R\text{-valued functions on } R^n$$

for every $n$, with the following properties:

(A1) $\iota_n(Y_k)(y_1, \ldots, y_n) = y_k$, for $k = 1, \ldots, n$;
In the role of $\mathbb{Z}[[t]]$ we consider a general noetherian ring $A$ with a distinguished ideal $\mathfrak{o}(A) \neq A$, and $A$ is $\mathfrak{o}(A)$-adically complete.

A ring $R$ has $A$-analytic structure if there is a ring morphism

$$\iota_n : A\langle Y_1, \ldots, Y_n \rangle \to \text{ring of } R\text{-valued functions on } R^n$$

for every $n$, with the following properties:

(A1) $\iota_n(Y_k)(y_1, \ldots, y_n) = y_k$, for $k = 1, \ldots, n$;

(A2) $\iota_{n+1}$ extends $\iota_n$. 
In the role of $\mathbb{Z}[[t]]$ we consider a general noetherian ring $A$ with a distinguished ideal $\mathcal{O}(A) \neq A$, and $A$ is $\mathcal{O}(A)$-adically complete.

A ring $R$ has $A$-analytic structure if there is a ring morphism

$$\iota_n : A[Y_1, \ldots, Y_n] \to \text{ring of } R\text{-valued functions on } R^n$$

for every $n$, with the following properties:

(A1) $\iota_n(Y_k)(y_1, \ldots, y_n) = y_k$, for $k = 1, \ldots, n$;

(A2) $\iota_{n+1}$ extends $\iota_n$.

We consider valuation rings with $A$-analytic structure and construe their fraction fields as $\mathcal{L}_{\text{val}}^A$-structures.
With $A := \mathbb{C}[[t]]$, construe $\mathbb{C}((t))$ as a $\mathcal{L}^{\mathbb{C}[[t]]}_{\text{val}}$-structure.
With $A := \mathbb{C}[[t]]$, construe $\mathbb{C}((t))$ as a $\mathcal{L}_{\text{val}}^{\mathbb{C}[[t]]}$-structure, $-\mathbb{C}((t))_{\text{an}}$. 
With $A := \mathbb{C}[[t]]$, construe $\mathbb{C}((t))$ as a $\mathcal{L}_{\text{val}}^{\mathbb{C}[[t]]}$-structure, $-\mathbb{C}((t))_{\text{an}}$.

In connection with a non-archimedean analogue of Pila-Wilkie type counting result, BCN consider a 3-sorted structure $\mathcal{M}$ comprised of:

- the analytic valued field $\mathbb{C}((t))_{\text{an}}$,
- the field $\mathbb{C}$,
- the ordered abelian group $\mathbb{Z}$.
With $A := \mathbb{C}[[t]]$, construe $\mathbb{C}((t))$ as a $\mathcal{L}^{\mathbb{C}[[t]]}_{\text{val}}$-structure, $-\mathbb{C}((t))_{\text{an}}$.

In connection with a non-archimedean analogue of Pila-Wilkie type counting result, BCN consider a 3-sorted structure $\mathcal{M}$ comprised of:

- the analytic valued field $\mathbb{C}((t))_{\text{an}}$,
- the field $\mathbb{C}$,
- the ordered abelian group $\mathbb{Z}$.

and the $\nu$ and $\overline{ac}$ maps relating the sorts.
With $A := \mathbb{C}[[t]]$, construe $\mathbb{C}((t))$ as a $\mathcal{L}_{\text{val}}^{\mathbb{C}[[t]]}$-structure, $-\mathbb{C}((t))_{\text{an}}$.

In connection with a non-archimedean analogue of Pila-Wilkie type counting result, BCN consider a 3-sorted structure $\mathcal{M}$ comprised of:

- the analytic valued field $\mathbb{C}((t))_{\text{an}}$,
- the field $\mathbb{C}$,
- the ordered abelian group $\mathbb{Z}$.

and the $\nu$ and $\overline{ac}$ maps relating the sorts.

**Proposition (Binyamini – Cluckers – Novikov, 2022)**

*If $P \subseteq \mathbb{C}((t))^n$ is definable in $\mathcal{M}$, then $P \cap \mathbb{C}^n$ is definable in the field $(\mathbb{C}; 0, 1, +, -, \cdot)$.***
With $A := \mathbb{C}[[t]]$, construe $\mathbb{C}((t))$ as a $\mathcal{L}^{\mathbb{C}[[t]]}_{\text{val}}$-structure, $-\mathbb{C}((t))_{\text{an}}$.

In connection with a non-archimedean analogue of Pila-Wilkie type counting result, BCN consider a 3-sorted structure $\mathcal{M}$ comprised of:

- the analytic valued field $\mathbb{C}((t))_{\text{an}}$,
- the field $\mathbb{C}$,
- the ordered abelian group $\mathbb{Z}$.

and the $\nu$ and $\overline{ac}$ maps relating the sorts.

**Proposition (Binyamini – Cluckers – Novikov, 2022)**

If $P \subseteq \mathbb{C}((t))^n$ is definable in $\mathcal{M}$, then $P \cap \mathbb{C}^n$ is definable in the field $(\mathbb{C}; 0, 1, +, -, \cdot)$.

Proof uses that $\mathcal{M}$ has quantifier elimination.
With $A := \mathbb{C}[[t]]$, construe $\mathbb{C}((t))$ as a $L_{\text{val}}^{\mathbb{C}[[t]]}$-structure, $-\mathbb{C}((t))_{\text{an}}$.

In connection with a non-archimedean analogue of Pila-Wilkie type counting result, BCN consider a 3-sorted structure $\mathcal{M}$ comprised of:

- the analytic valued field $\mathbb{C}((t))_{\text{an}}$,
- the field $\mathbb{C}$,
- the ordered abelian group $\mathbb{Z}$.

and the $\nu$ and $\overline{ac}$ maps relating the sorts.

**Proposition (Binyamini – Cluckers – Novikov, 2022)**

*If $P \subseteq \mathbb{C}((t))^n$ is definable in $\mathcal{M}$, then $P \cap \mathbb{C}^n$ is definable in the field $(\mathbb{C}; 0, 1, +, -, \cdot)$.*

Proof uses that $\mathcal{M}$ has quantifier elimination.

Lou’s “analytic AKE” results do give that any subset of $\mathbb{C}^n$ definable in $\mathcal{M}$ is definable in the field $\mathbb{C}$, but that’s not enough.
We consider the 1-sorted structure \((C((t))_{an}, C, t^\mathbb{Z})\), which includes lifts for the residue field and value group.
We consider the 1-sorted structure \((C((t))_{an}, C, t^\mathbb{Z})\), which includes lifts for the residue field and value group.

**Theorem (B. – van den Dries, 2022)**

- If \(X \subseteq C^m\) is definable in \((C((t))_{an}, C, t^\mathbb{Z})\), then \(X\) is even definable in the field \((C; 0, 1, +, -, \cdot)\).
We consider the 1-sorted structure \((C((t))_{an}, C, t^\mathbb{Z})\), which includes lifts for the residue field and value group.

**Theorem (B. – van den Dries, 2022)**

- If \(X \subseteq C^m\) is definable in \((C((t))_{an}, C, t^\mathbb{Z})\), then \(X\) is even definable in the field \((C; 0, 1, +, -, \cdot)\).

- Similarly, if \(Y \subseteq (t^\mathbb{Z})^n\) is definable in \((C((t))_{an}, C, t^\mathbb{Z})\), then \(Y\) is even definable in the ordered group \((t^\mathbb{Z}; 1, \cdot, \leq)\).
We consider the 1-sorted structure \((C((t))_\text{an}, C, t^\mathbb{Z})\), which includes lifts for the residue field and value group.

**Theorem (B. – van den Dries, 2022)**

- If \(X \subseteq C^m\) is definable in \((C((t))_\text{an}, C, t^\mathbb{Z})\), then \(X\) is even definable in the field \((C; 0, 1, +, -, \cdot)\).
- Similarly, if \(Y \subseteq (t^\mathbb{Z})^n\) is definable in \((C((t))_\text{an}, C, t^\mathbb{Z})\), then \(Y\) is even definable in the ordered group \((t^\mathbb{Z}; 1, \cdot, \leq)\).

The BCN proposition is a special case.
Sample induced structure results

We consider the 1-sorted structure \((C((t))_{an}, C, t^\mathbb{Z})\), which includes lifts for the residue field and value group.

**Theorem (B. – van den Dries, 2022)**

- If \(X \subseteq C^m\) is definable in \((C((t))_{an}, C, t^\mathbb{Z})\), then \(X\) is even definable in the field \((C; 0, 1, +, –, \cdot)\).
- Similarly, if \(Y \subseteq (t^\mathbb{Z})^n\) is definable in \((C((t))_{an}, C, t^\mathbb{Z})\), then \(Y\) is even definable in the ordered group \((t^\mathbb{Z}; 1, \cdot, \leq)\).

The BCN proposition is a special case. Note that subsets \(\mathbb{C}\) and \(t^\mathbb{Z}\) of \(C((t))\) are not definable in \(\mathcal{M}\).
Outline

1. Classical AKE.

2. Denef – van den Dries’ analytic expansion.

3. Some induced structure by Binyamini – Cluckers – Novikov.

4. Running the AKE program.
Our assumptions

Work with valuation rings with $A$-analytic structure.
Our assumptions

Work with valuation rings with $A$-analytic structure.

$R$ will denote a valuation $A$-ring.
Our assumptions

Work with valuation rings with $A$-analytic structure.

$R$ will denote a valuation $A$-ring. Then $K = \text{Frac}(R)$ is an $\mathcal{L}_\text{val}^A$-structure.
Our assumptions

Work with valuation rings with $A$-analytic structure.

$R$ will denote a valuation $A$-ring.
Then $K = \text{Frac}(R)$ is an $\mathcal{L}_\text{val}^A$-structure.

Assume from here on that $A$-ring $R$ is viable:
Our assumptions

Work with valuation rings with $A$-analytic structure.

$R$ will denote a valuation $A$-ring.
Then $K = \text{Frac}(R)$ is an $\mathcal{L}^A_{\text{val}}$-structure.

Assume from here on that $A$-ring $R$ is viable:
$\mathfrak{o}(R) = \rho R$ for some $\rho$, and $\rho \in \sqrt{\mathfrak{o}(A)R}$.
Our assumptions

Work with valuation rings with $A$-analytic structure.

$R$ will denote a valuation $A$-ring. Then $K = \text{Frac}(R)$ is an $L_{\text{val}}^A$-structure.

Assume from here on that $A$-ring $R$ is viable:

$\mathfrak{o}(R) = \rho R$ for some $\rho$, and $\rho \in \sqrt{\mathfrak{o}(A)R}$.

$R$ viable $\implies$ $R$ is henselian.
Our assumptions

Work with valuation rings with A-analytic structure.

$R$ will denote a valuation $A$-ring. Then $K = \text{Frac}(R)$ is an $\mathcal{L}_\text{val}^A$-structure.

Assume from here on that $A$-ring $R$ is viable:

\[ \phi(R) = \rho R \text{ for some } \rho, \text{ and } \rho \in \sqrt{\phi(A)R}. \]

$R$ viable $\implies$ $R$ is henselian.

Our assumptions give that viable valuation $A$-rings have:

piecewise uniform Weierstrass division with respect to parameters.
Let $L$ be an $L^A_{\text{val}}$-extension of $K$. 
Let $L$ be an $\mathcal{L}_\text{val}^A$-extension of $K$. $R_L$ not necessarily viable.
Let $L$ be an $L^A_{\text{val}}$-extension of $K$. $R_L$ not necessarily viable.

For $a \in L$, $K_a$ denotes the $L^A_{\text{val}}$-structure generated by $a$ over $K$. 
Let $L$ be an $\mathcal{L}^A_{\text{val}}$-extension of $K$. $R_L$ not necessarily viable.

For $a \in L$, $K_a$ denotes the $\mathcal{L}^A_{\text{val}}$-structure generated by $a$ over $K$.

Suppose $a \in L$ is algebraic over $K$. 
Algebraic is easy

Let $L$ be an $\mathcal{L}_\text{val}^A$-extension of $K$. $R_L$ not necessarily viable.

For $a \in L$, $K_a$ denotes the $\mathcal{L}_\text{val}^A$-structure generated by $a$ over $K$.

Suppose $a \in L$ is algebraic over $K$. Henselianity of $R$ gives $K_a = K(a)$. 
Let $L$ be an $\mathcal{L}_{\text{val}}^A$-extension of $K$. $R_L$ not necessarily viable.

For $a \in L$, $K_a$ denotes the $\mathcal{L}_{\text{val}}^A$-structure generated by $a$ over $K$.

Suppose $a \in L$ is algebraic over $K$. Henselianity of $R$ gives $K_a = K(a)$.

Want an isomorphism theory for $K_a$:

1. when $a \preceq 1$ and $\pi(a)$ is transcendental over $k_K$. 

Let $L$ be an $\mathcal{L}_{\text{val}}^A$-extension of $K$. $R_L$ not necessarily viable.

For $a \in L$, $K_a$ denotes the $\mathcal{L}_{\text{val}}^A$-structure generated by $a$ over $K$.

Suppose $a \in L$ is algebraic over $K$. Henselianity of $R$ gives $K_a = K(a)$.

Want an isomorphism theory for $K_a$:

1. when $a \leq 1$ and $\pi(a)$ is transcendental over $k_K$.
2. when $a \neq 0$ and $dv(a) \notin \Gamma_K$ for all $d \geq 1$. 
Let $L$ be an $\mathcal{L}_{\text{val}}^{A}$-extension of $K$. $R_{L}$ not necessarily viable.

For $a \in L$, $K_{a}$ denotes the $\mathcal{L}_{\text{val}}^{A}$-structure generated by $a$ over $K$.

Suppose $a \in L$ is algebraic over $K$. Henselianity of $R$ gives $K_{a} = K(a)$.

Want an isomorphism theory for $K_{a}$:

1. when $a \equiv 1$ and $\pi(a)$ is transcendental over $k_{K}$.
2. when $a \neq 0$ and $d\nu(a) \notin \Gamma_{K}$ for all $d \geq 1$.
3. when $K(a)$ is an immediate extension of $K$. 
Immediate extensions

Assume $\text{char } k_K = 0$. 
Immediate extensions

Assume $\text{char } k_K = 0$. So $K$ is algebraically maximal,
Immediate extensions

Assume \( \text{char } k_K = 0 \). So \( K \) is algebraically maximal, and we need only consider the case of a pc-sequence \((a_\rho)\) of transcendental type.
Immediate extensions

Assume $\text{char } k_K = 0$. So $K$ is algebraically maximal, and we need only consider the case of a pc-sequence $(a_\rho)$ of transcendental type.

Take $a_\rho \sim a, a \in R_L$. 
Immediate extensions

Assume \( \text{char } k_K = 0 \). So \( K \) is algebraically maximal, and we need only consider the case of a pc-sequence \((a_\rho)\) of transcendental type.

Take \( a_\rho \sim a, a \in R_L \). Is \( K_a \) an immediate extension of \( K \)?
Immediate extensions

Assume $\text{char } k_K = 0$. So $K$ is algebraically maximal, and we need only consider the case of a pc-sequence $(a_\rho)$ of transcendental type.

Take $a_\rho \sim a, a \in R_L$. Is $K_a$ an immediate extension of $K$?

Set $R(a) := \{g(a) : g \in R(Z)\}$
Immediate extensions

Assume $\text{char } k_K = 0$. So $K$ is algebraically maximal, and we need only consider the case of a pc-sequence $(a_\rho)$ of transcendental type.

Take $a_\rho \sim a$, $a \in R_L$. Is $K_a$ an immediate extension of $K$?

Set $R\langle a \rangle := \{ g(a) : g \in R\langle Z \rangle \} = \bigcup_n \{ f(a_1, \ldots, a_n, a) : f \in A\langle Y_1, \ldots, Y_n, Z \rangle \}$. 
Immediate extensions

Assume \( \text{char } k_K = 0 \). So \( K \) is algebraically maximal, and we need only consider the case of a pc-sequence \( (a_\rho) \) of transcendental type.

Take \( a_\rho \sim a, a \in R_L \). Is \( K_a \) an immediate extension of \( K \)?

Set \( R(a) := \{ g(a) : g \in R[Z] \} = \bigcup_n \{ f(a_1, \ldots, a_n, a) : f \in A[Y_1, \ldots, Y_n, Z] \} \)

\( R(a) \subseteq K_a, \)
Assume $\text{char } k_K = 0$. So $K$ is algebraically maximal, and we need only consider the case of a pc-sequence $(a_\rho)$ of transcendental type.

Take $a_\rho \sim a, a \in R_L$. Is $K_a$ an immediate extension of $K$?

Set $R(a) := \{ g(a) : g \in R[Z] \} = \bigcup_n \{ f(a_1, \ldots, a_n, a) : f \in A[Y_1, \ldots, Y_n, Z] \}$

$R(a) \subseteq K_a, K(a) \not\subseteq K_a$.
Immediate extensions

Assume \( \text{char } k_K = 0 \). So \( K \) is algebraically maximal, and we need only consider the case of a pc-sequence \((a_\rho)\) of transcendental type.

Take \( a_\rho \sim a, \ a \in R_L \). Is \( K_a \) an immediate extension of \( K \)?

Set \( R\{a\} := \{g(a) : g \in R\{Z\}\} = \bigcup_n \{f(a_1, \ldots, a_n, a) : f \in A\{Y_1, \ldots, Y_n, Z\}\} \)

\( R\{a\} \subseteq K_a, \ K(a) \not\subseteq K_a, \) and \( R\{a\} \) is not a valuation ring.
Immediate extensions

Assume \( \text{char } k_K = 0 \). So \( K \) is algebraically maximal, and we need only consider the case of a pc-sequence \((a_\rho)\) of transcendental type.

Take \( a_\rho \sim a, a \in R_L \). Is \( K_a \) an immediate extension of \( K \)?

Set \( R(a) := \{ g(a) : g \in R\langle Z \rangle \} = \bigcup_n \{ f(a_1, \ldots, a_n, a) : f \in A\langle Y_1, \ldots, Y_n, Z \rangle \} \)

\( R(a) \subseteq K_a, K(a) \not\subseteq K_a \), and \( R(a) \) is not a valuation ring.

Take an index \( \rho_0 \) such that for \( \rho > \rho_0 \),

\[
a = a_\rho + t_\rho u_\rho,
\]
Immediate extensions

Assume \( \text{char } k_K = 0 \). So \( K \) is algebraically maximal, and we need only consider the case of a pc-sequence \((a_\rho)\) of transcendental type.

Take \( a_\rho \sim a, a \in R_L \). Is \( K_a \) an immediate extension of \( K \)?

Set \( R\langle a \rangle := \{ g(a) : g \in R\langle Z \rangle \} = \bigcup_n \{ f(a_1, \ldots, a_n, a) : f \in A\langle Y_1, \ldots, Y_n, Z \rangle \} \)

\( R\langle a \rangle \subseteq K_a, K(a) \nsubseteq K_a, \) and \( R\langle a \rangle \) is not a valuation ring.

Take an index \( \rho_0 \) such that for \( \rho > \rho_0 \),

\[ a = a_\rho + t_\rho u_\rho, \quad t_\rho \in R, \]
Assume \( \text{char } k_K = 0 \). So \( K \) is algebraically maximal, and we need only consider the case of a pc-sequence \((a_\rho)\) of transcendental type.

Take \( a_\rho \sim a, a \in R_L \). Is \( K_a \) an immediate extension of \( K \)?

Set \( R(a) := \{ g(a) : g \in R(Z) \} = \bigcup_n \{ f(a_1, \ldots, a_n, a) : f \in A(Y_1, \ldots, Y_n, Z) \} \)

\( R(a) \subseteq K_a, K(a) \not\subseteq K_a \), and \( R(a) \) is not a valuation ring.

Take an index \( \rho_0 \) such that for \( \rho > \rho_0 \),

\[ a = a_\rho + t_\rho u_\rho, \quad t_\rho \in R, \quad v(u_\rho) = 0 \]
Immediate extensions

Assume \( \text{char } k_K = 0 \). So \( K \) is algebraically maximal, and we need only consider the case of a pc-sequence \((a_\rho)\) of transcendental type.

Take \( a_\rho \rightsquigarrow a, a \in R_L \). Is \( K_a \) an immediate extension of \( K \)?

Set \( R\{a\} := \{g(a) : g \in R\{Z\}\} = \bigcup_n \{ f(a_1, \ldots, a_n, a) : f \in A\{Y_1, \ldots, Y_n, Z\} \} \)
\( R\{a\} \subseteq K_a, K(a) \not\subseteq K_a \), and \( R\{a\} \) is not a valuation ring.

Take an index \( \rho_0 \) such that for \( \rho > \rho_0 \),

\[
a = a_\rho + t_\rho u_\rho, \quad t_\rho \in R, \quad v(u_\rho) = 0
\]

and \( v(t_\rho) \) is strictly increasing as a function of \( \rho > \rho_0 \).
Immediate extensions

Assume $\text{char } k = 0$. So $K$ is algebraically maximal, and we need only consider the case of a pc-sequence $(a_\rho)$ of transcendental type.

Take $a_\rho \sim a$, $a \in R_L$. Is $K_a$ an immediate extension of $K$?

Set $R\langle a \rangle := \{g(a) : g \in R\langle Z \rangle \} = \bigcup_n \{f(a_1, \ldots, a_n, a) : f \in A\langle Y_1, \ldots, Y_n, Z \rangle \}$

$R\langle a \rangle \subseteq K_a$, $K(a) \not\subseteq K_a$, and $R\langle a \rangle$ is not a valuation ring.

Take an index $\rho_0$ such that for $\rho > \rho_0$,

$$a = a_\rho + t_\rho u_\rho, \quad t_\rho \in R, \quad v(u_\rho) = 0$$

and $v(t_\rho)$ is strictly increasing as a function of $\rho > \rho_0$.

$R\langle a \rangle \subseteq R\langle u_\rho \rangle \subseteq R_a$,
Immediate extensions

Assume \( \text{char } k_K = 0 \). So \( K \) is algebraically maximal, and we need only consider the case of a pc-sequence \( (a_\rho) \) of transcendental type.

Take \( a_\rho \sim a, a \in R_L \). Is \( K_a \) an immediate extension of \( K \)?

Set \( R(a) := \{ g(a) : g \in R\langle Z \rangle \} = \bigcup_n \{ f(a_1, \ldots, a_n, a) : f \in A\langle Y_1, \ldots, Y_n, Z \rangle \} \)

\( R(a) \subseteq K_a, K(a) \not\subseteq K_a \), and \( R(a) \) is not a valuation ring.

Take an index \( \rho_0 \) such that for \( \rho > \rho_0 \),

\[
a = a_\rho + t_\rho u_\rho, \quad t_\rho \in R, \quad v(u_\rho) = 0
\]

and \( v(t_\rho) \) is strictly increasing as a function of \( \rho > \rho_0 \).

\( R(a) \subseteq R(u_\rho) \subseteq R_a \), and we discover that \( R_a = \bigcup_{\rho > \rho_0} R(u_\rho) \).
Completing the extension array

Let \( a \in L \).
Completing the extension array

Let $a \in L$. Weierstrass preparation for affinoid algebras gives a nice piecewise description of 1-variable terms,
Completing the extension array

Let $a \in L$.

Weierstrass preparation for \textit{affinoid algebras} gives a nice piecewise description of 1-variable terms, and we obtain:
Completing the extension array

Let $a \in L$.
Weierstrass preparation for *affinoid algebras* gives a nice piecewise description of 1-variable terms, and we obtain:

**Proposition**

The quantifier-free $\mathcal{L}_{\text{val}}^A$-type of $a$ over $K$ is completely determined by its quantifier-free $\mathcal{L}_{\text{val}}$-type over $K$. 

\[\text{Proposition}\]

19/22
Completing the extension array

Let \( a \in L \).
Weierstrass preparation for affinoid algebras gives a nice piecewise description of 1-variable terms, and we obtain:

**Proposition**

The quantifier-free \( \mathcal{L}_{\text{val}}^A \)-type of \( a \) over \( K \) is completely determined by its quantifier-free \( \mathcal{L}_{\text{val}} \)-type over \( K \).

**Lemma**

(i) If \( a \leq 1 \) and \( \pi(a) \) is transcendental over \( k_K \), then \( K_a \) is an immediate extension of \( K(a) \).
Completing the extension array

Let $a \in L$.
Weierstrass preparation for *affinoid algebras* gives a nice piecewise description of 1-variable terms, and we obtain:

**Proposition**

The quantifier-free $L^A_{\text{val}}$-type of $a$ over $K$ is completely determined by its quantifier-free $L_{\text{val}}$-type over $K$.

**Lemma**

(i) If $a \leq 1$ and $\pi(a)$ is transcendental over $k_K$, then $K_a$ is an immediate extension of $K(a)$.

(ii) If $a \neq 0$ and $dv(a) \notin \Gamma_K$ for all $d \geq 1$, then $K_a$ is an immediate extension of $K(a)$.
Completing the extension array

Let $a \in L$.

Weierstrass preparation for affinoid algebras gives a nice piecewise description of 1-variable terms, and we obtain:

**Proposition**

The quantifier-free $\mathcal{L}^A_{\text{val}}$-type of $a$ over $K$ is completely determined by its quantifier-free $\mathcal{L}_{\text{val}}$-type over $K$.

**Lemma**

(i) If $a \leq 1$ and $\pi(a)$ is transcendental over $k_K$, then $K_a$ is an immediate extension of $K(a)$.

(ii) If $a \neq 0$ and $dv(a) \notin \Gamma_K$ for all $d \geq 1$, then $K_a$ is an immediate extension of $K(a)$ provided $\Gamma_K$ is a $\mathbb{Z}$-group and $R_a$ is viable.
Completing the extension array

Let $a \in L$.

Weierstrass preparation for affinoid algebras gives a nice piecewise description of 1-variable terms, and we obtain:

**Proposition**

The quantifier-free $\mathcal{L}_\text{val}^A$-type of $a$ over $K$ is completely determined by its quantifier-free $\mathcal{L}_\text{val}$-type over $K$.

**Lemma**

(i) If $a \leq 1$ and $\pi(a)$ is transcendental over $k_K$, then $K_a$ is an immediate extension of $K(a)$.

(ii) If $a \neq 0$ and $dv(a) \notin \Gamma_K$ for all $d \geq 1$, then $K_a$ is an immediate extension of $K(a)$ provided $\Gamma_K$ is a $\mathbb{Z}$-group and $R_a$ is viable.

• Is $K_a$ always an immediate extension of $K(a)$?
Our analytic AKE equivalence

Let $A = \mathbb{Z}[[t]]$ and $\sigma(A) = t\mathbb{Z}[[t]]$. 
Our analytic AKE equivalence

Let $A = \mathbb{Z}[[t]]$ and $\sigma(A) = t\mathbb{Z}[[t]]$. Then for $\mathcal{L}_{\text{val}}^{\text{Acg}}$-structures $\mathcal{K} = (K_{\text{an}}, C_K, G_K)$ and $\mathcal{E} = (E_{\text{an}}, C_E, G_E)$:
Our analytic AKE equivalence

Let $A = \mathbb{Z}[[t]]$ and $\mathcal{O}(A) = t\mathbb{Z}[[t]]$.
Then for $\mathcal{L}_{val}^{A_{\text{cg}}}$-structures $\mathcal{K} = (K_{\text{an}}, C_K, G_K)$ and $\mathcal{E} = (E_{\text{an}}, C_E, G_E)$:

**Theorem (B. – van den Dries, 2022)**

Assume $\text{char } k_K = 0$ and $\Gamma_K$ is a $\mathbb{Z}$-group.
Our analytic AKE equivalence

Let \( A = \mathbb{Z}[[t]] \) and \( \sigma(A) = t\mathbb{Z}[[t]] \).
Then for \( L_{\text{val}}^{A_{\text{cg}}} \)-structures \( \mathcal{K} = (K_{\text{an}}, C_{\mathcal{K}}, G_{\mathcal{K}}) \) and \( \mathcal{E} = (E_{\text{an}}, C_{\mathcal{E}}, G_{\mathcal{E}}) \):

**Theorem (B. – van den Dries, 2022)**

Assume \( \text{char } k_{\mathcal{K}} = 0 \) and \( \Gamma_{\mathcal{K}} \) is a \( \mathbb{Z} \)-group. Suppose \( t \in G_{\mathcal{K}}, G_{\mathcal{L}} \).
Our analytic AKE equivalence

Let $A = \mathbb{Z}[[t]]$ and $\varphi(A) = t\mathbb{Z}[[t]]$. Then for $L_{val}^{A_{\text{cg}}}$-structures $\mathcal{K} = (K_{\text{an}}, C_K, G_K)$ and $\mathcal{E} = (E_{\text{an}}, C_E, G_E)$:

**Theorem (B. – van den Dries, 2022)**

Assume $\text{char } k_K = 0$ and $\Gamma_K$ is a $\mathbb{Z}$-group. Suppose $t \in G_K, G_L$. Then

$$\mathcal{K} \equiv \mathcal{E} \iff C_K \equiv C_E \text{ and } G_K \equiv_t G_E.$$
Our analytic AKE equivalence

Let $A = \mathbb{Z}[[t]]$ and $\phi(A) = t\mathbb{Z}[[t]]$.
Then for $\mathcal{L}_{\text{val}}^{\text{AEC}}$-structures $\mathcal{K} = (K_{\text{an}}, C_K, G_K)$ and $\mathcal{E} = (E_{\text{an}}, C_E, G_E)$:

**Theorem (B. – van den Dries, 2022)**

Assume $\text{char } k_K = 0$ and $\Gamma_K$ is a $\mathbb{Z}$-group. Suppose $t \in G_K, G_L$. Then

$$\mathcal{K} \equiv \mathcal{E} \iff C_K \equiv C_E \text{ and } G_K \equiv_t G_E.$$ 

Using an NIP transfer principle by Jahnke and Simon, we obtain:
Our analytic AKE equivalence

Let \( A = \mathbb{Z}[[t]] \) and \( \phi(A) = t\mathbb{Z}[[t]] \).

Then for \( \mathcal{L}^{A_{\mathrm{Acg}}} \)-structures \( \mathcal{K} = (K_{\mathrm{an}}, C_K, G_K) \) and \( \mathcal{E} = (E_{\mathrm{an}}, C_E, G_E) \):

**Theorem (B. – van den Dries, 2022)**

Assume \( \text{char } k_K = 0 \) and \( \Gamma_K \) is a \( \mathbb{Z} \)-group. Suppose \( t \in G_K, G_L \). Then

\[
\mathcal{K} \equiv \mathcal{E} \iff C_K \equiv C_E \text{ and } G_K \equiv_t G_E.
\]

Using an NIP transfer principle by Jahnke and Simon, we obtain:

**Proposition (B. – van den Dries, 2022)**

Let \( A \) be “general”. Assume \( \text{char } k_K = 0 \) and \( \Gamma_K \) is a \( \mathbb{Z} \)-group.
Our analytic AKE equivalence

Let $A = \mathbb{Z}[[t]]$ and $\varphi(A) = t\mathbb{Z}[[t]]$.
Then for $\mathcal{L}_{\text{val}}^{\text{Asg}}$-structures $\mathcal{K} = (K_{\text{an}}, C_K, G_K)$ and $\mathcal{E} = (E_{\text{an}}, C_E, G_E)$:

**Theorem (B. – van den Dries, 2022)**

Assume $\text{char } k_K = 0$ and $\Gamma_K$ is a $\mathbb{Z}$-group. Suppose $t \in G_K, G_L$. Then

$$\mathcal{K} \equiv \mathcal{E} \iff C_K \equiv C_E \text{ and } G_K \equiv_t G_E.$$ 

Using an NIP transfer principle by Jahnke and Simon, we obtain:

**Proposition (B. – van den Dries, 2022)**

Let $A$ be “general”. Assume $\text{char } k_K = 0$ and $\Gamma_K$ is a $\mathbb{Z}$-group. Then

the $\mathcal{L}_{\text{val}}^{\text{Asg}}$-structure $\mathcal{K}$ has NIP $\iff$ the ring $k_K$ has NIP.
References


Thank you!