An analytic AKE program

with induced structure results on coefficient field and monomial group

The Online Logic Seminar

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Weizmann Institute of Science

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- In parallel to the original theory of valued fields, we develop an extension theory in our framework.
- New is that in addition to AKE-type results for these structures, we obtain induced structure results for the coefficient field and monomial group.

Joint work with Lou van den Dries.

Outline

1. Classical AKE.
2. Denef – van den Dries' analytic expansion.
3. Some induced structure by Binyamini – Cluckers – Novikov.
4. Running the AKE program.

The AKE principle

All rings are commutative with unity.

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A valuation ring R is an integral domain such that for all $x \neq 0 \in K := \mathrm{Frac}(R)$, $x \in R$ or $x^{-1} \in R$.

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Let K and L be henselian valued fields of equicharacteristic 0. Then

$$K \equiv L \iff \boldsymbol{k}_K \equiv \boldsymbol{k}_L \text{ as fields, and } \Gamma_K \equiv \Gamma_L \text{ as ordered groups.}$$

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Let σ be any $\mathcal{L}_{\mathrm{val}}$-sentence. Then

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Gives relative elementarity, model completeness, elimination of quantifiers; and *induced structure results for lifts of the residue field and the value group.*

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Corollary

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Corollary

- *If $X \subseteq C_K^m$ is definable in (K, C_K, G_K), then X is even definable in the field $(C_K; 0, 1, +, -, \cdot)$.*
- *Similarly, if $Y \subseteq G_K^n$ is definable in (K, C_K, G_K), then Y is even definable in the ordered group $(G_K; 1, \cdot, \preccurlyeq)$.*

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Moreover, both $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$ are homomorphic images of $\mathbb{Z}[[t]]$:

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Can interpret the analytic structure on $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$ through a common language.

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Extend the language $\mathcal{L}_{\mathrm{val}}$ to $\mathcal{L}_{\mathrm{val}}^{\mathbb{Z}[[t]]}$ by augmenting an n-ary function symbol for each $f \in \mathbb{Z}[[t]]\langle Y_1, \ldots, Y_n\rangle$.

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Construe $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ as $\mathcal{L}_{\mathrm{val}}^{\mathbb{Z}[[t]]}$-structures.
$f \in \mathbb{Z}[[t]]\langle Y\rangle$ only takes values in $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$.

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Theorem (van den Dries, 1992)

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for all but finitely many primes p.

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Gives relative elementarity, model completeness, elimination of quantifiers, but **not** induced structure results for the coefficient field and monomial group.

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We consider valuation rings with A-analytic structure and construe their fraction fields as $\mathcal{L}^A_{\mathrm{val}}$-structures.

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In connection with a non-archimedean analogue of Pila-Wilkie type counting result, BCN consider a 3-sorted structure $\mathcal{M}$ comprised of:

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With $A := \mathbb{C}[[t]]$, construe $\mathbb{C}((t))$ as a $\mathcal{L}_{\mathrm{val}}^{\mathbb{C}[[t]]}$-structure, $-\mathbb{C}((t))_{\mathrm{an}}$.

In connection with a non-archimedean analogue of Pila-Wilkie type counting result, BCN consider a 3-sorted structure $\mathcal{M}$ comprised of:

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Lou's "analytic AKE" results do give that any subset of $\mathbb{C}^n$ definable in $\mathcal{M}$ is definable in the field $\mathbb{C}$, but that's not enough.

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The BCN proposition is a special case. Note that subsets $\mathbb{C}$ and t^{Z} of $\mathbb{C}((t))$ are not definable in $\mathcal{M}$.

Outline

1. Classical AKE.
2. Denef – van den Dries' analytic expansion.
3. Some induced structure by Binyamini – Cluckers – Novikov.
4. Running the AKE program.

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Our assumptions give that viable valuation A-rings have:

piecewise uniform Weierstrass division with respect to parameters.

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$R\langle a\rangle \subseteq R\langle u_\rho\rangle \subseteq R_a$, and we discover that $R_a = \bigcup_{\rho > \rho_0} R\langle u_\rho\rangle$.

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- Is K_a always an immediate extension of $K(a)$?

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Using an NIP transfer principle by Jahnke and Simon, we obtain:

Proposition (B. – van den Dries, 2022)

Let A be "general". Assume $\mathrm{char}\, \boldsymbol{k}_K = 0$ and Γ_K is a $\mathbb{Z}$-group. Then

$$\text{the } \mathcal{L}_{\mathrm{val}}^{\mathrm{Acg}}\text{-structure } \mathcal{K} \text{ has NIP} \iff \text{the ring } \boldsymbol{k}_K \text{ has NIP}.$$

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Thank you!