An analytic AKE program

with induced structure results on coefficient field and monomial group

The Online Logic Seminar

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Weizmann Institute of Science

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- New is that in addition to AKE-type results for these structures, we obtain induced structure results for the coefficient field and monomial group.

Joint work with Lou van den Dries.

Outline

- 1 Classical AKE.
- 2 Denef van den Dries' analytic expansion.
- 3 Some induced structure by Binyamini Cluckers Novikov
- 4 Running the AKE program.

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Let K and L be henselian valued fields of equicharacteristic 0. Then

 $K \equiv L \iff k_K \equiv k_L$ as fields, and $\Gamma_K \equiv \Gamma_L$ as ordered groups.

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Corollary

Let σ be any $\mathcal{L}_{\mathrm{val}}$ -sentence. Then

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The AKE program runs through an extension theory of valued fields.

Gives relative elementarity, model completeness, elimination of quantifiers; and induced structure results for lifts of the residue field and the value group.

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- ▶ If $X \subseteq C_K^m$ is definable in (K, C_K, G_K) , then X is even definable in the field $(C_K; 0, 1, +, -, \cdot)$.
- ▶ Similarly, if $Y \subseteq G_K^n$ is definable in (K, C_K, G_K) , then Y is even definable in the ordered group $(G_K; 1, \cdot, \leq)$.

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Moreover, both \mathbb{Z}_p and $\mathbb{F}_p[[t]]$ are homomorphic images of $\mathbb{Z}[[t]]$:

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Can interpret the analytic structure on \mathbb{Z}_p and $\mathbb{F}_p[[t]]$ through a common language.

Introducing restricted power series

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$$f = f(Y_1, \dots, Y_n) = \sum_{\nu} a_{\nu} Y_1^{\nu_1} \cdots Y_n^{\nu_n}, \qquad \nu = (\nu_1, \dots, \nu_n) \text{ ranging over } \mathbb{N}^n,$$

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Extend the language $\mathcal{L}_{\mathrm{val}}$ to $\mathcal{L}_{\mathrm{val}}^{\mathbb{Z}[[t]]}$ by augmenting an n-ary function symbol for each $f \in \mathbb{Z}[[t]]\langle Y_1, \ldots, Y_n \rangle$.

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Construe \mathbb{Q}_p and $\mathbb{F}_p((t))$ as $\mathcal{L}_{\mathrm{val}}^{\mathbb{Z}[[t]]}$ -structures. $f \in \mathbb{Z}[[t]]\langle Y \rangle$ only takes values in \mathbb{Z}_p and $\mathbb{F}_p[[t]]$.

Theorem (van den Dries, 1992)

Let σ be any $\mathcal{L}_{\mathrm{val}}^{\mathbb{Z}[[t]]}$ -sentence. Then

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We consider valuation rings with A-analytic structure and construe their fraction fields as $\mathcal{L}_{\mathrm{val}}^{A}$ -structures.

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In connection with a non-archimedean analogue of Pila-Wilkie type counting result, BCN consider a 3-sorted structure \mathcal{M} comprised of:

the analytic valued field $\mathbb{C}((t))_{\mathrm{an}},$ the field $\mathbb{C},$ the ordered abelian group $\mathbb{Z}.$

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Proposition (Binyamini – Cluckers – Novikov, 2022)

If $P \subseteq \mathbb{C}((t))^n$ is definable in \mathcal{M} , then $P \cap \mathbb{C}^n$ is definable in the field $(\mathbb{C}; 0, 1, +, -, \cdot)$.

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Proof uses that ${\mathcal M}$ has quantifier elimination.

With $A := \mathbb{C}[[t]]$, construe $\mathbb{C}((t))$ as a $\mathcal{L}_{\mathrm{val}}^{\mathbb{C}[[t]]}$ -structure, $-\mathbb{C}((t))_{\mathrm{an}}$.

In connection with a non-archimedean analogue of Pila-Wilkie type counting result, BCN consider a 3-sorted structure $\mathcal M$ comprised of:

the analytic valued field $\mathbb{C}((t))_{\mathrm{an}},$ the field $\mathbb{C},$ the ordered abelian group $\mathbb{Z}.$

and the v and \overline{ac} maps relating the sorts.

Proposition (Binyamini – Cluckers – Novikov, 2022)

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Lou's "analytic AKE" results do give that any subset of \mathbb{C}^n definable in \mathcal{M} is definable in the field \mathbb{C} , but that's not enough.

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The BCN proposition is a special case. Note that subsets \mathbb{C} and t^Z of $\mathbb{C}((t))$ are not definable in \mathcal{M} .

Outline

- Classical AKE
- 2 Denef van den Dries' analytic expansion.
- 3 Some induced structure by Binyamini Cluckers Novikov
- 4 Running the AKE program.

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Our assumptions give that viable valuation A-rings have:

piecewise uniform Weierstrass division with respect to parameters.

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 $R(a) \subseteq R(u_{\rho}) \subseteq R_a$, and we discover that $R_a = \bigcup_{\rho > \rho_0} R(u_{\rho})$.

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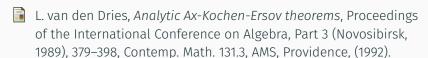
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The end

Thank you!