Some existential theories of fields

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Online Logic Seminar
Abstract

Building on previous work, I will discuss Turing reductions between various fragments of theories of fields. In particular, we exhibit several theories of fields Turing equivalent to the existential theory of the rational numbers. This is joint work with Arno Fehm, in progress.
Today

1. Motivation
2. Main theorem
   - Ingredient 1: henselian valued fields
   - Ingredient 2: power series fields
3. Equicharacteristic henselian nontrivially valued fields
   - Sketch proof of Corollary 3
4. Equicharacteristic henselian nontrivially ... (reprise)
5. Idea of proof of Theorem 1
   - Step 1
   - Step 2
     - Ingredient 3: large fields
   - Step 3
Motivation

Let $R$ be a ring.

**Hilbert’s Tenth Problem (H10) for $R$**

Give an algorithm to decide correctly, for each $f \in \mathbb{Z}[X_1, \ldots, X_n]$, whether the Diophantine equation

$$f(X_1, \ldots, X_n) = 0$$

has a solution in $R$.

Original version is $R = \mathbb{Z}$.
Let $R$ be a ring.

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**Theorem (Davis–Matiyasevich–Putnam–Robinson, 1949-70)**

H10 for $\mathbb{Z}$ is unsolvable, i.e. $\text{Th}_\exists(\mathbb{Z})$ is undecidable.

Three open decidability problems:

- $\text{Th}_\exists(\mathbb{Q})$
- $\text{Th}_\exists(\mathbb{C}(t), t)$
- $\text{Th}(\mathbb{F}_p((t)))$
Theorem 1 (A.–Fehm, 2021)

The following theories are Turing-equivalent:

1. The existential theory of $\mathbb{Q}$ in the language of rings.
2. The existential theory of $\mathbb{Q}((t))$ in the language of rings.
3. The existential theory of $\mathbb{Q}((t))$ in the language of valued fields.
4. The existential theory of $\mathbb{Q}((t))$ in the language of valued fields with constant $t$.
5. The existential theory of fields in the language of rings.
6. The existential theory of large fields in the language of rings.
7. The existential theory of large fields of characteristic zero in the language of rings.
Main theorem

Theorem 1 (A.–Fehm, 2021)

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6. The existential theory of large fields in the language of rings.
7. The existential theory of large fields of characteristic zero in the language of rings.

— None of these equivalences seems obvious, to me anyway!
— There is one equality, otherwise these are pairwise distinct
Ingredient 1: henselian valued fields

Definition

Valued field is a pair \((K, \nu)\), where \(K\) is a field and \(\nu : K \rightarrow \Gamma \cup \{\infty\}\), for (additive) ordered abelian group \(\Gamma\), such that

- \(\nu(x) = \infty\) iff \(x = 0\),
- \(\nu(xy) = \nu(x) + \nu(y)\), and
- \(\nu(x + y) \geq \min\{\nu(x), \nu(y)\}\).

- \(\Gamma\) value group
- \(\mathcal{O}_\nu = \{x \in K \mid \nu(x) \geq 0\}\) valuation ring
- \(m_\nu = \{x \in K \mid \nu(x) > 0\}\) maximal ideal
- \(K_\nu = \mathcal{O}_\nu / m_\nu\) residue field
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**Definition**

\((K, \nu)\) is henselian if for every monic \(f \in \mathcal{O}_\nu[X]\) and every \(a \in \mathcal{O}_\nu\) with \(\nu(f(a)) > 0\) and \(\nu(f'(a)) = 0\), there exists \(a' \in a + m_\nu\) with \(f(a') = 0\).
Let $F$ be any field, and $t$ be a ‘formal indeterminate’.

**Definition**

The **power series field** $F((t))$ is the set of formal series

$$\sum_{n \in \mathbb{Z}} a_n t^n,$$

where $a_n \in F$,

such that $\{n \in \mathbb{Z} : a_n \neq 0\}$ is empty or has minimum. We equip with addition and multiplication:

$$\left(\sum_{n} a_n t^n\right) + \left(\sum_{n} b_n t^n\right) := \sum_{n} (a_n + b_n) t^n,$$

$$\left(\sum_{m} a_m t^m\right) \cdot \left(\sum_{n} b_n t^n\right) := \sum_{k} \left(\sum_{m+n=k} a_m b_n\right) t^k.$$
We also equip $F((t))$ with the $t$-adic valuation:

$$v_t : F((t)) \rightarrow \mathbb{Z} \cup \{\infty\}$$

$$a = \sum_n a_n t^n \mapsto \begin{cases} \min\{n \mid a_n \neq 0\} & a \neq 0 \\ \infty & a = 0. \end{cases}$$

For $(F((t)), v_t)$,

- value group is $\mathbb{Z}$,
- valuation ring is $F[[t]] = \{a \in F((t)) \mid v_t(a) \geq 0\} = \{\sum_{n \geq 0} a_n t^n\}$,
- maximal ideal is $tF[[t]]$,
- residue field is $F[[t]]/tF[[t]] \cong F$. 
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$(F((t)), v_t)$ is henselian.
Equicharacteristic henselian nontrivially valued fields

— Language of rings is $\mathcal{L}_{\text{ring}} = \{+, \cdot, -, 0, 1\}$.

— Language of valued fields is $\mathcal{L}_{\text{vf}}$, which is three-sorted (field $K$, value group $\Gamma$, residue field $k$) with valuation map ($v : K \rightarrow \Gamma$) and residue map ($\text{res} : K \rightarrow k$).

**Theorem (Denef–Schoutens, [DS03])**

*Resolution of Singularities in characteristic $p$ implies that the existential theory of $\mathbb{F}_q((t))$ with constant for $t$ is decidable.*
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**Theorem 2 (A.–Fehm, [AF16])**

Equal characteristic henselian valued fields satisfy an existential-decidability Ax–Kochen/Ershov principle:

\[
\text{Th}_\exists(K, v) \text{ decidable} \iff \text{Th}_\exists(Kv) \text{ decidable}
\]
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**Corollary 3**

$\text{Th}_\exists(\mathbb{F}_q((t)), v_t)$ is decidable
Equicharacteristic henselian nontrivially valued fields

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Corollary 3

$\text{Th}_\exists(\mathbb{F}_q((t)), v_t)$ is decidable — no constant for $t$!
Sketch proof of Corollary 3

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$\text{Th}_\exists(\mathbb{F}_q((t)), \nu_t)$ is decidable.
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\( \text{Th}_\exists (\mathbb{F}_q((t)), \nu_t) \) is decidable.

The tame valued field \( \mathbb{F}_q((t)^Q) \) has decidable theory, by a theorem of F.-V. Kuhlmann, [Kuh14]. Denote by \( \mathbb{F}_q((t))^Q \) the relative algebraic closure of \( \mathbb{F}_q((t)) \) in \( \mathbb{F}_q((t^Q)) \). Then \( \mathbb{F}_q((t)) \) is a directed union of fields \( \mathbb{F}_q((s)) \), isomorphic copies of \( \mathbb{F}_q((t)) \). Moreover, \( \mathbb{F}_q((t))^Q \preceq \mathbb{F}_q((t^Q)) \) by another theorem of F.-V. Kuhlmann, [Kuh14].

\[
\mathbb{F}_q((t^Q)) \\
\downarrow \preceq \\
\mathbb{F}_q((t))^Q = \bigcup_s \mathbb{F}_q((s)) \\
\downarrow \preceq \exists - \emptyset \\
\mathbb{F}_q((t))
\]
Let $H^e'$ be a recursive axiomatisation of the $\mathcal{L}_{\text{vf}}$-theory of equicharacteristic henselian nontrivially valued fields. For any theory $R$ of fields, write

$$H^e'(R) = H^e' \cup \{\text{“}\varphi\text{ holds in residue field”} \mid \varphi \in R\}.$$
Definition

Let $\mathcal{H}'$ be a recursive axiomatisation of the $\mathcal{L}_{\text{vf}}$-theory of equicharacteristic henselian nontrivially valued fields. For any theory $R$ of fields, write

$$\mathcal{H}'(R) = \mathcal{H}' \cup \{\text{"$\varphi$ holds in residue field" | $\varphi \in R}\}.$$ 

Proposition 4

Let $F$ be any field. Then $\mathcal{H}'(\text{Th}(F))$ is existentially complete. That is, for every $\varphi \in \text{Sent}_\exists(\mathcal{L}_{\text{vf}})$, either $\mathcal{H}'(\text{Th}(F)) \models \varphi$ or $\mathcal{H}'(\text{Th}(F)) \models \neg \varphi$.

Corollary: Theorem 2.
Proposition 5 (Existential Elimination)

There is a recursive map

\[ \text{Sent}_\exists (\mathcal{L}_{vf}) \rightarrow \text{Sent}_\exists (\mathcal{L}_{\text{ring}}) \]

\[ \varphi \mapsto \varphi_k \]

such that

\[ (K, v) \models \varphi \iff Kv \models \varphi_k \]

for every \((K, v) \models H^{ef} \).
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Proof: Proposition 4 + Separation Lemma!
**Proposition 5 (Existential Elimination)**

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such that

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(K, v) \models \varphi \iff K^v \models \varphi_k
\]

for every \((K, v) \models \mathcal{H}^e\).  

**Proof**: Proposition 4 + Separation Lemma!

**Theorem 6 ("\(\mathcal{H}^e\) Theorem")**

For any \(\mathcal{L}_{\text{ring}}\)-theory \(R\) of fields, the set of existential consequences of \(R\) is Turing-equivalent to the set of existential sentences in the language of valued fields that hold in every equicharacteristic henselian non-trivially valued field with residue field a model of \(R\). That is:

\[
\mathcal{H}^e(\mathcal{R})_\exists \simeq_T \mathcal{R}_\exists
\]
Proposition 5 (Existential Elimination)

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\text{Sent}_\exists(\mathcal{L}_{vf}) \longrightarrow \text{Sent}_\exists(\mathcal{L}_{\text{ring}})
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\varphi \longmapsto \varphi_k
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such that

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(K, v) \models \varphi \iff Kv \models \varphi_k
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for every \((K, v) \models \text{He}'\).

Proof: Proposition 4 + Separation Lemma!

Theorem 6 (“\text{He}' Theorem”)

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\[
\text{He}'(R)_\exists \simeq_T R_\exists
\]

Proof: \(\varphi \in \text{He}'(R)_\exists \iff \varphi_k \in R_\exists\).
Let’s briefly recall the statement.

**Theorem 1**

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Idea of proof of Theorem 1

Step 1

\[(1) \leq_T (2) \simeq_T (3) \leq_T (4).\]

Reducts and interpretations

- \(\text{Th}_\exists(\mathbb{Q}) \leq_T \text{Th}_\exists(\mathbb{Q}(t), v_t) \leq_T \text{Th}_\exists(\mathbb{Q}(t), v_t, t)\)
- \(\text{Th}_\exists(\mathbb{Q}(t)) \leq_T \text{Th}_\exists(\mathbb{Q}(t), v_t)\)

Thus \((1) \leq_T (3) \leq_T (4)\) and \((2) \leq_T (3)\).
Idea of proof of Theorem 1

Step 1

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Reducts and interpretations

- \(\text{Th}_\exists(Q) \leq_T \text{Th}_\exists(Q((t)), v_t) \leq_T \text{Th}_\exists(Q((t)), v_t, t)\)
- \(\text{Th}_\exists(Q((t))) \leq_T \text{Th}_\exists(Q((t)), v_t)\)

Thus \((1) \leq_T (3) \leq_T (4)\) and \((2) \leq_T (3)\).

First we note that the valuation ring \(Q[[t]]\) is \(\exists\)- and \(\forall-\mathcal{L}_\text{ring}\)-definable in \(Q((t))\). For example

\[
Q[[t]] = \left\{ \frac{1}{z^2 - 2} + \frac{1}{1 + u^2 + v^2 + w^2 + x^2} \middle| u, v, w, x, z \in Q((t)) \right\},
\]

as in [AF17]. Therefore \(\text{Th}_\exists(Q((t))) \simeq_T \text{Th}_\exists(Q((t)), v_t)\). Thus \((2) \simeq_T (3)\).

\(\square\) (Step 1)

Theorem 2 already gives \((1) \simeq_T (3)\). Also see [San96, Remark, p. 23].
Idea of proof of Theorem 1

Step 2

\((4) \leq_T (1)\)

In fact we’ll simply show \((4) \simeq_T (1)\), i.e. \(\text{Th}_\exists(\mathbb{Q}((t))), v_t, t) \simeq_T \text{Th}_\exists(\mathbb{Q})\). We mimic the proof of \((3) \simeq_T (1)\) via Theorem 2. Expand the language with a new constant symbol:

\[
\mathcal{L}_{vf}(\pi) := \mathcal{L}_{vf} \cup \{\pi\}.
\]

Consider the \(\mathcal{L}_{vf}(\pi)\)-theory

\[
H^{e,\pi} := H^{e'} \cup \{\forall x \ (0 < v(x) \rightarrow v(\pi) \leq v(x))\}.
\]

Also \(H^{e,\pi}(R) = H^{e,\pi} \cup \{“\varphi \ holds \ in \ residue \ field” \mid \varphi \in R\}\.\)
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In fact we’ll simply show \( (4) \simeq_T (1) \), i.e. \( \text{Th}_\exists(\mathbb{Q}((t))), v_t, t \simeq_T \text{Th}_\exists(\mathbb{Q}) \). We mimic the proof of \((3) \simeq_T (1)\) via Theorem 2. Expand the language with a new constant symbol:

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\[ H^{e,\pi} := H^{e'} \cup \{\forall x \ (0 < v(x) \rightarrow v(\pi) \leq v(x))\}. \]

Also \( H^{e,\pi}(R) = H^{e,\pi} \cup \{“\varphi \text{ holds in residue field”} \ | \ \varphi \in R\} \).

Claim 1

\( H^{e,\pi}(\text{Th}(\mathbb{Q})) \) is existentially complete.

It follows that \( H^{e,\pi}(\text{Th}(\mathbb{Q}))_\exists = \text{Th}_\exists(\mathbb{Q}((t))), v_t, t \) and \( H^{e,\pi}(\text{Th}(\mathbb{Q}))_\exists \simeq_T \text{Th}_\exists(\mathbb{Q}) \).
Proof of Claim 1

Let $(K, \nu, \pi_\nu), (L, w, \pi_w) \models H_{\nu}^{\pi}(\text{Th}(\mathbb{Q}))$. Assuming sufficient saturation, there are sections of the residue maps: $K\nu \rightarrow K$ and $Lw \rightarrow L$; and moreover there is an embedding $K\nu \rightarrow Lw$ between the residue fields. In fact we get this picture:

![Diagram showing the relationships between the fields and residue maps]

The embedding on the left is existentially closed because $\text{char}(K\nu((s))) = 0$.

\(\Box\) (Step 2)

Observation ([AF16])

$\text{Th}_\exists(F((t)), \nu_t, \nu) \simeq_T \text{Th}_{\forall \exists}^1(F((t)), \nu_t)$. 
A field $L$ is *large* if

$$C(L) \neq \emptyset \implies |C(L)| = \infty,$$

for all smooth curves $C$ defined over $L$. 

Theorem (Pop, [Pop96])

TFAE

1. $L$ is large
2. for each smooth irreducible $L$-variety $V$, if $V(L) \neq \emptyset$ then $V(L)$ is Zariski dense in $V$.
3. $L \preceq \exists L((t))$
4. $L \preceq \exists L(t) h$

Theorem (Pop, [Pop96])

If $(L, w)$ is a henselian nontrivially valued field, then $L$ is large.
Definition

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Ingredient 3: large fields

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**Theorem (Pop, [Pop96])**

If $(L, w)$ is a henselian nontrivially valued field, then $L$ is large.
### Example

1. \( \mathbb{C}, \mathbb{Q}_{\text{alg}}, \mathbb{F}_{p}^{\text{alg}} \), any algebraically closed field
2. \( \mathbb{R}, \mathbb{R}_{\text{alg}}, \), any real closed field
3. \( \mathbb{Q}_{p}, \mathbb{Q}_{p,\text{alg}} \), any \( p \)-adically closed field
4. PAC/PRC/PpC/PCC
**Example**

1. $\mathbb{C}, \mathbb{Q}_{\text{alg}}, \overline{\mathbb{F}}_p$, any algebraically closed field
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**Non-examples**

1. $\mathbb{F}_{p^k}$, 
2. number fields
3. function fields
Example

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4. PAC/PRC/PpC/PCC

Non-examples

1. \( \mathbb{F}_{p^k} \),
2. number fields
3. function fields

Definition

- \( \mathbb{L}_{\exists} \) is the common existential \( \mathbb{L}_{\text{ring}} \)-theory of large fields
- \( \mathbb{L}_{\exists,0} \) (resp. \( \mathbb{L}_{\exists,p} \)) is ... of characteristic 0 (resp. characteristic \( p \))
- \( \mathbb{L}_{\exists,>0} \) is ... of positive characteristic
- \( \mathbb{L}_{\exists,\gg0} \) is ... of sufficiently high positive characteristic.
Step 3

\[(6) \cong_T (7) = (2).\]

Claim 2 (Sander, [San96, Proposition 2.25])

\[\text{Th}_\exists(\mathbb{Q}((t))) = L_{\exists,0}.\]
Idea of proof of Theorem 1

Step 3

\[(6) \simeq_T (7) = (2).\]

Claim 2 (Sander, [San96, Proposition 2.25])

\[\text{Th}_\exists(\mathbb{Q}((t))) = \mathbb{L}_{\exists,0}.\]

Proof of Claim 2

Since \(\mathbb{Q}((t))\) is nontrivially valued henselian, it is large. Thus \(\text{Th}_\exists(\mathbb{Q}((t))) \models \mathbb{L}_{\exists,0}.\)

In the other direction, let \(L\) be a large field of characteristic 0. Then \(L \preceq_\exists L((t)) \supseteq \mathbb{Q}((t)).\) Thus \(L \models \text{Th}_\exists(\mathbb{Q}((t))).\) Therefore \(\mathbb{L}_{\exists,0} \models \text{Th}_\exists(\mathbb{Q}((t))).\)

Thus \((2) = (7).\)

Similarly: \(\text{Th}_\exists(\mathbb{F}_p((t))) = \mathbb{L}_{\exists,p}\) for each prime number \(p.\)
Claim 3

Both $L_{\exists, > 0}$ and $L_{\exists, \gg 0}$ are decidable.
Claim 3
Both $L_{∃,>0}$ and $L_{∃,≫0}$ are decidable.

Proof of Claim 3
We use the theory $\text{Fin}$ of finite fields, and the theory $\text{Fin}_{≫0}$ of finite fields of sufficiently high characteristic. Both are decidable by Ax.

Building on Claim 2, we have

- $L_{∃,>0} = H^{e_t}(\text{Fin}_∃)_{∃} \cap \text{Sent}(\mathcal{L}_{\text{ring}})$
- $L_{∃,≫0} = H^{e_t}(\text{Fin}_∃,≫0)_{∃} \cap \text{Sent}(\mathcal{L}_{\text{ring}})$.

By the $H^{e_t}$ Theorem:

- $L_{∃,>0} \leq_T \text{Fin}_∃$
- $L_{∃,≫0} \leq_T \text{Fin}_∃,≫0$. 
Claim 4

\[ L_\exists \cong T L_{\exists,0}. \]

Proof of Claim 4

First:

\[ L_\exists \models \forall \iff L_\exists \models \forall_0 \& L_\exists \models \forall_0 \models [T_h \exists (F p (\varphi))] \cong (T_h \exists (F p)) \]

is decidable. We outline an algorithm to decide \( L_\exists \models_0 \) given an oracle for \( L_\exists \models_0 \).

1. Decide whether or not \( L_\exists \models_0 \models \forall \). If ‘NO’ then \( L_\exists \models_0 \not\models \forall \). If ‘YES’ then continue.

2. For each \( n \geq 2 \), let \( \chi_n \models = \bigvee_{p = 1}^n 1 \). Let \( \varphi_n \models = \varphi \lor \chi_n \). Since \( L_\exists \models_0 \models \forall \), there is some \( n \) such that \( L_\exists \models_0 \models \chi_n \). Apply decision procedure for \( L_\exists \models_0 \) to \( \varphi_n \) successively until we have found minimal \( n_0 \) with \( L_\exists \models_0 \models \chi_n \).

3. Decide whether or not \( L_\exists \models_0 \models \forall \). If ‘YES’ then we have \( L_\exists \models_0 \models \forall \). Thus \( L_\exists \models_0 \models \forall \). Output ‘YES’ and finish. If ‘NO’ then \( L_\exists \models_0 \not\models \forall \). We already know that \( L_\exists \models_0 \models \forall \). Thus \( L_\exists \models_0 \not\models \forall \). Output ‘NO’ and finish.
Claim 4

$L_\exists \simeq_T L_{\exists,0}$.

Proof of Claim 4

First: $L_\exists \models \varphi \iff L_{\exists,0} \models \varphi \& L_{\exists,>0} \models \varphi$. Thus $L_\exists \leq_T L_{\exists,0}$. For the other direction, we know that $L_\exists, p = \text{Th}_\exists(\mathbb{F}_p((t))) \simeq_T \text{Th}_\exists(\mathbb{F}_p)$ is decidable. We outline an algorithm to decide $L_{\exists,0}$ given an oracle for $L_\exists$. Let $\varphi \in \text{Sent}_\exists(\mathcal{L}_{\text{ring}})$.

1. Decide whether or not $L_{\exists,\gg 0} \models \varphi$.

   If ‘NO’ then $L_{\exists,0} \not\models \varphi$. If ‘YES’ then continue.

2. For each $n \geq 2$, let $\chi_n \models \bigvee_{p \leq n} 1 + \ldots + 1 = 0$. Let $\varphi_n \models \varphi \lor \chi_n$. Since $L_{\exists,\gg 0} \models \varphi$, there is some $n$ such that $L_{\exists,>0} \models \varphi_n$. Apply decision procedure for $L_{\exists,>0}$ to $\varphi_n$ successively until we have found minimal $n_0$ with $L_{\exists,>0} \models \varphi_{n_0}$.

3. Decide whether or not $L_\exists \models \varphi_{n_0}$.

   If ‘YES’ then we have $L_{\exists,0} \models \varphi_{n_0}$. Thus $L_{\exists,0} \models \varphi$. Output ‘YES’ and finish. If ‘NO’ then $L_\exists \not\models \varphi_{n_0}$. We already know that $L_{\exists,>0} \models \varphi_{n_0}$. Thus $L_{\exists,0} \not\models \varphi_{n_0}$. Output ‘NO’ and finish.

\square (Step 3)
Thank you for listening. Questions are very welcome!
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