Some existential theories of fields

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Online Logic Seminar

Abstract

Building on previous work, I will discuss Turing reductions between various fragments of theories of fields. In particular, we exhibit several theories of fields Turing equivalent to the existential theory of the rational numbers. This is joint work with Arno Fehm, in progress.

Today

Motivation

Let R be a ring.

Hilbert's Tenth Problem (**H10**) for R

Give an algorithm to decide correctly, for each $f \in \mathbb{Z}[X_1, \ldots, X_n]$, whether the Diophantine equation

$$f(X_1, ..., X_n) = 0$$

has a solution in R.

Original version is $R = \mathbb{Z}$.

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Theorem (Davis–Matiyasevich–Putnam–Robinson, 1949-70)

H10 *for* $\mathbb{Z}$ *is unsolvable, i.e.* $\mathrm{Th}_\exists(\mathbb{Z})$ *is undecidable.*

Three open decidability problems:

- $\mathrm{Th}_\exists(\mathbb{Q})$
- $\mathrm{Th}_\exists(\mathbb{C}(t), t)$
- $\mathrm{Th}(\mathbb{F}_p((t)))$

Main theorem

Theorem 1 (A.–Fehm, 2021)

The following theories are Turing-equivalent:

1. The existential theory of $\mathbb{Q}$ in the language of rings.
2. The existential theory of $\mathbb{Q}((t))$ in the language of rings.
3. The existential theory of $\mathbb{Q}((t))$ in the language of valued fields.
4. The existential theory of $\mathbb{Q}((t))$ in the language of valued fields with constant t.
5. The existential theory of fields in the language of rings.
6. The existential theory of large fields in the language of rings.
7. The existential theory of large fields of characteristic zero in the language of rings.

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— None of these equivalences seems obvious, to me anyway!

— There is one equality, otherwise these are pairwise distinct

Ingredient 1: henselian valued fields

Definition

Valued field is a pair (K, v), where K is a field and $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$, for (additive) ordered abelian group Γ, such that

- $v(x) = \infty$ iff $x = 0$,
- $v(xy) = v(x) + v(y)$, and
- $v(x + y) \geq \min\{v(x), v(y)\}$.

- Γ value group
- $\mathcal{O}_v = \{x \in K \mid v(x) \geq 0\}$ valuation ring
- $\mathfrak{m}_v = \{x \in K \mid v(x) > 0\}$ maximal ideal
- $Kv = \mathcal{O}_v/\mathfrak{m}_v$ residue field

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Definition

(K, v) is *henselian* if for every monic $f \in \mathcal{O}_v[X]$ and every $a \in \mathcal{O}_v$ with $v(f(a)) > 0$ and $v(f'(a)) = 0$, there exists $a' \in a + \mathfrak{m}_v$ with $f(a') = 0$.

Ingredient 2: power series fields

Let F be any field, and t be a 'formal indeterminate'.

Definition

The *power series field* $F((t))$ is the set of formal series

$$\sum_{n\in\mathbb{Z}} a_n t^n, \qquad \text{where } a_n \in F,$$

such that $\{n \in \mathbb{Z} : a_n \neq 0\}$ is empty or has minimum. We equip with addition and multiplication:

$$\Big(\sum_n a_n t^n\Big) + \Big(\sum_n b_n t^n\Big) := \sum_n (a_n + b_n) t^n,$$
$$\Big(\sum_m a_m t^m\Big) \cdot \Big(\sum_n b_n t^n\Big) := \sum_k \Big(\sum_{m+n=k} a_m b_n\Big) t^k.$$

We also equip $F((t))$ with the *t-adic valuation*:

$$\begin{aligned} v_t : F((t)) &\twoheadrightarrow \mathbb{Z} \cup \{\infty\} \\ a = \sum_n a_n t^n &\longmapsto \begin{cases} \min\{n \mid a_n \neq 0\} & a \neq 0 \\ \infty & a = 0. \end{cases} \end{aligned}$$

For $(F((t)), v_t)$,

- value group is $\mathbb{Z}$,
- valuation ring is $F[[t]] = \{a \in F((t)) \mid v_t(a) \geq 0\} = \{\sum_{n \geq 0} a_n t^n\}$,
- maximal ideal is $tF[[t]]$,
- residue field is $F[[t]]/tF[[t]] \cong F$.

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$(F((t)), v_t)$ is henselian.

Equicharacteristic henselian nontrivially valued fields

- Language of rings is $\mathfrak{L}_{\text{ring}} = \{+, \cdot, -, 0, 1\}$.
- Language of valued fields is $\mathfrak{L}_{\text{vf}}$, which is three-sorted (field K, value group Γ, residue field k) with valuation map ($v : K \longrightarrow \Gamma$) and residue map ($\text{res} : K \longrightarrow k$).

Theorem (Denef–Schoutens, [DS03])

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Theorem 2 (A.–Fehm, [AF16])

Equal characteristic henselian valued fields satisfy an existential-decidability Ax–Kochen/Ershov principle:

$$\mathrm{Th}_{\exists}(K, v) \text{ decidable} \Leftrightarrow \mathrm{Th}_{\exists}(Kv) \text{ decidable}$$

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$\mathrm{Th}_\exists(\mathbb{F}_q((t)), v_t)$ is decidable — no constant for t!

Sketch proof of Corollary 3

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The tame valued field $\mathbb{F}_q((t^{\mathbb{Q}}))$ has decidable theory, by a theorem of F.-V. Kuhlmann, [Kuh14]. Denote by $\mathbb{F}_q((t))^{\mathbb{Q}}$ the relative algebraic closure of $\mathbb{F}_q((t))$ in $\mathbb{F}_q((t^{\mathbb{Q}}))$. Then $\mathbb{F}_q((t))$ is a directed union of fields $\mathbb{F}_q((s))$, isomorphic copies of $\mathbb{F}_q((t))$. Moreover, $\mathbb{F}_q((t))^{\mathbb{Q}} \preceq \mathbb{F}_q((t^{\mathbb{Q}}))$ by another theorem of F.-V. Kuhlmann, [Kuh14].

$$\begin{tikzcd}
\mathbb{F}_q((t^{\mathbb{Q}})) \arrow[d, dash, "\preceq"] \\
\mathbb{F}_q((t))^{\mathbb{Q}} = \bigcup_s \mathbb{F}_q((s)) \arrow[d, dash, "\preceq_{\exists-\emptyset}"] \\
\mathbb{F}_q((t))
\end{tikzcd}$$

Equicharacteristic henselian nontrivially ... (reprise)

Definition

Let $\mathbf{H}^{e\prime}$ be a recursive axiomatisation of the $\mathfrak{L}_{\mathrm{vf}}$-theory of equicharacteristic henselian nontrivially valued fields. For any theory R of fields, write

$$\mathbf{H}^{e\prime}(R) = \mathbf{H}^{e\prime} \cup \{\text{“}\varphi \text{ holds in residue field”} \mid \varphi \in R\}.$$

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Proposition 4

Let F be any field. Then $\mathbf{H}^{e\prime}(\mathrm{Th}(F))$ is existentially complete. That is, for every $\varphi \in \mathrm{Sent}_{\exists}(\mathfrak{L}_{\mathrm{vf}})$, either $\mathbf{H}^{e\prime}(\mathrm{Th}(F)) \models \varphi$ of $\mathbf{H}^{e\prime}(\mathrm{Th}(F)) \models \neg\varphi$.

Corollary: Theorem 2.

Proposition 5 (Existential Elimination)

There is a recursive map

$$\begin{aligned} \mathrm{Sent}_{\exists}(\mathfrak{L}_{\mathrm{vf}}) &\longrightarrow \mathrm{Sent}_{\exists}(\mathfrak{L}_{\mathrm{ring}}) \\ \varphi &\longmapsto \varphi_{\mathbf{k}} \end{aligned}$$

such that

$$(K, v) \models \varphi \iff Kv \models \varphi_{\mathbf{k}}$$

for every $(K, v) \models \mathbf{H}^{e\prime}$.

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Proof: Proposition 4 + Separation Lemma!

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Theorem 6 ("$\mathbf{H}^{e\prime}$ Theorem")

For any $\mathfrak{L}_{\mathrm{ring}}$-theory R of fields, the set of existential consequences of R is Turing-equivalent to the set of existential sentences in the language of valued fields that hold in every equicharacterstic henselian non-trivially valued field with residue field a model of R. That is:

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Proof: $\varphi \in \mathbf{H}^{e\prime}(R)_\exists \iff \varphi_{\mathbf{k}} \in R_\exists$.

Idea of proof of Theorem 1

Let's briefly recall the statement.

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Step 1

(1) $\leq_T$ (2) $\simeq_T$ (3) $\leq_T$ (4).

Reducts and interpretations

- $\mathrm{Th}_\exists(\mathbb{Q}) \leq_T \mathrm{Th}_\exists(\mathbb{Q}((t)), v_t) \leq_T \mathrm{Th}_\exists(\mathbb{Q}((t)), v_t, t)$
- $\mathrm{Th}_\exists(\mathbb{Q}((t))) \leq_T \mathrm{Th}_\exists(\mathbb{Q}((t)), v_t)$

Thus (1) $\leq_T$ (3) $\leq_T$ (4) and (2) $\leq_T$ (3).

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Thus (1) $\leq_T$ (3) $\leq_T$ (4) and (2) $\leq_T$ (3).

First we note that the valuation ring $\mathbb{Q}[[t]]$ is $\exists$- and $\forall$-$\mathfrak{L}_{\mathrm{ring}}$-definable in $\mathbb{Q}((t))$. For example

$$\mathbb{Q}[[t]] = \left\{ \frac{1}{z^2 - 2} + \frac{1}{1 + u^2 + v^2 + w^2 + x^2} \;\middle|\; u, v, w, x, z \in \mathbb{Q}((t)) \right\},$$

as in [AF17]. Therefore $\mathrm{Th}_\exists(\mathbb{Q}((t))) \simeq_T \mathrm{Th}_\exists(\mathbb{Q}((t)), v_t)$. Thus (2) $\simeq_T$ (3).

$\square$(Step 1)

Theorem 2 already gives (1) $\simeq_T$ (3). Also see [San96, Remark, p. 23].

Idea of proof of Theorem 1

Step 2

(4) $\leq_T$ (1)

In fact we'll simply show (4) $\simeq_T$ (1), i.e. $\mathrm{Th}_\exists(\mathbb{Q}((t)), v_t, t) \simeq_T \mathrm{Th}_\exists(\mathbb{Q})$. We mimic the proof of (3) $\simeq_T$ (1) via Theorem 2. Expand the language with a new constant symbol:

$$\mathfrak{L}_{\mathrm{vf}}(\pi) := \mathfrak{L}_{\mathrm{vf}} \cup \{\pi\}.$$

Consider the $\mathfrak{L}_{\mathrm{vf}}(\pi)$-theory

$$\mathbf{H}^{e,\pi} := \mathbf{H}^{e\prime} \cup \{\forall x\ (0 < v(x) \longrightarrow v(\pi) \leq v(x))\}.$$

Also $\mathbf{H}^{e,\pi}(R) = \mathbf{H}^{e,\pi} \cup \{\text{“}\varphi \text{ holds in residue field”} \mid \varphi \in R\}$.

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Also $\mathbf{H}^{e,\pi}(R) = \mathbf{H}^{e,\pi} \cup \{$"$\varphi$ holds in residue field" $\mid \varphi \in R\}$.

Claim 1

$\mathbf{H}^{e,\pi}(\mathrm{Th}(\mathbb{Q}))$ is existentially complete.

It follows that $\mathbf{H}^{e,\pi}(\mathrm{Th}(\mathbb{Q}))_\exists = \mathrm{Th}_\exists(\mathbb{Q}((t)), v_t, t)$ and $\mathbf{H}^{e,\pi}(\mathrm{Th}(\mathbb{Q}))_\exists \simeq_T \mathrm{Th}_\exists(\mathbb{Q})$.

Proof of Claim 1

Let $(K, v, \pi_v), (L, w, \pi_w) \models \mathbf{H}^{e,\pi}(\mathrm{Th}(\mathbb{Q}))$. Assuming sufficient saturation, there are sections of the residue maps: $Kv \longrightarrow K$ and $Lw \longrightarrow L$; and moreover there is an embedding $Kv \longrightarrow Lw$ between the residue fields. In fact we get this picture:

$$
\begin{tikzcd}
 & & (L, w, \pi_w) \arrow[d, dash] \\
(K, v, \pi_v) & & (Lw((t)), v_t, t) \\
(Kv((s)), v_s, s) \arrow[u, "\preceq_\exists"'] \arrow[urr, dash] & &
\end{tikzcd}
$$

The embedding on the left is existentially closed because $\mathrm{char}(Kv((s))) = 0$.

$\square$(Step 2)

Observation ([AF16])

$\mathrm{Th}_\exists(F((t)), v_t, v) \simeq_T \mathrm{Th}_{\forall^1\exists}(F((t)), v_t)$.

Ingredient 3: large fields

Definition

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Theorem (Pop, [Pop96])

TFAE

1. *L is large*
2. *for each smooth irreducible L-variety V, if $V(L) \neq \emptyset$ then $V(L)$ is Zariski dense in V.*
3. $L \preceq_{\exists} L((t))$
4. $L \preceq_{\exists} L(t)^h$

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Theorem (Pop, [Pop96])

If (L, w) is a henselian nontrivially valued field, then L is large.

Example

1. $\mathbb{C}$, $\mathbb{Q}_{\mathrm{alg}}$, $\mathbb{F}_p^{\mathrm{alg}}$, any algebraically closed field
2. $\mathbb{R}$, $\mathbb{R}_{\mathrm{alg}}$, any real closed field
3. $\mathbb{Q}_p$, $\mathbb{Q}_{p,\mathrm{alg}}$ any p-adically closed field
4. PAC/PRC/PpC/PCC

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Definition

- $\mathbf{L}_{\exists}$ is the common existential $\mathfrak{L}_{\mathrm{ring}}$-theory of large fields
- $\mathbf{L}_{\exists,0}$ (resp. $\mathbf{L}_{\exists,p}$) is ... of characteristic 0 (resp. characteristic p)
- $\mathbf{L}_{\exists,>0}$ is ... of positive characteristic
- $\mathbf{L}_{\exists,\gg 0}$ is ... of sufficiently high positive characteristic.

Idea of proof of Theorem 1

Step 3

(6) $\simeq_T$ (7) $=$ (2).

Claim 2 (Sander, [San96, Propostion 2.25])

$\mathrm{Th}_\exists(\mathbb{Q}((t))) = \mathbf{L}_{\exists,0}$.

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Claim 2 (Sander, [San96, Propostion 2.25])

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Proof of Claim 2

Since $\mathbb{Q}((t))$ is nontrivially valued henselian, it is large. Thus $\mathrm{Th}_\exists(\mathbb{Q}((t))) \models \mathbf{L}_{\exists,0}$.

In the other direction, let L be a large field of characteristic 0. Then $L \preceq_\exists L((t)) \supseteq \mathbb{Q}((t))$. Thus $L \models \mathrm{Th}_\exists(\mathbb{Q}((t))$. Therefore $\mathbf{L}_{\exists,0} \models \mathrm{Th}_\exists(\mathbb{Q}((t)))$.

Thus (2) $=$ (7).

Similarly: $\mathrm{Th}_\exists(\mathbb{F}_p((t))) = \mathbf{L}_{\exists,p}$ for each prime number p.

Claim 3

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Proof of Claim 3

We use the theory **Fin** of finite fields, and the theory $\mathbf{Fin}_{\gg 0}$ of finite fields of sufficiently high characteristic. Both are decidable by Ax.

Building on Claim 2, we have

- $\mathbf{L}_{\exists,>0} = \mathbf{H}^{e'}(\mathbf{Fin}_{\exists})_{\exists} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$
- $\mathbf{L}_{\exists,\gg 0} = \mathbf{H}^{e'}(\mathbf{Fin}_{\exists,\gg 0})_{\exists} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$.

By the $\mathbf{H}^{e'}$ Theorem:

- $\mathbf{L}_{\exists,>0} \leq_T \mathbf{Fin}_{\exists}$
- $\mathbf{L}_{\exists,\gg 0} \leq_T \mathbf{Fin}_{\exists,\gg 0}$.

Claim 4

$\mathbf{L}_{\exists} \simeq_T \mathbf{L}_{\exists,0}$.

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Proof of Claim 4

First: $\mathbf{L}_\exists \models \varphi \iff \mathbf{L}_{\exists,0} \models \varphi$ & $\mathbf{L}_{\exists,>0} \models \varphi$. Thus $\mathbf{L}_\exists \leq_T \mathbf{L}_{\exists,0}$. For the other direction, we know that $\mathbf{L}_{\exists,p} = \mathrm{Th}_\exists(\mathbb{F}_p((t))) \simeq_T \mathrm{Th}_\exists(\mathbb{F}_p)$ is decidable. We outline an algorithm to decide $\mathbf{L}_{\exists,0}$ given an oracle for $\mathbf{L}_\exists$. Let $\varphi \in \mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{ring}})$.

1. Decide whether or not
$$\mathbf{L}_{\exists,\gg 0} \models \varphi.$$
If 'NO' then $\mathbf{L}_{\exists,0} \not\models \varphi$. If 'YES' then continue.

2. For each $n \geq 2$, let $\chi_n \doteq \bigvee_{p \leq n} \underbrace{1 + \ldots + 1}_{p} = 0$. Let $\varphi_n \doteq \varphi \vee \chi_n$. Since $\mathbf{L}_{\exists,\gg 0} \models \varphi$, there is some n such that $\mathbf{L}_{\exists,>0} \models \varphi_n$. Apply decision procedure for $\mathbf{L}_{\exists,>0}$ to φ_n successively until we have found minimal n_0 with $\mathbf{L}_{\exists,>0} \models \varphi_{n_0}$.

3. Decide whether or not
$$\mathbf{L}_\exists \models \varphi_{n_0}.$$
If 'YES' then we have $\mathbf{L}_{\exists,0} \models \varphi_{n_0}$. Thus $\mathbf{L}_{\exists,0} \models \varphi$. Output 'YES' and finish. If 'NO' then $\mathbf{L}_\exists \not\models \varphi_{n_0}$. We already know that $\mathbf{L}_{\exists,>0} \models \varphi_{n_0}$. Thus $\mathbf{L}_{\exists,0} \not\models \varphi_{n_0}$. Output 'NO' and finish.

$\square$(Step 3)

Thank you for listening. Questions are very welcome!

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