CHAPTER 5

Pseudofinite Objects and 0–1 Laws

5.1. 0–1 Laws

5.1.1. Theories of Random Graphs. The set of all graphs with exactly e edges on vertex set $\{0,\dots,n-1\}$ is, of course, finite, and has a uniform measure in which each graph (up to identity) has probability $\frac{1}{\binom{\binom{n}{2}}{N}}$. Of course, when we ask about the probability of a graph up to isomorphism, the situation will be more subtle. At a more basic level, we can ask for the probability that a graph chosen randomly from this distribution is, for instance, connected. In a foundational 1959 paper [**162**], Erdös and Rényi considered this sort of problem in the limit as the number of vertices grows arbitrarily large.

Proposition 5.1.1. *Let c be a constant real number, and*

$$N(n,c) = \left\lfloor \frac{1}{2} n \log n + cn \right\rfloor .$$

Let $P(n,N)$ be the probability that a uniformly randomly chosen graph on vertices $V = \{0,\dots,n-1\}$ with $N(n,c)$ edges is connected. Then

$$\lim_{n\to\infty} P\left(n, N(n,c)\right) = e^{-e^{-2c}}.$$

We conventionally denote the probability space of graphs on n vertices with M edges by $\mathcal{G}(n,M)$. Often we may choose M to be some function of n.

Of course, there are many approaches to random graphs. Instead of considering all graphs with a fixed number of vertices and a fixed number of edges, giving each equal probability, we could instead consider n vertices and make each pair of vertices adjacent independently with probability p. The set of graphs on n vertices, with the probability measure induced by this independent adjacency construction is denoted by $\mathcal{G}(n,p)$. Again, p is often a function of n. For many properties, the two models are interchangeable. Cases of probability 1 and 0 are of special interest.

Theorem 5.1.2. *Let P be a set of graphs on that is closed under isomorphism and such that whenever $F \subseteq G \subseteq H$ are graphs with $F,H \in P$, then we also have $G \in P$. Suppose also that p is a function of n such that $\lim_{n\to\infty} p(1-p)n = \infty$ and that $\lim_{n\to\infty} p(1-p)\binom{n}{2} = \infty$. Let $M(x) = \left\lfloor p\binom{n}{2} + x\sqrt{p(p-1)\binom{n}{2}}\right\rfloor$. Then the following are equivalent:*

1. *Almost every graph in $\mathcal{G}(n,p)$ is an element of P.*
2. *For every fixed x, almost every graph in $\mathcal{G}(n,M)$ is an element of P.*

Proof. We outline here a proof that is given in more detail as part of Theorem 2.2 in [**70**]. The number of edes in a graph in $\mathcal{G}(n,p)$ is a binomial random variable with expected value $p\binom{n}{2}$. On the one hand, if for every fixed x almost every graph in $\mathcal{G}(n, M(x))$ is an element of P, then sufficiently large choices of x would require arbitrarily close agreement between the probability of P in $\mathcal{G}(n,p)$ and in $\mathcal{G}(n,M)$. On the other hand, suppose that P is almost sure in $\mathcal{G}(n,p)$. Then for each x, the probability that an element of $\mathcal{G}(n,p)$ has exactly $M(x)$ edges is positive, so that P must be almost sure in $\mathcal{G}(n, M(x))$. □

For the classical treatment of random graphs in model theory, the almost sure properties are enough. However, something is known about the sets with probability strictly between 0 and 1. A careful analysis of the details of the proof (again, see [**70**]) shows that from sufficient information about the probability in various spaces $\mathcal{G}(n,M)$ we can infer the probability in $\mathcal{G}(n,p)$. The other direction is considerably more fraught; for instance, the set of graphs with an even number of edges will generally have probability $1/2$ in $\mathcal{G}(n,p)$ (as long as n is sufficiently large and p is chosen appropriately).

Another approach, pursued by Erdős and Rényi in [**161**], views the *construction* of a random graph in a more direct way. We consider G_0 to be a discrete graph (no edges) on n vertices. G_{t+1} is formed by choosing, uniformly at random, a pair (a,b) of vertices not adjacent in G_t, and adding an edge between a and b. We denote by $\tilde{\mathcal{G}}$ the set of all sequences of this kind. Note that each element of $\tilde{\mathcal{G}}$ is a sequence of graphs of length $\binom{n}{2}$. Now the map taking each sequence to its Mth element is a measure-preserving map from $\tilde{\mathcal{G}}$ to $\mathcal{G}(n,M)$. In this sense, the construction, or "random graph process" view matches the probability space $\mathcal{G}(n,M)$.

Of course, we may consider random graphs from some collection smaller than all graphs of a particular size. One major derandomization problem for many years was the construction of so-called *expander* graphs. These graphs have applications in error correcting codes, in interactive proofs, and so-called "hardness of approximation" (in which one proves that even approximating a certain calculation to a specified precision has high complexity), as well as the derandomization of other algorithms.

To define this class of graphs, we give some preliminary definitions, which will be useful in some other results on random graphs.

DEFINITION 5.1.3. Let G be a d-regular graph on n vertices.

(1) Let A_G be the matrix with entries

$$[A_G]_{ij} = \begin{cases} \frac{1}{d} & \text{if } iEj \\ 0 & \text{otherwise} \end{cases} .$$

(2) For any vector $\mathbf{v}$, let $\| \mathbf{v} \|_2 = \sqrt{\sum_{i=1}^{n} v_i^2}$.

(3) For any vector $\mathbf{v}$, let $\mathbf{v}^\perp$ be the set of all vectors orthogonal to $\mathbf{v}$.

(4) Let $\mathbf{1}_n$ denote the vector $(1/n, 1/n, \ldots, 1/n)$.

(5) Let $\lambda(G)$ denote

$$\max_{\mathbf{v} \in \mathbf{1}^\perp, \|\mathbf{v}\|_2 = 1} \| A\mathbf{v} \|_2 .$$

We now proceed to give two equivalent definitions of expander graphs. We give the following result without proof.

THEOREM 5.1.4. *The following conditions on a finite, d-regular graph $G = (V,E)$, with $|V| = n$ are equivalent:*

(1) *Every subset* $T \subseteq V$ *with* $|T| < \frac{|V|}{2}$ *has the property that the set of edges from* T *to the complement of* T *has size* $\Omega(|T|)$.
(2) *The graph* G *has the property that* $\lambda(G) \leq 1 - \epsilon$ *for some constant* $\epsilon > 0$.

We now define an *expander* graph to be a graph satisfying the conditions of Theorem 5.1.4. Explicit constructions for expander graphs (especially graphs where ϵ is explicitly known) have been given, but were at first elusive ([**28**] gives some detail on this history). Quite early in the study of these objects, however, a probabilistic construction was made.

The idea of the so-called *probabilistic method* is that one proves existence of an object (e.g. a graph) with a certain property P by proving that a random object has property P with some positive probability. This was the original motivation for the work of Erdős and Rényi.

The original probabilistic proof of the existence of expanders was in [**368**]. Pinsker carried this out by bounding from above the number of non-expander graphs, showing that some d-regular graphs on n vertices remain, which must be expanders.

Consider a randomized algorithm with error probability p that uses m random bits. Suppose that we have an explicitly constructed d-regular expander graph with 2^m vertices which we can identify with the strings in $\{0,1\}^m$, and such that $\lambda(G) < 1/10$. Then take a random walk of length k on G starting with a randomly selected vertex v_1 and at each stage v_i choosing stage v_{i+1} uniformly at random from the neighbors of v_i. Then in a randomized algorithm, we can use $v_1, \ldots, v_k$ as the random bits for each of k computations with the algorithm, and output a majority vote. Since G was an expander with $\lambda(G) < 1/10$, one can prove that the error probability of this enhanced algorithm is similar to that achieved by using mk random bits, but requires only $m + O(k)$ random bits. A detailed explanation of expander graphs and their uses in randomized computation, error-correcting codes, and interactive proofs is given in [**28**].

In modern use, random graphs are often used to model the development of informational, social, biochemical, or other networks. In these situations, uniform selection of random graphs from all graphs of the same size (even with some extra stated properties) is not always a good model for the application. One alternative, frequently seen in social and informational networks, is the *preferential attachment* model. In this model, we take G_0 to be some starting graph (often a single vertex with a self-loop. We also fix at the outset a probability p. Then at each stage s we act (independently of all other steps) as follows to produce G_{s+1}:

(1) With probability p, we will add a new vertex v, and add an edge from v to some existing vertex u, where u is chosen at random, with probability proportional to its degree in G_s.
(2) Otherwise, we add a new edge between two vertices $u, v \in G_s$ which are independently chosen at random with probability proportional to their degree in G_s.

This model is of some antiquity. It is often attributed to Simon [**410**], but I have been unable to find it there. It seems to have come to its modern prominence after its use in a 1999 paper by Barabási and Albert [**41**] to model the world wide web.

One of the more conspicuous features of the preferential attachment random network is the distinctive degree distribution.

THEOREM 5.1.5. *Let $(G_t)_{t\in\mathbb{N}}$ be a preferential attachment random graph process, and let $m_{k,t}$ be the number of vertices in G_t with degree k. Then for each k, the limit $\lim_{t\to\infty} E(m_{k,t})$ exists, and this limit is proportional to*

$$M_k = k^{-\left(2+\frac{p}{2-p}\right)}.$$

By contrast, the degree distribution of Erdős-Rényi random graphs is generally Poisson distributed.

Again, one may find a particular finite random graph by stopping the process at some particular finite stage, although we should note that the natural mappings to the Erdős-Rényi models are emphatically not measure-preserving.

Another model of random graph development has been suggested to model biological networks, whose degree distribution is also a power law distribution, like preferential attachment. An important motivation for this model is that many biochemical networks tend to have degree distributions proportional to $M_k = k^\beta$ with $1 < \beta < 2$, while preferential attachment gives $2 \leq \beta \leq 3$.

We start with some initial graph G_0, often taken to be a discrete graph of a single vertex. We also fix a probability p. Then at each stage s, we act (independently of all other stages) as follows to produce G_{s+1}: We select a vertex $u \in G_s$ uniformly at random. We introduce a new vertex v, and set it adjacent to u. Then, for each neighbor w of u, we make v adjacent to w independently with probability p.

This *duplication model* was introduced by Chung and others [**112**], and also has a power law degree distribution, with values of β more consistent with biochemical observation.

An extensive collection of results on preferential attachment, duplication, and other complex graph process models can be found in [**111**].

5.1.2. The Almost Sure Theory. From a model-theoretic perspective, the most important fact about random graphs is that, under the right circumstances, first-order sentences are either almost surely true or almost surely false, and the set of almost surely true sentences forms a complete consistent theory.

THEOREM 5.1.6 (Fagin [**165**], Glebskii–Kogan–Liogonkii–Talanov [**200**]). *There is a complete consistent theory T such that T consists of exactly the sentences in the language of graphs that are almost always true in $G(n, \frac{1}{2})$.*

PROOF. For each natural number m, consider a set of variables $x_1, \ldots, x_m$, and for each map $\sigma : \{1, \ldots, m\}^2 \to \{0, 1\}$, set $D_{m,\sigma}(x_1, \ldots, x_m)$ to be the conjunction of $E(x_i, x_j)$ for each (i, j) where $\sigma(i, j) = 0$ and $\neg E(x_i, x_j)$ each (i, j) where $\sigma(i, j) = 1$.

We then compose, for each m, for each σ, and for each $\tau : \{1, \ldots, m + 1\} \to \{0, 1\}$ extending σ, the sentence

$$\varphi_{m,\sigma,\tau} = \forall x_1, \ldots, x_m \left(\left(\left(\left(\bigwedge_{i\neq j} x_i \neq x_j\right) \wedge D_{m,\sigma}(x_1, \ldots, x_m) \to \right.\right.\right.$$
$$\left.\left.\left. \to \exists y \left(\left(\bigwedge_{i=1}^{m} y \neq x_i\right) \wedge D_{m,\tau}(x_1, \ldots x_m, y)\right)\right)\right)\right).$$

Now we define T to be the closure under derivation of the set

$$\{\varphi_{m,\sigma,\tau} : m \in \mathbb{N},\, \tau \supseteq \sigma\}\,.$$

LEMMA 5.1.7. *For each m and each $\tau \supseteq \sigma$, the sentence $\varphi_{m,\sigma,\tau}$ is almost surely true in $\mathcal{G}\left(n, \frac{1}{2}\right)$.*

PROOF. For any sentence ψ, let $P_n(\psi)$ denote the probability of σ in $\mathcal{G}(n,m)$. Now we calculate $P_n(\neg\varphi_{m,\sigma,\tau})$ This quantity matches $P_n(\theta)$, where

$$\theta = \exists x_1, \ldots, x_m \left(\left(\bigwedge_{i \neq j} x_i \neq x_j \right) \wedge D_{m,\sigma}(x_1, \ldots, x_m) \right) \wedge$$
$$\wedge \forall y \left(\left(\bigwedge_{i=1}^{m} y \neq x_i \right) \wedge \neg D_{m,\tau}(x_1, \ldots x_m, y) \right).$$

We set

$$\tilde{\theta}(\bar{x}) = D_{m,\sigma}(x_1, \ldots, x_m) \wedge \forall y \left(\left(\bigwedge_{i=1}^{m} y \neq x_i \right) \wedge \neg D_{m,\tau}(x_1, \ldots x_m, y) \right).$$

Now $P_n(\theta)$ is bounded by the sum of $P_n(\tilde{\theta}(\bar{a}))$, where $\bar{a}$ ranges over all possible substitutions of distinct elements. There are $\frac{n}{m}$ such substitutions, each of which (independently) has probability $1 - \frac{1}{2^{2m+1}}$. This bound approaches zero as n increases. Consequently, $P_n(\varphi_{m,\sigma,\tau})$ approaches 1. □

We now show that this set of axioms is complete and consistent.

LEMMA 5.1.8. *T is a complete consistent theory.*

PROOF. Toward consistency, let S be a finite subset of T. Then let U_S be the conjunction of all elements of S, and we will show that this sentence has a model. Since the sentence U_S is true in almost all finite graphs, there is some finite graph in which it is true. By compactness, then, T is consistent.

Toward completeness, we first note that T has no finite models, for if T had a model of size n, it would not satisfy any of the sentences $\varphi_{n,\sigma,\tau}$, since they all imply the existence of at least $n+1$ distinct elements. We can prove that T is $\aleph_0$-categorical by a back-and-forth argument. Indeed, let G_1, G_2 be countable models of T. We construct an isomorphism as follows. Let f_{-1} be the empty function. At stage $2s$, we find a new element x_{2s} of G_1, and find σ and τ such that $D_{s,\sigma}$ is the atomic diagram of $dom(f_{2s})$ and $D_{s+1,\tau}$ is the atomic diagram of $dom(f_{2s}) \cup \{x_s\}$. Then to form f_{2s+1}, note that $G_2 \models \varphi_{s,\sigma,\tau}$, so there exists some y such that $f_{2s} \cup (x_{2s}, y_{2s})$ is a partial isomorphism from $G_1 \to G_2$. Similarly, at stage $2s+1$, we start with an element of G_2 and extend. In the limit, we achieve an isomorphism from G_1 to G_2.

Now by the Łoś-Vaught test, since T has no finite models and is $\aleph_0$-categorical, we conclude that it is complete. □

This concludes the proof of the theorem. □

COROLLARY 5.1.9. *Every first-order sentence in the language of graphs is either almost surely true or almost surely false in finite graphs.*

PROOF. Let φ be a first order sentence. Either $T \vdash \varphi$ or $T \vdash \neg\varphi$. If $T \vdash \varphi$, then all of the axioms necessary to prove φ are true in almost all finite graphs, so that φ must be true in this set of graphs. If $T \vdash \neg\varphi$, then $\neg\varphi$ must be true in almost all finite graphs, so that φ must be false in almost all finite graphs. □

Indeed, this theorem, referred to as the 0–1 law, generalizes in almost every direction. To begin, the original proofs show that the result holds for finite structures in any finite relational language. It is not hard to modify the proof to replace the constant fraction $\frac{1}{2}$ with any other $p \in (0, 1)$. Another natural generalization is to cases where p is not a constant probability, but a more general function of n, a very common setting in random graphs. Shelah and Spencer took up this question in [**404**]. Of particular interst are the so-called *sparse* situations, where $\lim_{n\to\infty} p(n) = 0$ and (the term is often used vaguely) usually this convergence is fast.

We write $P_{n,p(n)}(\varphi)$ for the probability that φ is satisfied in $\mathcal{G}(n, p(n))$.

THEOREM 5.1.10 (Shelah-Spencer). *Suppose that one of the following holds:*

(1) *For all* $\epsilon > 0$*, we have* $\lim_{n\to\infty} n^\epsilon p(n) = 0$
(2) $\lim_{n\to\infty} n^2 p(n) = 0$
(3) *For some integer* k*, both* $\lim_{n\to\infty} n^{1+1/(k-1)} p(n) = 0$ *and* $\lim_{n\to\infty} n^{1+1/k} p(n) = \infty$
(4) *For all* $\epsilon > 0$*, both* $\lim_{n\to\infty} np(n) = 0$ *and* $\lim_{n\to\infty} n^{1+\epsilon} p(n) = \infty$
(5) *Both* $\lim_{n\to\infty} \frac{np(n)}{\log n} = 0$ *and* $\lim_{n\to\infty} pn = \infty$
(6) *For all* $\epsilon > 0$*, both* $\lim_{n\to\infty} \frac{(\log n)(p(n))}{n} = 0$ *and* $\lim_{n\to\infty} n^{1-\epsilon} p(n) = 0$.

Then for each first order sentence φ *in the language of graphs,* $\lim_{n\to\infty} P_{n,p(n)}(\varphi)$ *exists and is either 0 or 1.*

PROOF. The proof outline is, in each case, similar to Fagin's. Case 1 can be proved exactly like Fagin's theorem, except that the quantity at the end of the proof whose convergence to zero must be checked is a different one. In Case 2, we axiomatize the theory of a discrete graph (one with no edges) and show that it is complete and satisfied with probability 1. The proof for Case 3 is similar, with a more involved axiomatization.

The remaining cases are more subtle, in that the resulting theories are not $\aleph_0$-categorical. It is possible, however, in each case, to give an almost-sure complete axiomatization consisting of certain required and forbidden configurations. □

An additional result of Shelah and Spencer, from the same paper, has been the subject of considerable subsequent interest.

THEOREM 5.1.11 (Shelah-Spencer). *Let* α *be an irrational number. Then for each first order sentence* φ *in the language of graphs,* $\lim_{n\to\infty} P_{n,n^{-\alpha}}(\varphi)$ *exists and is either 0 or 1.*

Again, the proof is by formulating axioms that are almost surely true in the appropriate space of structures, and then proving that they constitute a complete consistent theory. A few axiomatizations have been given for this theory. Baldwin and Shelah [**39**] point out the similarity of the theory to Hrushovski's construction of an $\aleph_0$-categorical strictly stable pseudoplane. The one given here, due to Laskowski

[**297**], has several advantages. It is relatively simple, it is an $\forall\exists$ theory, and it allows a clarification of several model-theoretic properties of the theory.

Proof. Let α be an irrational number. For each finite graph $\mathcal{G} = (V, E)$, we set $\delta(\mathcal{G}) = |V| - \alpha|E|$. Now we define the class $\overline{K}_\alpha$ to be the set of all graphs (finite or not) $\mathcal{G}$ such that for each finite (complete) subgraph $\mathcal{H} \subseteq \mathcal{G}$ we have $\delta(\mathcal{H}) \geq 0$, and we set K_α to be the set of finite graphs in $\overline{K}_\alpha$. We note that membership in $\overline{K}_\alpha$ is elementary since, for each n, we may write that for each n-tuple, every relation of the form "there are at most m edges," with $m \leq \frac{n}{\alpha}$, holds.

We now arrive at our axiomatization. We take sentences guaranteeing membership in $\overline{K}_\alpha$, and guaranteeing that for every pair of finite graphs $\mathcal{G}, \mathcal{H}$ and every embedding $f : \mathcal{G} \to \mathcal{M}$ in a model $\mathcal{M}$, if $\mathcal{G}$ is a complete subgraph of $\mathcal{H}$ with $\delta(\mathcal{H}) - \delta(\mathcal{G}) \geq 0$ and both are in $\overline{K}_\alpha$, then f extends to an embedding of $\mathcal{H}$ in $\mathcal{M}$.

To prove that this is a complete theory, we prove that every formula is provably equivalent to one of a specified form. An extension formula is a formula $\psi_{\mathcal{G},\mathcal{H}}$, where $\mathcal{G} \subseteq \mathcal{H}$ is a complete subgraph, with $\mathcal{G} \in K_\alpha$, of the form $D_{\mathcal{G}}(\bar{x}) \wedge \exists \bar{y}\, D_{\mathcal{H}}(\bar{x}\bar{y})$, where $D_{\mathcal{G}}, D_{\mathcal{H}}$ are the atomic diagrams of $\mathcal{G}, \mathcal{H}$, respectively. It turns out that for any formula φ, the theory T_α proves that φ is equivalent to some boolean combination of extension formulas. Now since the empty structure is contained in every element of K_α, the theory T_α decides every extension formula with no free variables, so that T_α is complete. It follows from the original work of Shelah and Spencer (using Chebyshev's Inequality) that each of the axioms of T_α holds almost surely. $\square$

Interestingly, Laskowski proves that the theory T_α is stable, but not superstable. By contrast, the Fagin theory for constant-proportion random graphs is not only unstable, but even has the independence property (see Section 5.3.2). Brody and Laskowski [**84**] show that certain theories closely related to T_α interpret a sufficient fragment of arithmetic to be essentially undecidable and others are $\aleph_0$-stable. It would be interesting to know for what funtions $p(n)$ the almost sure theory of $\mathcal{G}(n, p(n))$ is stable, or has NIP. Of course, not every such structure even has an almost sure theory (there are functions for which the zero-one law fails).

It is important to note than in each of these cases, the zero-one law is established by giving a complete effective axiomatization of the almost sure theory, so that the almost sure theory is decidable. It is not, *a priori*, obvious even that the almost surely true and almost surely false sentences should be computably separable, and in cases where the zero-one law fails (the example of Brody and Laskowski, for instance), it is sometimes known that they are not computably separable.

There are some negative results on zero-one laws, but less than a complete characterization. Shelah and Spencer had one case.

Theorem 5.1.12 (Shelah-Spencer). *Let $q(n)$ be such that $n^{1/\log_5 n} < q(n) < \frac{\log n}{\log_5 n}$, and let $p(n) = n^{-1/7}\left(q(n)\right)^{1/7}$. Then there is a first-order sentence φ such that $\left(P_{n,p(n)}(\varphi) : n \in \mathbb{N}\right)$ has no limit*

Proof. We begin by some preliminary formulas toward the goal of defining φ. For all of these preliminary formulas, we will have global variables $x_1, \ldots, x_7, y, y_1, \ldots, y_7$. We say that S holds of a 7-tuple when the elements of the 7-tuple are distinct, and there is no other vertex adjacent to all of them. We say that $N(z, x_1, x_2, \ldots, x_7)$ holds when all of the variables are distinct and z is adjacent to all of the x_i.

Continuing, we let $\Sigma(x_1, \ldots, x_7, y, y_1, \ldots, y_6)$ be the assertion that

(1) For all z_1 for which $N(z_1, \bar{x})$ holds, there exists a unique tuple $(z_2, \ldots, z_6$ of distinct vertices such that $S(y, z_1, \ldots, z_6)$ holds,
(2) For all z_1 for which $N(z_1, \bar{x})$ holds, there is a unique i such that there exist distinct $z_2, \ldots, z_6$ with $S(y_i, z_1, \ldots, z_6)$, and
(3) If $S(y, z_1, \ldots, z_6)$ for distinct z_j and i_j is the i guaranteed by item 2 for z_j in the place of z_1, then all of the i_j are distinct.

This Σ will be a conjunct, and so we will freely make reference to the i guaranteed by Σ for a particular z_1.

We will now write an interpretation of arithmetic. For the universe of the interpretation of arithmetic, we take Z to be the set of q such that $N(q, \bar{x})$ holds and such that the i guaranteed by Σ is 1. Further, for each $q \in Z$, we set $q^{(1)} = q$ and $q^{(i)}$ to be the unique element r such that there are distinct $z_3, \ldots, z_6$ such that $S(y, q, r, z_3, \ldots, z_6)$. We then let $\Pi(\bar{x}, y, \bar{y})$ be the statement that there is a labeling $1, 2, \ldots, s$ of Z such that $z_1 + z_2 = z_3$ and $z_4 z_5 = z_6$ if and only if $S(x, z_1^{(1)}, z_2^{(2)}, \ldots, z_6^{(6)})$.

We aim for our eventual sentence to describe a "largest" possible arithmetic, so we introduce a formula to compare the size of two definable sets that could potentially play the role of Z in the formula Π. For each pair $(\bar{x}, y, \bar{y}), (\bar{x'}, y', \bar{y'})$, we write Γ for the statement that there exist distinct vertices $v_3, \ldots, v_7$ such that $f_{\bar{v}} := \{(x, f(x)) : S(x, f(x), \bar{v})\}$ is an injection from the Z defined from $(\bar{x}, y, \bar{y})$ to the Z defined from $(\bar{x'}, y', \bar{y'})$, but not a bijection. We can then formulate $M(\bar{x}, y, \bar{y})$ as the statement that there does not exist a tuple $(\bar{x'}, y', \bar{y'})$ such that $\Sigma \wedge \Pi \wedge \Gamma$ holds.

Within this arithmetic, we will formulate additional properties. The iterated logarithm of n, denoted $\log^* n$, is the least k such that a k-fold composition of log evaluated on n is less than or equal to 1. On the other hand, the tower function $t(n)$ is defined inductively by $t(1) = 2$ and $t(n+1) = 2^{t(n)}$. We can then define $L(x, \bar{x}, y, \bar{y}) = \exists k\, [t(x) = k] \wedge \forall \ell\, [t(x+1) \neq \ell]$ where x ranges over Z and the arithmetic is that defined on Z. We now define the final conjunct of φ by setting $\Lambda(\bar{x}, y, \bar{y})$ denote

$$\forall(x) \left[L(x, \bar{x}, y, \bar{y}) \to \exists q \in Z \left[\bigvee_{i=1}^{50} (x = 100q + i) \right] \right].$$

Now the desired sentence φ is given by

$$\varphi = \exists \bar{x}, y, \bar{y}\, [\Sigma \wedge \Pi \wedge M \wedge \Lambda]\,.$$

The failure of the zero-one law for this sentence can be calculated by observing its behavior on a certain almost sure set of graphs. Indeed, on one such set, φ holds of all graphs on n vertices where $\log^* n \equiv 25 \bmod 100$ and fails on all graphs on n vertices where $\log^* n \equiv 50 \bmod 100$. Thus, the sequence $\left(P_{n,p(n)}(\varphi) : n \in \mathbb{N}\right)$ has subsequences converging to each of 0 and 1. □

The question of which functions $p(n)$ satisfy a zero-one law is an important one. Łuczak and Spencer [**310**] made several additional characterizations, as well as the following interesting observation:

THEOREM 5.1.13 (Łuczak–Spencer). *There is no recursive function p such that $p(n) < n^{-1/7}$ and $p(n) = n^{-\frac{1}{7}+o(1)}$ where $\mathcal{G}(n, p(n))$ satisfies the zero-one law.*

Shelah and Spencer pose the problem of characterizing the *threshold spectra* of first-order sentences. It is common as early as the 1960 work of Erdős and Rényi on the evolution of random graphs [**161**] to identify $p(n)$ as a *threshold function* for some property φ if and only if for all functions $\tilde{p}(n)$, we have this dichotomy:

- If $\lim\limits_{n\to\infty} \frac{\tilde{p}(n)}{p(n)} = 1$ then φ is almost sure, and
- If $\lim\limits_{n\to\infty} \frac{\tilde{p}(n)}{p(n)} = 1$ then $\neg\varphi$ is almost sure.

The spectrum of a sentence φ is the set of all $a > 0$ such that for all $\epsilon > 0$ there is no $v \in \{0,1\}$ such that for all $p(n)$ with $n^{a-\epsilon} < p(n) < n^{a+\epsilon}$ we have $\lim\limits_{n\to\infty} P_{n,p(n)} = v$. Given the positive results of Shelah and Spencer on zero-one laws, the points of the spectrum of φ represent discontinuities in $\lim\limits_{n\to\infty} P_{n,p(n)}$ as it depends on $p(n)$. The initial Shelah-Spencer paper [**404**] gave some initial properties of possible spectra of setences, and Spencer used Ehrenfeucht-Fraïssé games [**415**] and other techniques [**414**], and a full characterization, even in restricted cases, is still an area of active research [**466, 418, 330**].

Another direction for zero-one laws is the logic from which the formulas come. It would certainly be too much to suggest that every second order sentence was either almost surely true or almost surely false even on $\mathcal{G}(n, \frac{1}{2})$. Indeed, Compton [**114**] uses the classification of Shelah and Spencer to identify cases where even a monadic second order zero-one law fails. However, Compton sumarizes several other results showing that zero-one laws hold for TC and LFP logics, both important benchmarks for the descriptive complexity theory described in Section 4.1.3.

THEOREM 5.1.14 (Kolaitis-Vardi [**285**]). *Let L be the set of Σ^1_1 sentences φ in the language of graphs such that φ is of the form*

$$\exists \mathbf{S} \exists x_1, \ldots, x_k \forall y \exists z_1, \ldots, z_k \; R(\bar{x}, y, \bar{Z}, S)$$

where R is quantifier free in the language of graphs with a unary predicate for S. Then each formula of L is either almost surely true or almost surely false on $\mathcal{G}\left(n, \frac{1}{2}\right)$.

To prove this theorem, we adopt a slightly different perspective on Fagin's result. From the Kolaitis-Vardi perspective, the important point was not the construction of a *theory* of almost sure sentences, but the construction of a *structure* (which they termed the *random structure*) whose theory was the almost sure theory. This structure should be the unique countable model of an appropriate set of "extension axioms." As it turns out, the first-order extension axioms of Fagin suffice, so that the unique countable model of those axioms has the property that it satisfies, among φ of the form given in the theorem, exactly those φ which are almost sure. Similar reasoning also allowed Kolaitis and Vardi to prove the following theorem.

THEOREM 5.1.15 (Kolaitis-Vardi [**286**]). *Let $\varphi \in L_{\infty\omega}$ be an infinitary sentence in which a finite number of variables occur. Then φ is either almost surely true or almost surely false on $\mathcal{G}\left(n, \frac{1}{2}\right)$.*

5.2. Fraïssé Limits

5.2.1. Fraïssé's Theorem. Kolaitis and Vardi were perhaps not the first to refer to the unique countable model of the almost sure theory as "the random structure." However, this terminology makes an important link between the random behavior of finite structures and a standard construction technique. Before the emergence of zero-one laws as a standard area of research, Fraïssé [**176**] proved the following.

THEOREM 5.2.1 (Fraïssé). *Let L be a countable language, and K be a set, at most countable, of finite L-structures with the following properties:*

(1) *If $\mathcal{M}_1 \in K$ and $\mathcal{M}_2$ is a substructure of $\mathcal{M}_1$, then $\mathcal{M}_2$ has an isomorphic copy in K,*
(2) *If $\mathcal{M}_1, \mathcal{M}_2 \in K$, then there is some $\mathcal{M}_3 \in K$ such that each of $\mathcal{M}_1$ and $\mathcal{M}_2$ embeds in $\mathcal{M}_3$, (the* Joint Embedding Property*), and*
(3) *If $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_3 \in K$, with embeddings $e_i : \mathcal{M}_0 \to \mathcal{M}_i$, then there exists some $\mathcal{M}_* \in K$ with embeddings $f_i : \mathcal{M}_i \to \mathcal{M}_*$ such that $f_1 \circ e_2 = f_2 \circ e_2$ (the* Amalgamation Property*).*

Then there is a unique L-structure $\mathcal{A}$ of cardinality at most ω such that

(1) *K is the class of all finitely generated structures that can be embedded in $\mathcal{A}$ (we say that K is the* age *of $\mathcal{A}$), and*
(2) *Every isomorphism between finitely generated substructures of $\mathcal{A}$ extends to an automorphism of $\mathcal{A}$ (we say that $\mathcal{A}$ is ultrahomogeneous).*

PROOF. Suppose that K satisfies the hypotheses of the theorem. Let $A_0 \in K$. At stage $s+1$, we set A_{s+1} to be an element of K (whose existence is guaranteed by condition 3) such that for all (f, B, C) where $B \subseteq C$ and $f : B \to D_k$ is an embedding, the function f extends to an embedding of C in D_{s+1}. We take $\mathcal{A} = \bigcup_{s \in \mathbb{N}} A_k$.

Note first that every finitely generated substructue of $\mathcal{A}$ is in K, by condition 1. Suppose that $B \in K$, and take $C \supseteq B$ in which A_0 embeds. Then by the construction of A_{s+1}, there is some s such that C embeds in D_s, so that B is a substructure of $\mathcal{A}$ and K is the class of all finitely gnerated structures that can be embedded in $\mathcal{A}$, as required.

Toward the homogeneity condition, notice that any embedding of a finite substructure B in $\mathcal{A}$ extends along any inclusion $B \subseteq C$. Now ultrahomogeneity follows by a back-and-forth argument.

It remains to show uniqueness. Ultrahomogeneity implies that any embedding of a finite substructure B in $\mathcal{A}$ extends along any inclusion $B \subseteq C$. This permits a back-and-forth argument showing that any two countable ultrahomogeneous structures with the same age must be isomorphic. □

We call the structure $\mathcal{A}$ whose existence is guaranteed by Fraïssé's theorem the *Fraïssé limit of K*. As an important example, consider the random graph. Let K be the class of all finite graphs. Clearly all of the hypotheses are satisfied. We claim that the unique countable model $\mathcal{A}$ of the almost sure theory of $\mathcal{G}\left(n, \frac{1}{2}\right)$ is the Fraïssé limit of K. Clearly K is the age of $\mathcal{A}$. It remains only to prove that $\mathcal{A}$ is ultrahomogeneous.

PROPOSITION 5.2.2. *The unique countable random graph $\mathcal{A}$ is ultrahomogeneous.*

PROOF. Note that a finitely generated sbustructure is, in this case, just finite. Let $\bar{a}$ and $\bar{b}$ be two finite tuples of vertices of $\mathcal{A}$ such that $f_0 : \bar{a} \to \bar{b}$ is an isomorphism. At stage $2s+1$, we let x be the least vertex of $\mathcal{A}$ which is not in the domain of f_{2s}. The axioms of the almost sure theory of $\mathcal{G}\left(n, \frac{1}{2}\right)$ guarantees that there is some vertex y of $\mathcal{A}$ such that the atomic diagram of $(\bar{a}, x)$ exactly matches that of $(\bar{b}, y)$. We then set $f_{2s+1} = f_{2s} \cup \{(x, y)\}$. At stage $2s+2$, we act symmetrically, extending so that the range of f_{2s+2} contains the first new element. Now $f := \bigcup_s f_s$ is an isomorphism from $(\mathcal{A}, \bar{a})$ to $(\mathcal{A}, \bar{b})$. □

It is not obvious that the Fraïssé limit should always satisfy the almost sure theory of a class — if such a theory even exists, which may itself be uncertain depending on what measures are chosen on the age and what structures are included in the age. A notorious challenge to this interpretation of the Fraïssé limit as a random structure was posed by the so-called Henson graph, the Fraïssé limit of the class of finite triangle-free graphs.

Lemma 5.2.3. *The class of finite triangle-free graphs has a Fraïssé limit.*

Proof. Any subgraph of a triangle free graph is itself triangle free. Similarly, the disjoint union of two triangle-free graphs is triangle-free. The principal challenge, then, is amalgamation. Let G_0, G_1, G_2 be finite triangle-free graphs with embeddings $e_i : G_0 \to G_i$. Let G_* be defined as follows. The vertices of G_* are the equivalence classes of vertices of $G_1 \uplus G_2$ under the relation $x \sim y$ if and only if $x = e_1(v)$ and $y = e_2(v)$ for some fixed $v \in G_0$. If x, y are two vertices of G_*, then we put an edge from x to y if and only if there are alements of x and y joined by an edge in $G_1 \uplus G_2$.

We first observe that the natural embeddings $f_i : G_i \to G_*$ are graph homomorphisms. Indeed, if x_1 is adjacent to x_2 in G_i, then it is obvious that $f_i(x_1)$ and $f_i(x_2)$ will be adjacent. For the converse, suppose that $f_i(x_1)$ and $f_i(x_2)$ are adjacent in G_*. Then there are x_1', x_2' such that $x_j \sim x_j'$ and x_1' is adjacent to x_2' in $G_1 \uplus G_2$. To be adjacent, they must be in the same G_1, so that their preimages are adjacent.

The property that $f_1 \circ e_2 = f_2 \circ e_2$ follows from the definition of G_*. □

We call this limit the Henson graph.

Proposition 5.2.4. *The Henson graph is not a model of the almost sure theory of triangle-free graphs in the uniform measure.*

Proof. The bipartite graphs have measure 1 among the triangle-free graphs [**159**]. However, bipartite graphs are 2-colorable and the Henson graph has infinite chromatic number [**226**]. Since "bipartite" is a first-order property, a sentence asserting it must be provable in the almost-sure theory. □

However, the uniform measure is not the only possible measure. A more recent result [**365**] showed that the Henson graph *is* a model of the almost sure theory in some invariant measure.

Theorem 5.2.5. *There is an invariant probability measure on the class of countable triangle-free graphs such that the isomorphism type of the Henson graph has measure 1.*

The proof of this will be deferred until the discussion in Section 6.1.2 of the much more general results in this same direction of Ackerman, Freer, and Patel. For now it suffices that Fraïssé's theorem does guarantee the existence, under some circumstances, of structures that are universal in a certain sense and that are random — sometimes in a way that is not intuitive at all.

If Fraïssé limits are interesting, it is helpful to know their effectiveness properties. One thinks quickly of the expander graphs, whose existence was proved in the abstract long before there was any concrete construction, when the concrete construction is exactly what is most needed. While the complexity-theoretic structure seems largely unexplored, something is known about the computability-theoretic content.

We say that a set K of finitely generated structures of the same language is a *computable age* if and only if $K = ((\mathcal{A}_i, \bar{a}_i) : i \in \mathbb{N})$ is uniformly computable, with $\mathcal{A}_i$ generated by $\bar{a}_i$. We say that such a system K has the *computable extension property* if there is a partial computable function which, given $i \in \mathbb{N}$ and a quantifier free formula $\theta(\bar{a}, \bar{x})$, behaves as follows:

- Returns a triple $(j, e, \bar{b})$, where $\varphi_e : \mathcal{A}_i \to \mathcal{A}_j$ is an embedding, and $\mathcal{A}_j \models \theta\left(\varphi_e\left(\bar{a}\right), \bar{b}\right)$ if such a triple exists, and
- Otherwise does not halt.

THEOREM 5.2.6 ([**126**]). *Let K be a computable age that satisfies the amalgamation property. Then K has a computable Fraïssé limit if and only if K has a computable representation which has the computable extension property.*

There are two fundamental questions about the effectiveness of Fraïssé's theorem: the existence and the uniqueness. Theorem 5.2.6 addresses the existence side: under certain circumstances, there will be a computable structure which is (isomorphic to) the Fraïssé limit. Fraïssé's theorem also posits the *uniqueness* of the limit, up to isomorphism. The best possible result, from a computable standpoint, is that any two isomorphic copies of the Fraïssé limit must be isomorphic by a *computable* isomorphism (we say that the structure is *computably isomorphic.* This is a little stronger than the truth. However, the following result says that any two copies must be isomorphic by a Δ^0_2 isomorphism.

THEOREM 5.2.7 ([**169**]). *Let $\mathcal{A}$ be a computable structure which is a Fraïssé limit. Then $\mathcal{A}$ is relatively Δ^0_2-categorical.*

PROOF. Given $\mathcal{B} \cong \mathcal{B}$, the isomorphism can be constructed by a back-and-forth argument, which depends only on determinations of whether, for given $\bar{a}$ and $\bar{b}$, whether there is an isomorphism from the structure generated by $\bar{a}$ to that generated by $\bar{b}$ that takes a_i to b_i. This can be determined by checking all atomic formulas, possible with a $\mathcal{B}'$ oracle. □

5.2.2. Ehrenfeucht-Fraïssé Games and 0–1 Laws. A somewhat different view of the proof of Theorem 5.1.6 is given by Ehrenfeucht-Fraïssé games. Again, for concreteness, we carry out the analysis on graphs, although the results are valid much more broadly.

Consider two finite graphs, G_1 and G_2. the Ehrenfeucht-Fraïssé game of length k on G_1, G_2, denoted $EF_k(G_1, G_2)$, is played by two players, E and A, over k rounds. In round i, player A first chooses either G_1 or G_2, then chooses one vertex of it to label i. Then player E chooses one vertex of the other graph to label i. The vertex labeled i in G_1 is called x_i and the one in G_2 is called y_i.

Player E wins if for all $i, j < k$, we have x_i adjacent to x_j if and only if y_i is adjacent to y_j. Otherwise player A wins. Since $EF_k(G_1, G_2)$ is a finite game of perfect information, it is determined; that is, exactly one of the players has strategy to guarantee a win. We say $G_1 \equiv_k G_2$ exactly when E has a winning strategy. It is routine to show that $\equiv_k$ is an equivalence relation.

This game, while first formulated as a game by Ehrenfeucht, amounts to a formalization of the back-and-forth proofs used by Fraïssé to prove Theorem 5.2.1.

DEFINITION 5.2.8. We define the quantifier rank of a first-order formula φ, denoted $qr(\varphi)$, as follows:

(1) If φ is atomic, then $qr(\varphi) = 0$.
(2) If $\varphi = \psi_1 \wedge \psi_2$, then $qr(\varphi) = max\,(qr(\psi_1), qr(\psi_2))$.
(3) If $\varphi = \psi_1 \vee \psi_2$, then $qr(\varphi) = max\,(qr(\psi_1), qr(\psi_2))$.
(4) If $\varphi = \neg\psi$, then $qr(\varphi) = qr(\psi)$.
(5) If $varphi = \exists x \psi$ then $qr(\varphi) = qr(\psi) + 1$
(6) If $varphi = \forall x \psi$ then $qr(\varphi) = qr(\psi) + 1$

Those accustomed to easy contraction of multiple like quantifiers will need to take careful note that every quantifier counts here.

The fundamental result of Ehrenfeucht-Fraïssé games is the following:

THEOREM 5.2.9. *The following are equivalent:*

(1) $G_1 \equiv_k G_2$
(2) G_1 *and* G_2 *satisfy the same sentences of quantifier depth at most* k.

One standard application of these games is to show that a given property is not first-order: If we can, for any k, find $G_1 \equiv_k G_2$ which differ on the property in question, then no first-order sentence can capture the property.

For any fixed k, suppose now that G_1 and G_2 are chosen at random from $G(n, \frac{1}{2})$. We claim that, as n becomes arbitrarily large, with high probability in the pair (G_1, G_2), player E has a winning strategy. Indeed, in round i of the game, we suppose without loss of generality that A determines x_i. Then E should choose y_i so that for all $j < i$ we have y_i adjacent to y_j if and only if x_i is adjacent to x_j. By similar analysis to the earlier proof, the probability that this strategy can be carried out goes to 1. This is certainly a winning strategy for E, so with probability approaching 1, we have $G_1 \equiv_k G_2$, providing an alternate proof of Theorem 5.1.6. This is the approach taken, for instance, in [**248**], and is also described as an alternate approach in [**416**]. It retains the flexibility of the original approach, as well; in [**415**], the results of [**404**] are derived using this approach.

An important difference from the interactive proof games of Section 4.3 is that here the *play* is completely deterministic. Only the particular game is chosen at random, and that is chosen once for all, at the outset of the game.

This alternate proof, though, has allowed a finer analysis of the speed of convergence of the zero-one law. For a fixed random graph model $\mathcal{G}(n, p(n))$, which is suppressed in the notation, we define the *tenacity function* $T_\epsilon(n)$ be the maximum k such that E has, with probability at least $1 - \epsilon$, a winning strategy for $EF_k(G_1, G_2)$, where G_1, G_2 are drawn independently at random from $\mathcal{G}(n, p(n))$. We expect, in general, that $T_\epsilon(n)$ will grow with n; heuristically, larger graphs can have more subtle differences than small ones. Indeed, [**415**] shows that $T_\epsilon(n) \to \infty$. However, this growth can be very slow.

Theorem 5.2.10 ([**417**]). *Fix $\epsilon > 0$ and let $p(n) \gg n^{-1/2}$ and $1-p(n) \gg n^{-1/2}$. Then for any $n \leq Tower(k)$, we have $T_\epsilon(n) > 5k+1$.*

What this means is that, with high probability, the quantifier depth necessary to distinguish random graphs is a very slow-growing function of n. Close at hand here is the notion of a *Scott sentence*. For any finite structure $\mathcal{M}$ there is a first-order sentence φ whose models are exactly the isomorphic copies of $\mathcal{M}$. This is also true for a countable structure, if we allow φ to be a sentence of $L_{\omega_1\omega}$.

For other graph distributions, including the one of constant probability, [**280**] quantifies the growth of T_ϵ (equivalently, the quantifier depth of Scott sentences for finite graphs). In particular, the results of [**280**] show that the results of [**417**] depended importantly on the particular function $p(n)$ selected.

Theorem 5.2.11 ([**280**]). *Let $p(n) = p$ be constant, with $0 < p \leq \frac{1}{2}$. Let G be selected at random from $\mathcal{G}(n,p)$, and let φ_G be a Scott sentence for G of minimal quantifier depth among all Scott sentences for G. Then the condition*

$$-O(1) \leq qd(\varphi_G) - \log_{1/p} n + 2\log_{1/p} \ln n \leq (2+o(1))\,\frac{\ln\ln n}{-p\ln p - (1-p)\ln(1-p)}$$

holds with probability $1-o(1)$ as $n \to \infty$.

5.3. Model Theory of Pseudofinite Structures

5.3.1. Pseudofinite Fields. We mentioned already that graphs were not unique in the existence of a zero-one law, although they are, of course, well-studied. It is natural to ask, in more familiar categories, what a random element of the category looks like. This question can often be a good deal more subtle, and is, in a sense, the subject of the next chapter. On the other hand, the question of what happens in a typical finite structure from the category is now within the range of our consideration.

This question is well-studied in fields. In the natural language, at least, fields are not the class of all structures in their language, so the application of the Glebskii–Fagin Theorem is not direct. However, it is an observation, arising largely from the work of Ax, that there is at least a large, if not quite a complete, theory that consists of the sentences true in almost all (that is, all but finitely many) finite fields.

Recall that finite fields are relatively rare, and there is a standard classification of them: there is exactly one field $\mathbb{F}_{p^d}$ for each prime p and natural d, and it has the structure $\mathbb{Z}_p[X]/(q(X))$ for some polynomial q. Now for each n, consider the sentence

$$\underbrace{1+1+\cdots+1}_{n \text{ times}} = 0.$$

This sentence is false in all but finitely many finite fields. In that sense, each particular positive characteristic is not the characteristic of a "typical" finite structure. Indeed, what is typical is that all of these sentences fail, so that the "typical" behavior is to have characteristic zero. The fields of large positive characteristic approximate more and more correctly the typical case of characteristic zero.

We need a few algebraic definitions here.

Definition 5.3.1. Let F be a field. We say that F is perfect in either of the following cases:

(1) F has characteristic zero, or
(2) F has characteristic p and every element of p has a pth root in F.

Given a finite sequence of polynomials $f_1, \ldots, f_m \in F[X_1, \ldots, X_n]$, we consider the variety
$$V = V(f_1, \ldots, f_m) = \{\bar{x} \in acl(F)^n : \forall i \ f_i(\bar{x}) = 0\}.$$
It is also conventional to consider, for a variety V over any field K, the set $I_K(V) = \{f \in K[X_1, \ldots, X_n] : \forall \bar{x} \in V \ \ [f(\bar{x} = 0]\}$.

We say that V is defined over F if and only if $I_{acl(F)}(V)$ is generated by elements of $F[X_1, \ldots, X_n]$. We say that V has a K-rational point if there is some $\bar{x} \in V \cap K^n$.

Definition 5.3.2. Let F be a field. We say that F is pseudo algebraically closed if every non-empty variety defined over F has an F-rational point.

We first observe that every algebraically closed field is pseudoalgebraically closed, although the converse is not true.

Now for any field F, in a fixed algebraic closure $\tilde{F}$ of F, there is a unique separable extension F^s of F, containing all separable extensions of F within $\tilde{F}$. Further, $F^s = \tilde{F}$ if and only if F is perfect.

In any case, we consider the Galois group $Gal(F) = Gal(F^s/F)$. Toward characterizing the typical behavior of finite fields, we note that the field $\mathbb{F}_p$ has a unique extension of each degree n. This extension has a cyclic Galois group. By following the interactions of this tower of extensions, we find that $Gal(\mathbb{F}_p) = \lim_{\leftarrow} \mathbb{Z}/n\mathbb{Z}$. We call this group $\hat{\mathbb{Z}}$.

We now proceed, after a definition, to the first major result.

Definition 5.3.3. The first order theory of finite fields is the set of sentences which are true of almost all finite fields.

Theorem 5.3.4 (Ax [**36**]). *The following are equivalent*

(1) *F is an infinite field satisfying all sentences that are satisfied by all but finitely many fields of characteristic p.*
(2) *F is perfect, pseudo algebraically closed, of characteristic $p > 0$, and $Gal(F) = \hat{\mathbb{Z}}$.*

Proof. The proof goes through the mechanism of ultraproducts. Given a nonempty set S, an *ultrafilter* on S is a nonempty subset $\mathcal{U} \subseteq P(S)$ satisfying the following properties:

(1) If $A \in \mathcal{U}$ and $A \subseteq B$, then $B \in \mathcal{U}$.
(2) If $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$.
(3) $\emptyset \notin \mathcal{U}$.
(4) For each $A \in P(S)$, exactly one of $\{A, S - A\}$ is in $\mathcal{U}$.

The ultrafilter $\mathcal{U}$ is said to be principal if and only if it is of the form $\{A : B \subseteq A\}$ for some fixed B. Otherwise it is said to be *non-principal*.

Now let $(F_i : i \in I)$ be a family of non-empty fields, and $\mathcal{U}$ some ultrafilter on I. We form a field as a quotient of the usual cartesian product $\prod_{i \in I} F_i$ in the following way. We first set $P = \prod_{i \in I} F_i$, and define an equivalence relation $\sim$ on P by $f \sim g$ if and only if the set of $i \in I$ where f and g agree is an element of $\mathcal{U}$. Now $P/\sim$ will be the domain of our new field. We can define a natural structure on this set, taking 0 and 1 to be the equivalence class of $(0_i : i \in I)$ and $(1_i : i \in I)$, respectively, and with addition and multiplication defined similarly (e.g. taking the equivalence class of the sum for the sum of the equivalence classes). It takes checking to be sure that this is well defined, but it is. The resulting structure is called the *ultraproduct* $\prod_{i \in I} F_i/\mathcal{U}$

Even granting that the structure has been well-defined, it is not obvious even that it should be a field. The following fundamental result of Łoś, which we state here without proof, is important.

LEMMA 5.3.5 (Łoś's Theorem). *Let L be a first-order signature, $(\mathcal{A}_i : i \in I)$ a family of non-empty L-structures, and $\mathcal{U}$ an ultrafilter over I. Then for any L-sentence φ, the following are equivalent:*

(1) $\prod_{i \in I} F_i/\mathcal{U} \models \varphi$
(2) *The set of i for which $\mathcal{A}_i \models \varphi$ is an element of $\mathcal{U}$.*

Now suppose that φ is true in all but finitely many finite fields, and let F be perfect, pseudo algebraically closed, of characteristic p, and $Gal(F) = \hat{\mathbb{Z}}$. Now φ must certainly be true of all but finitely many of the fields $\mathbb{F}_{p^n}$. We construct an ultraproduct $D = \prod_{n \in \mathbb{N}} \mathbb{F}_{p^n}/\mathcal{U}$, where $\mathcal{U}$ is still to be specified.

It is a standard result that if we have a family of subsets of $\mathbb{N}$ such that any finite intersection of members of this family must be infinite, then there is a nonprincipal ultrafilter containing this family. We invoke this result on the family

$$\alpha_d := \left\{ n \in \mathbb{N} : \mathbb{F}_{p^n} \cap \mathbb{F}_{p^d} = acl(\mathbb{F}_p) \cap \mathbb{F}_{p^d} \right\}.$$

We choose the resulting nonprincipal ultrafilter as $\mathcal{U}$.

We now have $D \cap acl(\mathbb{F}_p) = F \cap acl(\mathbb{F}_p)$. Since the properties of being perfect, pseudo algebraically closed, and having absolute Galois group $\hat{\mathbb{Z}}$ are all elementary, as is the specification of a characteristic, so that D must inherit all of these properties (via Łoś's Theorem) from its factors. In combination with having $D \cap acl(\mathbb{F}_p) = F \cap acl(\mathbb{F}_p)$, this is enough to guarantee that F and D are elementarily equivalent. Since $D \models \varphi$ by design, it follows that $F \models \varphi$.

Conversely, suppose that we have an infinite sequence F_i of finite fields of characteristic p, all satisfying $\neg\varphi$. Now for any nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we have $\prod_{i \in \mathbb{N}} F_i/\mathcal{U} \models \neg\varphi$, and this ultraproduct is perfect, pseudo algebraically closed, and has absolute Galois group $\hat{\mathbb{Z}}$ and characteristic p. □

A similar argument shows the following:

THEOREM 5.3.6 (Ax [**36**]). *The following are equivalent*

(1) *F is an infinite field satisfying all sentences that are satisfied by all but finitely many finite fields.*
(2) *F is perfect, pseudo algebraically closed, of characteristic 0, and $Gal(F) = \hat{\mathbb{Z}}$.*

DEFINITION 5.3.7. We say that a field F is pseudofinite if and only if it is perfect, pseudo algebraically closed, and $Gal(F) = \hat{\mathbb{Z}}$.

Intuitively, these fields resemble random graphs in that the property of satisfying all sentences satisfied by "almost all" finite structures in the class resembles a zero-one law. They also resemble Fräissé limits in that the ultraproduct construction in some ways resembles the extension and amalgamation process of the existence proof for Fräissé's theorem.

For Ax, the main point of this was that the "theory of finite fields" (that is, the set of sentences satisfied by all but finitely many finite fields) is decidable. Theorem 5.3.6, together with his classification of pseudofinite fields up to elementary equivalence, establishes this fact. We have already used this classification, without proof, to establish Theorem 5.3.4, since its proof is not central to the argument of the present book, but we do state the result here.

PROPOSITION 5.3.8 (Ax [**36**]). *Let E and F be pseudofinite fields with prime fields E_0, F_0, respectively. Then $E \equiv F$ if and only if $E \cap acl(E_0) \cong F \cap acl(F_0)$.*

These pseudofinite fields are ubiquitous among fields, but it is difficult to give more than a few concrete examples. Probably the most comprehesnive reference on the algebraic aspects of these fields is [**180**].

Pseudofinite fields have also been extensively studied from the perspective of model theory. One often talks about the "theory of pseudofinite fields," although, as we have seen, this theory is not complete. However, we do understand its completions very well. There are natural known languages in which the theory of pseudofinite fields admits quantifier elimination, or is model complete. Fundamental for more recent results is the following result.

THEOREM 5.3.9 ([**105**]). *Let $\varphi(\bar{x}, \bar{y})$ be a formula in the language of rings. Then there is a finite set $D \subseteq \{0, 1, \ldots, n\} \times \mathbb{Q}^{>0} \cup \{(0,0)\}$ of pairs (d, μ), along with a positive constant C such that for any q and each $\bar{x} \in \mathbb{Q}^m$, if the set of $\bar{y} \in \mathbb{F}_q^n$ such that $\varphi(\bar{x}, \bar{y})$ is nonempty, then there is some (d, μ) such that*

$$\left| |\varphi(\bar{x}, \mathbb{F}_q| - \mu q^d \right| \leq C q^{d-1/2}.$$

The point of comparison here is the bound of [**296**].

THEOREM 5.3.10. *There exists a constant A, depending only on n, d, and r such that for any variety V of dimension r and degree d in $\mathbb{P}^n$ defined over a finite field k, we compute $\delta = (d-1)(d-2)$, and denote by N the number of points of V which are in k. The following inequality holds:*

$$|N - q^r| \leq \delta q^{r-\frac{1}{2}} + A q^{r-1}.$$

Even more can be said about the definable groups. The following result, originally due to Hrushovski and Pillay [**246**], was simplified by Hodges [**240**].

THEOREM 5.3.11. *Let F be a pseudofinite field, and G a d-dimensional affine group defined over F. For each $i \in I$, let $X(i) \subseteq G$ be an irreducible F-definable set containing 1_G. Let A be the subgroup of $G(F)$ generated by $\bigcup_{i \in I} X(i)$, and let B be the Zariski closure of A. Then the following hold:*

(1) *B is connected,*
(2) *A has finite index in $B(F)$, and*
(3) *There are $i_1, \ldots, i_m \in I$ and $\epsilon_1 \ldots, \epsilon_m \in \{-1, 1\}$ such that*
   (a) *$A = X(i_1)^{\epsilon_1} \cdots X(i_m)^{\epsilon_m}$*
   (b) *$B = \overline{X(i_1)}^{\epsilon_1} \cdots \overline{X(i_m)}^{\epsilon_m}$.*

PROOF. We take $i_1, \ldots, i_k$ so that the set $\overline{X(i_1)} \cdots \overline{X(i_k)}$ has maximal dimension in G. The rest, by various dimension couting arguments, boils down to the following result, of independent interest.

THEOREM 5.3.12. *Let F be a pseudofinite subfield of K, and G an affine group definable over F. Let U be a subset of $G(F)$ definable in F which contains 1_G and has the same dimension as $G(F)$. Then the subgroup of $G(F)$ generated by U is definable in F, and has the form $U^{\nu_1} \cdots U^{\nu_k}$ for some $\nu_1, \ldots, \nu_k \in \{-1, 1\}$.*

PROOF. Let W be the set of all $x \in G$ such that $dim(xU \cap U) = dim(G(F))$. For any $g \in W$, there must then be $u_1, u_2 \in U$ such that $gu_1 = u_2$, so that $W \subseteq UU^{-1}$.

We let a_0 be the identity, and at stage $s+1$ we choose some a_{s+1} in the subgroup of $G(F)$ generated by U such that $a_{s+1} \notin a_0W \cdots a_sW$. By a dimensional argument, this process must halt, and then the decomposition of the subgroup of $G(F)$ generated by U is chosen according to the sequence of a_i. □

□

The importance of these results is that the very tight control these fields have on their definable groups is similar to what is seen in algebraically closed fields. In general fields, there may be quite a lot of definable things, very few of which correspond nicely to affine algebraic groups defined *over* the field. The tight relationship here is given by the following result of Hrushovski and Pillay.

THEOREM 5.3.13 ([**245**]). *Let G be a group definable in a pseudofinite field F. Then there is a definable subgroup $G_1 \subseteq G$, an algebraic group H defined over F, and a definable group homomorphism $G_1 \to H(F)$ with finite kernel.*

Beyond the status of pseudofinite fields as models of an almost sure theory in their own right, an interesting result of Beyarslan shows that certain random structures (in the sense of Section 5.1) can be interpreted in them.

THEOREM 5.3.14 ([**149**]). *Let K be a pseudofinite field which is not separably closed. Then the inifinite random graph is interpretable in K.*

PROOF. We take the set of field elements as vertices. For any two vertices a, b, we make a and b adjacent if there is a pth root of $a + b$. The graph defined in this way satisfies the axioms of a random graph. □

COROLLARY 5.3.15. *Pseudofinite fields have the independence property, and so are unstable.*

More recently, Beyarslan has taken up the extension of this line of thought to random hypergraphs.

DEFINITION 5.3.16. A k-hypergraph is a structure (V, H), such that H is a k-ary relation with the following properties:

(1) For any permutation $\pi \in S_k$ and for any $a_1, \ldots, a_k \in V$, we have $H(a_1 \ldots, a_k)$ if and only if we have $H(a_{\pi(1)}, \ldots, a_{\pi(k)})$.
(2) For any $a_1, \ldots, a_k \in V$ and any $i, j \leq k$, if $a_i = a_j$, then we do not have $H(a_1, \ldots, a_k)$.

Hypergraphs generalize graphs, of course, in that a 2-hypergraph is exactly a graph. If $H(a_1,\ldots,a_k)$ holds, we say taht there is a hyperedge containing vertices $a_1,\ldots,a_k$. These structures see frequent applications, including the phenomenon of *folksonomy*, in which a group of individuals collaboratively annotate a data set, as in the application of tags in social media. One way to model this phenomenon is to have G consist of all users, resources, and tags, and to represent a tag of a resource by a user as a 3-hyperedge [**197**]. There is now a significant body of combinatorial and probabilistic literature on random hypergraphs, similar to that on random graphs [**127, 197, 122**].

It is not hard to imagine a set of extension axioms for a theory of random hypergraphs, and indeed it is common to refer, by a *random k-hypergraph*, to a hypergraph in wich for all distinct $a_1,\ldots,a_m,b_1,\ldots,b_n \in V^{k-1}$ (that is, each a_i and each b_i is a $k-1$-sequence), there is some $c \in V$ such that

$$\left(\bigwedge_{i\leq m} H(a_i,c)\right) \wedge \left(\bigwedge_{i\leq n} \neg H(b_i,c)\right).$$

Matushkin has announced work in the tradition of [**404**] toward a broader class of zero-one laws for random hypergraphs [**329**].

A result of Hrushovski showed that random k-hypergraphs are more complex in increasing values of k, in the sense that one cannot interpret a random $(k+1)$-hypergraph in a random k-hypergraph [**242**]. In particular, this means that Duret's result on interpreting random graphs does not directly give interpretability of random k-hypergraphs for $k>2$.

THEOREM 5.3.17 ([**64**]). *Let F be a pseudofinite field and $k \in \mathbb{N}$. Then there is a random k-hypergraph interpretable in F.*

PROOF. Suppose that we can construct some polynomial $g(T,Y_1,\ldots,Y_k) \in F[T,Y_1,\ldots,Y_k]$ which is symmetric in $Y_1,\ldots,Y_k$, and which satisfies the following properties. (For each $\bar{a} \in F^{k-1}$, we let $L_{\bar{a}}$ be the splitting field of $g(t,\bar{a},x)$ over $F(x)$.)

(1) $Gal\,(L_{\bar{a}}/F(x))$ is non-Abelian and simple.
(2) $L_{\bar{a}}$ is a regular extension of F.
(3) $L_{\bar{a}} = L_{\bar{b}}$ if and only if $\bar{a}=\bar{b}$.

We then define a k-hypergraph on F by setting $H(a_1,\ldots,a_k)$ if and only if F has a root of $g(T,a_1,\ldots,a_n)$.

To verify the success of this construction, it suffices to show that the extension axioms are satisfied in an elementary extension (since they would then have to be satisfied in F itself). Given a sequence of distinct $a_1,\ldots,a_m,b_1,\ldots,b_n \in F^{k-1}$, we observe that the fields L_z are linearly disjoint as z ranges over $a_1,\ldots,a_m,b_1,\ldots,b_n$. We can then find an automorphism μ of the join of all of the L_z which moves all roots of $g(T,b_i,x)$ for each i, but restricts to the identity on each L_{a_i}.

Let $\sigma \in Gal(F)$ be such that the topological closure of $\langle\sigma\rangle$ is $Gal(F)$, and let $\tau \in Gal\,(F(x))$ be a common extension of σ and μ. Let M be the fixed field of τ, and note that $Gal(M)=\hat{\mathbb{Z}}$. Thus, there is a pseudofinite E containing M with $acl(M)\cap E = M$, so that E is an elementary extension of F. By construction, $x \in E$ witnesses the extension axioms showing that (F,H) is a random k-hypergraph.

All that remains is to construct the polynomial g. This can, in fact be done, completing the proof. □

The reverse interpretation is not possible. Any structure interpretable in an $\aleph_0$-categorical theory must also be $\aleph_0$-categorical, so no infinite field can be interpreted in a random graph. From an alternate viewpoint, if a pseudofinite field could be interpreted in a random k-hypergraph, then one could interpret a random $(k+1)$-hypergraph in that field, which would give an interpretation of a random $(k+1)$-hypergraph in a random k-hypergraph, a contradiction.

In this sense, pseudofinite fields have quite a lot of randomness. They are themselves models of an almost sure theory, and interpret many other models of almost sure theories. This gives rise to the following problem.

PROBLEM 5.3.18. Is there a theory T such that if (K, μ) is a probability space of finite structures in a finite relational language satisfying a zero-one law, with almost sure theory T_K, and $\mathcal{M}$ is a model of T_K, then there is some model of T that interprets $\mathcal{M}$?

5.3.2. Pseudofinite Groups. The analogous class of groups, the pseudofinite groups, is defined differently. A significant part of the reason for this is that the good model-theoretic behavior of groups under this definition is analogous to that of the pseudofinite fields already defined. A second part is their characterization, originally due to Wilson, but strengthened by Ryten as groups of Lie type over pseudofinite fields.

PROPOSITION 5.3.19. *Given a group G, the following are equivaelnt:*

(1) *G is an infinite group and G satisfies every first-order sentence that holds in all finite groups.*
(2) *G is an infinite group such that every first-order sentence true of G is also true of some finite group.*
(3) *G is elementarily equivalent to an infinite ultraproduct of finite groups.*

DEFINITION 5.3.20. A group G is said to be pseudofinite if and only if it satisfies one of the equivalent conditions of Proposition 5.3.19.

As we might expect, pseudofinite groups are not quite as neatly classified as pseudofinite fields. However, the simple pseudofinite groups are well-understood (see [**207**]).

DEFINITION 5.3.21. A simple group of Lie type is a simple non-Abelian composition factor of the centralizer in an algebraic group G over a field F by a surjective endomorphism of G.

In partiular, all simple groups of Lie type are subgroups of linear algebraic groups. The following theorem was proved, in its original form, by [**460**], and was strengthened to the present statement in [**388**]. Much of the reasoning in the proof appears already in [**168**].

THEOREM 5.3.22. *Every simple pseudofinite group is isomorphic to a group of Lie type, possibly twisted, over a pseudofinite field.*

PROOF. We give an outline of the proof. Suppose that G is simple and is elementarily equivalent to an ultraproduct $\prod_{i \in I} G_i/\mathcal{U}$ of finite groups. We consider each factor G_i. There is some integer k such that each element of G_i is a product of k commutators. One can also show that for each i, the group G_i is either simple or has the property that the set of all products of $k+3$ commutators is a proper normal subgroup. This property is first-order, so the set of i on which it holds cannot belong to $\mathcal{U}$. Thus, there is a set $I' \subseteq I$ such that $I' \in \mathcal{U}$, such that G_i is simple for all $i \in I'$, and such that $G \equiv \prod_{i \in I'} G_i/\mathcal{U}$. A similar argument shows that if G is elementarily equivalent to an ultraproduct of groups of Lie type, then it must be elementarily equivalent to an ultraproduct of groups of the *same* Lie type. An ultraproduct of alternating groups must be finite. Consequently, G must be elementarily euqivalent to an ultraproduct of finite simple groups of the same Lie type, thus, a group of Lie type over a pseudofinite field F.

If we can now replace elementary equivalence with isomorphism, the theorem is established. For this, it suffices to show that there is a field F or difference field (F, σ) (according to the Lie type of G), such that F or (F, σ) is uniformly bi-interpretable with G. If this is true, then the theory of G will state that there is a field F or a difference field (F, σ) and a group H of Lie type over F hich is definably isomorphic to G. We outline the non-twisted case.

Define H to be the subgroup of G generated by the elements of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Note that this group is definable in G.

We shall also need the *root group*, which for most readers of this book will require some explanation. Let $\mathfrak{G}$ be the Lie algebra associated with G, and let $\mathfrak{H}$ be a subalgera of $\mathfrak{G}$ such that $\mathfrak{H}$ is nilpotent (interation of the Lie bracket terminates) and such that if $[x, h] \in \mathfrak{H}$ for all $h \in \mathfrak{H}$, then $x \in \mathfrak{H}$. We can then decompose $\mathfrak{G}$ as a direct sum of $\mathfrak{H}$-invariant subspaces $\mathfrak{G} = \mathfrak{H} \oplus \left(\bigoplus_{i=1}^{k} \mathfrak{L}_i \right)$, where $\mathfrak{L}_i$ is 1-dimensional for each i. We let e_i be a non-zero element of $\mathfrak{L}_i$. We define

$$exp(\delta) = \sum_{j=0}^{\infty} \frac{\delta^j}{j}$$

and let $x_i(t) = exp\,(te_r^*)$, where $*$ denotes the adjoint operation. We note that this series does, in fact, terminate, by the construction of the direct sum decomposition. Now for each i, we have a subgroup $X_i := \{x_i(t) : t \in F\}$ of G, which we call the *root subgroup*.

The proof works more generally, but suppose that H acts transitively on the root subgroup X_i. Now to multiply $x_i(a)$ and $x_i(b)$, we define the element h_z such that $h_z x_i(1) = x_i(z)$. Now $x_i(a) \otimes x_i(b)$ can be defined as $h_a h_b x_i(1)$. Now X_i was already isomorphic to the additive subgroup of F, and it can be seen that the operation $\otimes$ defines a field on X_i which is isomorphic to F. □

Naturally, the situation beyond simple groups is a good deal more complex. We will consider NIP theories in more detail in Section , but since that condition is important in the literature of pseudofinite groups, we define it here.

Definition 5.3.23. Let T be a first-order L-theory.

(1) Let $\varphi(\bar{x}; \bar{y})$ a first-order L-formula, and S a set of n-tuples. Then we say that S is *shattered by* $\varphi(\bar{x}; \bar{y})$ if and only if there is a family $(\bar{b}_i : i \in I)$ such that for each $V \subseteq S$ we have some i such that $\varphi(\bar{a}; \bar{b}_i)$ holds exactly of those $\bar{a}$ with $\bar{a} \in S$.

(2) A formula $\varphi(\bar{x};\bar{y})$ is said to be NIP if no infinite set S is shattered by $\varphi(\bar{x};\bar{y})$.
(3) A theory T is said to be NIP if and only if all formulas $\varphi(x;y) \in L$ are NIP.

THEOREM 5.3.24 ([**316**]). *Let G be a pseudofinite group with NIP theory, and suppose that there is a natural number n such that there is no sequence of sets $F_1, \ldots, F_{n+1} \subset G$ with $C_G(F_1) < C_G(F_2) < \cdots < C_G(F_{n+1})$. Then G has a solvable definable normal subgroup of finite index.*

More recently, Conant and Pillay have shown that in the context of psuedofinite groups one can apply NIP theory locally, a key feature of stability that does not easily generalize to the broader approach of NIP [**115**].

In addition to the asymptotic behavior of satisfaction of sentences, there is also an approach to the asymptotic study of realization of formulas.

DEFINITION 5.3.25 ([**156, 314**]). Let N be a positive integer and $\mathcal{C}$ a class of finite structures in a common signature L. We say that $\mathcal{C}$ is an *N-dimensional asymptotic class* if and only if for every L-formula $\varphi(\bar{x},\bar{y})$ there is a finite set $D \subseteq \left(\{0,\ldots,N\cdot\ell(\bar{x})\}\times\mathbb{R}^{>0}\right)\cup\{(0,0)\}$, and for each pair $(d,\mu)\in D$ a collection $\Phi_{(d,\mu)}$ of pairs $(M,\bar{a})$ such that

(1) $M \in \mathcal{C}$
(2) $\bar{a} \in M^{\ell(\bar{y})}$
(3) $\left\{\Phi_{(d,\mu)} : (d,\mu)\in D\right\}$ is a partition of $\left\{(M,\bar{a}) : M\in\mathcal{C}, \bar{a}\in M^{\ell(\bar{y})}\right\}$, and
(4) When M ranges over $\mathcal{C}$, we have

$$\lim_{|M|\to\infty}\frac{\left|\left|\varphi\left(M^{\ell(\bar{x})},\bar{a}\right)\right| - \mu\left|M\right|^{d/N}\right|}{\left|M\right|^{d/N}} = 0$$

(5) The family $\left\{\bar{a} : (M,\bar{a})\in\Phi_{(d,\mu)}\right\}$ is uniformly $\emptyset$-definable.

This definition, especially in Clause 4, parallels the Lang-Weil bound of Theorem 5.3.10. The difference from a zero-one law may be exemplified by the case of finite fields. Theorem 5.3.6 gives the behavior of finite fields that most nearly matches a zero-one law. On the other hand, Theorem **??** shows that finite fields constitute a 1-dimensional asymptotic class. The proof of Theorem 5.3.22 demonstrates that any family of finite simple groups of fixed Lie type constitutes an N-dimensional asymptotic class for some N.

5.3.3. Classes of Finite Structures and Pseudofinite Structures. The general definition of pseudofinite structures is similar to that for pseudofinite groups.

DEFINITION 5.3.26. A structure $\mathcal{M}$ is said to be pseudofinite if for every sentence φ such that $\mathcal{M}\models\varphi$, there is some finite structure $\mathcal{N}$ with $\mathcal{N}\models\varphi$.

In this framework, certain parts of nonstandard analysis will become useful. It is common to consider a standard model $\mathbb{V}=(V,\in)$ of set theory, and a "large saturated" elementary extension $\mathbb{V}^*=(V^*,\in^*)$ of $\mathbb{V}$). Of course, precise formulation of a "large saturated" model of set theory is more subtle, but the subtleties are not central to our needs here. We say that an object in $\mathbb{V}^*$ is *internal* if and only if it is definable with parameters in $\mathbb{V}^*$. We can then define a pseudofinite object (set, structure, etc.) to be one which is finite in the sense of $\mathbb{V}^*$ (recall that $\mathbb{V}^*$

will have many functions that $\mathbb{V}$ does not have, which may witness bijections that may not exist in $\mathbb{V}$). If we have a pseudofinite structure satisfying a sentence in $\mathbb{V}^*$, we can use the fact that $\mathbb{V}*$ is an elementary extension of $\mathbb{V}$ to find a finite structure satisfying the same sentence. Conversely, we can construct, by compactness, a non-principal type in $\mathbb{V}$ whose realization in $\mathbb{V}^*$ is a $\mathbb{V}^*$-finite structure satisfies the conditions of Definition 5.3.26.

We first explore a particular family of strengthenings of the pseudofinite condition that give rise to highly regular model theory. In particular, the existence of a chain of finite submodels with strong regularity properties is equivalent to a decomposition into a very limited class of structures — enough, at least, to imply $\aleph_0$-categoricity.

Definition 5.3.27. Let $\mathcal{M}$ be a structure.

(1) $\mathcal{M}$ is said to be smoothly approximable by finite structures if and only if it is countable and $\aleph_0$-categorical and there is a chain

$$\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \cdots$$

of finite substructures such that
  (a) $\mathcal{M} = \bigcup_{i \in \mathbb{N}} \mathcal{M}_i$
  (b) For each i, if $\bar{a}, \bar{b}$ are sequences from $\mathcal{N}$ of finite length, then they lie in the same $Aut(\mathcal{M})$ orbit if and only if they lie in the same orbit of the setwise stabilizer of N in $Aut(\mathcal{M})$.

(2) $\mathcal{M}$ is said to be strongly k-quasifinite if and only if in a nonstandard extension of the set-theoretical universe, there is an internally finite structure $\mathcal{N} \equiv \mathcal{M}$ with a finite number of internal k-types which coincide with the k-types.

The major work of [**106**] is to associate these properties of quasifiniteness and smooth approximability with certain geometric properties, and to exploit this equivalence. We now define a list of structures collectively called *geometries*. We will then define a class of structures whose key feature is that they can be decomposed as an assembly of geometries. Finally, we will see the main theorem just described.

Definition 5.3.28. Geometries are defined as follows:

(1) A linear geometry is an expansion of one of the following types of structures $\mathcal{M}$ by a set of algebraic elements in $\mathcal{M}^{eq}$:
  (a) A pure set
  (b) A pure vector space over a finite field, with a sort for the field and scaling as a binary function
  (c) $(V \cup W, K, L, \beta)$, where K is a finite field, L, V, and W are vector spaces over K where L has dimension 1, and $\beta : V \times W \to L$ is a nondegenerate bilinear map.
  (d) An inner product space over a finite field
  (e) (V, K, L, q), where K is a finite field, L and V vector spaces over K where L has dimension 1, and q is a quadratic form $V \to L$ whose associated bilinear form is nondegenerate
  (f) A finite field K of characteristic 2 with a vector space K and nondegenerate symplectic bilinear form β_V, along with a set Q of quadratic forms on V, with some additional structure $(+_Q, -_Q, \beta_Q, \omega)$ on Q.

(2) A projective geometry is the quotient of a linear geometry by the relation $acl(x) = acl(y)$.

DEFINITION 5.3.29. A structure $\mathcal{M}$ is said to be Lie coordinatizable if and only if it is bi-interpretable with a structure $\mathcal{M}'$ such that

(1) $\mathcal{M}'$ has finitely many 1-types.
(2) $\mathcal{M}'$ has a definable tree structure of finite height with an $\emptyset$-definable root, with the following properties:
   (a) For each $a \in \mathcal{M}$ which is not the root, either a is algebraic over its predecessor or there is $b < a$ and a b-definable projective geometry J_b with representative c_b in $\mathcal{M}'^{eq}$, such that
      (i) The $\emptyset$-definable relations of J_b are the relations on $\mathcal{M}$ which are c_b-definable in $\mathcal{M}'$.
      (ii) The set of $\mathcal{M}'$-definable relations on J_b is uniformly definable in J_b.
      (iii) One of the following holds:
         (A) $a \in J_b$, or
         (B) There is $b' \in \mathcal{M}'$ with $b < b' < a$ and a b'-definable affine or quadratic geometry $(J_{b'}, A_{b'})$ such that $a \in A_{b'}$ and the projectivization of $J_{b'}$ is J_b.
   (b) If $tp(a) = tp(b)$ and are associated with geometries J_a and J_b of type f, then any definable map preserves all elements of the geometric structure except ω, then it must also preserve ω.

Lie coordinatizability is related to the notion, defined earlier, of asymptotic classes (see Definition 5.3.25).

PROPOSITION 5.3.30 ([**156**]). *Let $\mathcal{M}$ be a Lie coordinatized structure. Then there exists some N and some N-dimensional asymptotic class $\mathcal{C}$ such that $\mathcal{C}$ smoothly approximates $\mathcal{M}$.*

It is not obvious, but true, that every Lie coordinatizable structure is $\aleph_0$-categorical. We now state the main result of [**106**].

THEOREM 5.3.31. *The following are equivalent.*

(1) *$\mathcal{M}$ is smoothly approximable.*
(2) *$\mathcal{M}$ is strongly 4-quasifinite.*
(3) *$\mathcal{M}$ is Lie coordinatizable.*

PROOF. The implication $1 \to 2$ is straightforward from the definitions. Toward $3 \to 1$, we note that in a Lie coordinatizable structure one can construct a subset known as an envelope. A dimension function is a function μ whose domain is the set of equivalence classes of $\emptyset$-definable functions mapping a complete type to a family of projective geometries and which takes values in isomorphism types of finite or countable dimensional geometries of the same type. If $\mathcal{M}$ is Lie coordinatizable, then relative to a particular choice of μ, we can construct, for any sufficiently regular algebraically closed $E_0 \subseteq \mathcal{M}$, a set $E \subseteq \mathcal{M}$ containing E_0 which is algebraically closed in $\mathcal{M}$, and with the following two properties:

(1) For any $c \in \mathcal{M} - E$ we have
   (a) a $\emptyset$-definable function J mapping a complete type A to a family of projective geometries, and

(b) an element $b \in A \cap E$ such that $acl(E) \cap J_b \subsetneq acl(E, c) \cap J_b$.

(2) For any such function J and any element $b \in A \cap E$, the isomorphism type of $J_b \cap E$ is given by $\mu(J)$.

While it is not elementary, it can be shown that this construction is always possible, and that if μ is always finite, then E is finite, as well. Finally, it can be shown that E has the necessary homogeneity to satisfy the conditions of witnessing that $\mathcal{M}$ is smoothly approximable.

It remains to show that $2 \to 3$. Suppose that $\mathcal{M}$ is strongly 4-quasifinite. Then we can construct a finite cover $\mathcal{M}^c$ of $\mathcal{M}$ (so that $\mathcal{M}^c$ is bi-interpretable with $\mathcal{M}$) which satisfies the necessary geometric properties. □

Definition 5.3.25 has limited application to ordered structures, in the sense that every o-minimal structure arising from linearly ordered finite structures must be discretely ordered [**367**]. To this end, Macpherson and Steinhorn offered the following definition:

Definition 5.3.32 ([**315**]). A *robust chain* of finite L-structures consists of a sequence $(M_i : i \in \mathbb{N})$ such that

(1) For each i, we have $M_i \subseteq M_{i+1}$
(2) For each L-formula $\varphi(\bar{x})$, there is a function $f_\varphi : \mathbb{N} \to \mathbb{N}$ such that for each $i \in \mathbb{N}$, for each $\bar{a} \subseteq M_i$, and each $j \geq f_\varphi(i)$, we have $M_{f_\varphi(i)} \models \varphi(\bar{a})$ if and only if $M_j \models \varphi(\bar{a})$.

An initial indication that this definition has appropriate scope is the following result:

Proposition 5.3.33 ([**324**]). *Let $\mathcal{M}$ be a countable infinite structure such that every finite subset of $\mathcal{M}$ is contained in a finite substructure of $\mathcal{M}$. Then $\mathcal{M}$ is the direct limit of a robust chain of finite structures.*

In previous constructions we have observed the existence of an almost sure theory and regarded its (often unique) model as the limit of the collection of finite models. Thinking strictly in the logic topology, this idea has merit. However, in the context of robust chains and their direct limits, "direct limit" is already a construction of a particular model, and it is necessary to assess agreement between the theory T^{lim} of the direct limit of the chain and, on the other hand, the almost sure theory T^{as} of the chain itself.

Proposition 5.3.34 ([**315**]). *Let $\mathcal{C}$ be a robust chain of models, with almost sure theory T^{as} and limit theory T^{lim}. If T^{as} is $\forall\exists$-axiomatized, then $T^{lim} \models T^{as}$.*

Proof. Every $\forall\exists$ sentence in T^{as} must hold in all M_i for sufficiently large i. Consequently, they must hold in the direct limit. □

On the other hand, let $\mathcal{C}$ be a robust chain of finite fields of characteristic p, having union $\overline{\mathbb{F}_p}$. Now the limit theory T^{lim} is a theory of algebraically closed fields, but the almost sure theory has the independence property, so that these two theories differ dramatically. In exchange for this limitation on the construction method, Macpherson and Steinhorn proved that there is a robust chain of finite structures whose limit theory is the theory of divisible ordered Abelian groups.

In light of Theorem **??** and Definition 5.3.25, MacPherson and Steinhorn proposed the following definition.

DEFINITION 5.3.35. An infinite L-structure $\mathcal{M}$ is said to be *measurable* if and only if there is, for every definable set X, a pair $\delta(X) = (d, \mu) \in \mathbb{N} \times \mathbb{R}$ with the following properties:

(1) For each formula $\varphi(\bar{x}, \bar{y})$, there is a finite set D_φ such that for any $\bar{a}$, we have $\delta(\varphi(\bar{x}, \bar{a})) \in D_\varphi$.
(2) If X is finite, then $\delta(X) = (0, |X|)$.
(3) For each L-formula $\varphi(\bar{x}, \bar{y})$ and each fixed pair (d, μ), the set of parameters $\bar{a}$ such that $\delta(\varphi(\bar{x}, \bar{a})) = (d, \mu)$ is definable without parameters.
(4) Let X, Y be definable sets of $\mathcal{M}$ and $f : X \to Y$ be a definable surjection. Then
   (a) There are finitely many pairs (d_i, μ_i) such that the sets
$$Y_i := \left\{\bar{y} \in Y : \delta\left(f^{-1}(\bar{y})\right) = (d_i, \mu_i)\right\}$$
   partition Y, and
   (b) If $\delta(Y_i) = (e_i, \nu_i)$, then $\delta(H) = (c, \theta)$, where $c = \max\{d_i + e_i\}$, where this maximum is attained by those i with $i \leq s$, and $\theta = \sum_{i \leq s} \mu_i \nu_i$.

In particular, every non-principal ultraproduct of an N-dimensional asymptotic class is measurable, via the pairs (d, μ) arising in Definition 5.3.25. However, there are measurable structures arising in other ways, too. One can get this simply by taking a vector space over an infinite field. Alternately, Elwes shows that such an example arises by taking the "Hrushovski fusion" of two algebraically closed fields of different positive characteristic [**156**].

Pseudofinite structures admit a particulary useful class of measures that mirror the counting measures on finite structures. We note that for any real $x \in \mathbb{V}^*$ there is a unique real $st(x) \in \mathbb{V}$ such that $|st(x) - x| < \frac{1}{n}$ for all standard natural numbers n. Then for any pseudofinite structure $\mathcal{M}$ and any definable set $X \subseteq M^n$, we may define
$$\mu_X(Y) := st\left(\frac{|Y|}{|X|}\right)$$
where $|\cdot|$ denotes the $\mathbb{V}^*$ cardinality and Y ranges over definable subsets of X. This function μ_X is a finitely additive probability measure on the definable subsets of X. That is, it is a Keisler measure.

Keisler measures (that is, finitely additive probability measures on a class of definable sets) generalize types in that a type can be interpreted as a $\{0, 1\}$ measure by giving measeure 0 to formulas not in the type and measure 1 tor formulas in the type. We will see more about such measures in Section 5.3.2, and the pseudofinite counting measure just described, in particular, will play a role in the discussion of the Szemeredi Regularity Lemma in Section 8.7.1.

5.4. The Lovasz Local Lemma

5.4.1. The Local Lemma and the probabilisitic Method. While there is considerable classical logical interest in zero-one laws, there are also techniques available for intermediate probabilities. Perhaps the best known example is the Lovász Local Lemma, an estimate of the probability of a certain Boolean combination of events.

The following result, now standard, was first proved in [**160**].

THEOREM 5.4.1 (Lovász Local Lemma). *Let $A_1, \ldots, A_n$ be events. Let E be the edge relation of a directed graph structure on $\{1, \ldots, n\}$ such that A_i is mutually independent of all events $\{A_j : (i,j) \notin E\}$. Finally, let $\{x_i : i \in \{1, \ldots, n\}\}$ be real numbers in the unit interval such that*

$$P(A_i) \leq x_i \prod_{(i,j)\in E} (1 - x_j).$$

Then the intersection of the complements of all of the A_i has positive probability bounded from below by $\prod_{i=1}^{n}(1 - x_i)$.

Before proceeding to a proof or to applications of this result, we should first observe the relationship of the directed graph E to the Bayesian networks of Section 2.3.1. The subtle difference in these graphs is that in Bayesian network, each A_i should be *conditionally* independent of $\{A_j : (i,j) \notin E\}$ over the events $\{A_j : (i,j) \in E\}$. Here, we disregard the conditionality — events on which A_i depends may be direct predecessors even if their influence is factored through other intermediate nodes in the network.

PROOF. We observe that $P\left(\bigwedge_{i=1}^{n} \overline{A_i}\right)$ can be computed by

$$\prod_{i=1}^{n} \left(1 - P\left(A_i | \bigwedge_{j=1}^{i-1}\right)\right).$$

We will now show that this quantity satisfies the claimed inequality.

Certainly, it holds that $P(A_i) \leq x_i$, since $(1 - x_j) < 1$ for all j. We proceed by induction to show that this estimate remains valid when the probability is conditioned on $\bigwedge_{i\in S} \overline{A_i}$ for some set $S \subseteq \{1, \ldots, n\}$. We have established, of course, that this holds for $|S| = 0$. Suppose that it holds for $|S| < s$, and separate the dependent events $S_{i,1} = \{j \in S : (i,j) \in E\}$ from the others $S_{i,2} = S - S_{i,1}$. For the immediate calculation, i is fixed, so we suppress it in our notation. We consider

$$P\left(A_i | \bigwedge_{j\in S} \overline{A_j}\right) = \frac{P\left(A_i \wedge \bigwedge_{j\in S_1} \overline{A_j} | \left(\bigwedge_{j\in S_2} \overline{A_j}\right)\right)}{P\left(\bigwedge_{j\in S_1} \overline{A_j} | \bigwedge_{j\in S_2} \overline{A_j}\right)}.$$

Now the numerator is bounded by $P\left(A_i | \bigwedge_{j\in S_2} \overline{A_j}\right)$, which, by the independence hypothesis is equal to

$$P(A_i) \leq x_i \leq x_i \prod_{(i,j)\in E} (1 - x_j).$$

We estimate the denominator by noting that it is equal to

$$\prod_{j\in S_1} \left(1 - P\left(A_j | \left(\bigwedge_{\substack{\ell\in S_1 \\ \ell<j}} \overline{A_\ell}\right) \wedge \left(\bigwedge_{\ell\in S_2} \overline{A_\ell}\right)\right)\right).$$

By induction, this quantity must be bounded from below by

$$\prod_{j\in S_1}(1-x_j)=\prod_{(i,j)\in E}(1-x_j).$$

In that case,

$$P\left(A_i|\bigwedge_{j\in S}\overline{A_j}\right)\leq\frac{x_i\prod_{(i,j)\in E}(1-x_j)}{\prod_{(i,j)\in E}(1-x_j)}=x_i.$$

Now we conclude the proof by noting that

$$\prod_{i=1}^{n}\left(1-P\left(A_i|\bigwedge_{j=1}^{i-1}\right)\right)\geq\prod_{i=1}^{n}(1-x_i).$$

□

A standard deployment of this result is in the use of the probabilistic method, which we have already seen in Section 5.1.1: to prove the existence of an object with prescribed properties (in Section 5.1.1, an expander graph) by showing that a random element of some class would have these properties with positive probability. A detailed treatment of the probabilistic method, including the Lovász Local Lemma, is found in [**18**].

COROLLARY 5.4.2 ([**413, 18**]). *Denoting by $R(k,\ell)$ the Ramsey number — that is, the least R such that every 2-coloring of the complete graph on R vertices (say, by Red and Green) contains either a red clique of k vertices or a green clique of ℓ vertices — we have $R(k,4)>k^{\frac{5}{2}+o(1)}$.*

PROOF. We randomly color the edges of K_n, coloring each edge red independently with probability p. For each set S of k vertices, let R_S be the event that every edge between elements of S is red, and for each set T of 4 elements let G_T be the event that every edge between elements of T is green. We can construct a dependency graph E, as in the theorem, by joining two events sharing an edge. We can calculate the probabilities of these two sets, as well as the necessary contributions of dependencies. For a given value of n, we calculate the probability that our random coloring produces no monochromatic cliques of the appropriate size.

The probability of R_S is $p^{\binom{k}{2}}$, and the probability of G_T is $(1-p)^{\binom{4}{2}}=(1-p)^6$. For any S of size k, there are at most $\binom{n}{k}$ other sets S_i size k such that R_S is adjacent to R_{S_i}, and there are at most $\binom{k}{2}\binom{n-2}{2}$ sets of T size 4 such that R_S is adjacent to G_T. Similarly, for any T of size 4, we have at most $\binom{n}{k}$ sets S of size k such that R_S is adjacent to G_T, and at most $\binom{4}{2}\binom{n}{2}=6\binom{n}{2}$.

In formulating the appropriate instance of the hypotheses of Theorem 5.4.1, we note that we would need

$$P(R_S)\leq x_1(1-x_1)^{\binom{n}{k}}(1-x_2)^{\binom{k}{2}\binom{n-2}{2}}$$

and

$$P(G_T)\leq x_2(1-x_1)^{\binom{n}{k}}(1-x_2)^{6\binom{n}{2}}.$$

We can achieve a choice of p,x_1,x_2 satisfying these inequalities whenever $n\leq k^{\frac{5}{2}+o(1)}$. In that case, Theorem 5.4.1 applies, producing a 2-coloring of K_n which has neither a Red k-clique nor a green 4-clique. □

5.4.2. The Computable Lovász Local Lemma. The Lovász Local Lemma is another example of the probabilistic method: it guarantees the existence of objects — indeed, a positive fraction of them — but it does not constructively give one. One direction of recent work has been to explore algorithmic aspects of this theorem.

Starting with the seminal 1991 paper of Beck, there have been algorithms developed that will, in certain cases, construct an element of the intersection of the complements of all of the events A_i in bounded time [**51**]. A seminal advance of [**348**] resulted in an algorithmic approach that covers most known applications of the Lemma. The major restriction remaining is that the general dependency graph of Theorem 5.4.1 is replaced by a slightly more restrictive system of dependency.

THEOREM 5.4.3 ([**348**]). *There is a randomized algorithm M with the following property. Let $\mathcal{P}$ be a finite set of mutually independent computable random variables, and let $A_1, \ldots, A_n$ each be a computable $\{0,1\}$-valued function of a finite number of random variables from $\mathcal{P}$. Let E be the edge relation of the directed graph structure on $\{1, \ldots, n\}$ such that A_i does not share any variables from $\mathcal{P}$ with any of the events $\{A_j : (i,j) \notin E\}$. Finally, let $\{x_i : i \in \{1, \ldots, n\}\}$ be real numbers in the unit interval such that*

$$P(A_i = 1) \leq x_i \prod_{(i,j)\in E} (1 - x_j).$$

Then M will return a set of values $\bar{p}$ for the elements of $\mathcal{P}$ such that for every i, we have $A_i(\bar{p}) = 0$.

In considering the hypotheses for this effective version of the theorem, consider the example of Corollary 5.4.2. The random variables $\mathcal{P}$ are the colors of the edges in K_n. The events R_S and G_T are, for each S and T, functions of only the edges included in S and T, and the dependencies are exactly between sets S and T that share an edge. Consequently, the Moser-Tardos Effective Local Lemma would, for each $n \leq k^{\frac{5}{2}+o(1)}$, produce a 2-coloring of K_n which has neither a red k-clique nor a green 4-clique.

PROOF OF THEOREM 5.4.3. The algorithm is as follows: First, generate a random assignment of values $\bar{p}$ to the elements of $\mathcal{P}$. Then, check to see if the assignment satisfies $A_i(\bar{p}) = 0$ for all i. If so, we are done. If not, pick the least i such that $A_i(\bar{p}) = 1$, and update $\bar{p}$ by generating a new independent random assignment of values to the variables on which A_i depends, and check satisfaction again. We continue until we find an assignment that is accepted.

Of course, if this algorithm ever halts, it will produce a satisfying evaluation of $\mathcal{P}$. Also, with probability one this algorithm will result in a satisfying assignment in *some* finite time, following what Russell Miller has whimsically called the "Are you my mother?" algorithm [**151**] (more prosaically known as exhaustive search). More practical, though, is to bound the number of "resamplings" (random reassignments of values) necessary for convergence. Moser and Tardos show that the expected number of resamplings is $\sum_{i=1}^{n} \frac{x_i}{1-x_i}$. Then, as in the proof of Proposition 4.2.6, we can use the Markov-Chebyshev Inequality to translate this expected run-time into time T by which, with high probability, M will terminate successfully.

To get this expectation for the number of resamplings, we will consider the ordered "log" of the events that we resample to correct, and, for each one, consider a tree, where each node in the gree is resampled because of its children (with the root being the resampling actually occurring in the log. We will then use the probability of appearance of each of these trees to bound the expected number of resamplings.

We now consider a random branching process (a Galton-Watson process) in which these witness trees are randomly generated.At each round, we take a vertex produced in the previous round, and, for each vertex adjacent to it in the dependency graph (counting every vertex as self-adjacent for this purpose), we add a node labeled by that vertex with probability determined by the related x.

The expected number of resamplings, then, is the sum over all possible witness trees of the probability that each one appears in the log. A routine, but laborious, calculation bounds this expectation below $\sum_{i=1}^{n} \frac{x_i}{1-x_i}$. □

In some sense, this does not fully answer the challenge: a non-deterministic algorithm is still used. While Moser and Tardos gave some conditions under which their result could be carried out by a deterministic algorithm, Chandrasekaran gave broad criteria under which one can run a deterministic version of this algorithm, giving, in some cases, a deterministic computable version of the Local Lemma. In another strengthening that has found use in computability theory, [**384, 385**] have given the following infinite computable version.

Theorem 5.4.4 ([**384, 385**]). *Suppose $\epsilon \in (0,1)$, suppose that we have a set of events $\mathcal{A} = \{A_0, A_1, \dots\}$ with graph structure E as in Theorem 5.4.3, and there is a uniformly computable function $x : \mathcal{A} \to (0,1)$ such that for each $A \in \mathcal{A}$, we have*

$$P(A_i) \leq (1-\epsilon)x(A_i) \prod_{(i,j)\in E} (1 - x(A_j)).$$

Then there exists a computable function σ assigning values to every variable occurring in $\mathcal{A}$ in such a way that every A_i evaluates to zero.

These effective versions of the Lovász Local Lemma became imporant in the investigation of a problem in reverse mathematics that is not obviously linked to probability. Work in the program of reverse mathematics takes some (true) combinatorial principle and attempts to classify it by its "proof-theoretic strength;" that is, over some weak base theory T_0, typically a fragment of second-order arithmetic, to situate it in an equivalence class of theorems under the relation

$$(\varphi_1 \sim \varphi_2) \Leftrightarrow ((T_0, \varphi_1 \vdash \varphi_2) \wedge (T_0, \varphi_2 \vdash \varphi_1)).$$

Hindman's theorem is a standard deterministic result in colorings, stating that for every coloring of $\mathbb{N}$ with finitely many colors, there is an infinite set H such that all nonempty sums of distinct elements of H have the same color. A natural weakening of this principle involves restricting the statement to sums of a specific number or range of distinct elements. The case of sums of exactly two elements was particularly difficult to classify in the system of reverse mathematics, and the eventual solution, published in [**113**], makes use of Rumyantsev and Shen's effective Lovász Local Lemma.

Proposition 5.4.5 ([**113**]). *There exists a computable coloring of $\mathbb{N}$ with exactly two colors, such that there is no computable infinite set H in which the set $\{(a+b) : a \neq b \in H\}$ is not homogeneous.*

PROOF. We consider an enumeration $\{W_i : i \in \mathbb{N}\}$ of the computably enumerable sets. We pick appropriate large values of k_i for each i, and set $E_i = W_{i,k_i}$. Now for each $s \in \mathbb{N}$, the event that $E_i + s$ is homogeneous has low probability and is independent of most other events of the form "$E_j + t$ is homogeneous." We then use Theorem 5.4.4 to produce a coloring avoiding all of these events. □

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