Mathematical Logic and Probability

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Preface

In the late 19th and early 20th centuries, logic and probability were frequently treated as closely related disciplines. Each has, in an important sense, gone its own way, so that neither, in its modern form, is in any proper sense a systematization of the "Laws of Thought," as Boole called them.

However, the last four decades have seen a remarkable rapproachment. On the most obvious level, the various probability logics have developed as formal systems of reasoning in the modern sense of logic.

At a deeper level, though, attempts have been made to formulate logics in which model theory of random variables, stochastic processes, and randomized structures can be explored from the perspective of model theory. Continuous first-order logic as a context for stability theory on metric structures is perhaps the most conspicuous example, but others exist.

At the same time, algorithmic randomness in its various forms has come to play a core role in computability theory, while probabilistic computation of various kinds (randomized computation, interactive proofs, and others) has come to dominate major parts of computational complexity. The older recursion-theoretic program of machine learning, initiated by Gold in the 1960s, has become much more important thanks to Valiant's reformulation in probabilistic terms to allow for reasonable errors.

The model theory of random objects, Fraïssé limits, and pseudofinite structures, each of which embodies some important aspect of 0-1 laws, has been important for longer, but advances in stability, simplicity, and the transition from finite to infinite model theory have enriched this subject.

In set theory, too, the study of dynamics that respect probability measures has played a central role in the study of equivalence relations. Probability is frequently at the center of modern descriptive set theory.

Nor have these developments been independent. The PAC learning theory of Valiant is inextricably linked to the model theory of NIP theories. The dynamics of computable Polish spaces have become an important emerging area in computability. Randomized computation is the natural computation on metric structures. Notions of random structures have become intertwined with algorithmic randomness, and are naturally described in continuous first order logic.

Many of these developments have been adequately treated in isolation by various books. Probability logic has been discussed at length from various perspectives in [**10, 239, 242, 394, 419**]. Bayesian networks are well-covered, for instance, in [**219, 397, 398**], and a monograph on adapted distributions also exists [**180**]. Randomized computation has a detailed treatment in [**30**]. Algorithmic randomness is the subject of three relatively recent books, [**158, 387, 194**]. Zero-one laws are treated at length in [**165, 230**], and other places, and [**266**] includes an extended treatment of Fraïssé limits. Random graphs are extensively covered in [**74, 125, 338**]. The definitive reference on PAC learning is [**297**]. In the field of set-theoretic dynamics, there have been several treatments at several levels of detail, among which [**55, 260, 298, 303**] merit special mention. There is no shortage of book-length treatments of subjects within the range of this book.

However, a reader in a well-stocked library might well pass all these separate books without knowing that they had anything substantive in common. Indeed, one could read most of them in detail — in addition to the long papers that give strong expositions on many related subjects (the seminal paper [**61**] on continuous first-order logic comes to mind) — without finding a commonality.

It is true that [**238**] describes connections between probability logics and Bayesian networks. However, it is silent on the rest of these issues.

The present book, then, attempts to take a unified — or, at least, unifying — approach to this subject. The expanding literature in each of these fields has seen more interaction between them, so that a model theorist might well want to know more about the frontier of probabilistic work in set theory, or a computability theorist more about the relevant work in model theory.

We focus here on *mathematical* logic and probability. Probability logic and its relatives seem frequently to arise as works of *philosophical* logic, and this has implications for the questions that are asked about it. Frequently it is seen in connection with the theory of rational decision, as in [**242**]. Mathematical logic, by contrast, asks about computability and undecidability; about theories and their models; about reducibilities and regularity of sets. Alternate logics are of interest to mathematical logic inasmuch as they provide the necessary infrastructure for carrying out this program in interesting settings. Applications of logic to artificial intelligence and other modeling contexts are important, but they arise as applications of the theory, not as its defining elements.

Chapter 1 begins to lay out the central thesis of the book: that all the other chapters have something to say to one another. This is done by identifying several important cross-cutting themes that come up in several of the other chapters.

In the next chapter, we begin the technical section of the book by describing the various logics useful for probability. Continuous first-order logic has a central role, not least because it generalizes many others. Probability logic is extensively studied, and is explored here as well, as are some other approaches.

In a third chapter, we will consider the theory of algorithmic randomness, with special attention to normal numbers, Martin-Löf randomness, and their relation to computation. This treatment will not be complete, of course — the subject is well-covered elsewhere. Rather, the focus will be on those aspects of algorithmic randomness that interact with other areas of advance in the logic and probability community.

The chapter on randomized computation involves the leap of reasoning that computability and complexity still have something to say to one another. Recent work on generic and coarse computability, as well as that on derandomization, descriptive complexity, and continuous first-order logic support this hypothesis.

The following two chapters will take up the various approaches to random structures. The investigation of random structures seems to have arisen historically from the study of random graphs, which invited generalization to 0-1 laws, and which connected with the earlier beginnings of Fraïssé limits. More recent approaches consider the "random" structure as a single structure that somehow embodies the possible variation — graphons, Keisler randomizations, invariant random subgroups, and the like. Others use algorithmic randomness to define the structure.

In taking up the problem of learning theory, there is a fair viewpoint from which learning after the tradition of Gold, probably approximately correct (PAC) learning after the tradition of Valiant, and the model theory of NIP structures are wildly different fields. The chapter devoted to these topics takes the opposite view. Valiant's definition is a natural extension of Gold's framework, and the theory of Vapnik-Chervonenkis dimension governs both PAC learning and NIP theories.

The final chapter surveys the general area of dynamics. An introduction to orbit equivalence relations and Borel cardinality is given, and several topics on the relation of measure to equivalence relations are considered, including the implications of ergodicity and Hjorth's notion of turbulence. Recent model-theoretic approaches to Szemeredi Regularity and Furstenberg Correspondence belong here, too, as does the characterization of 1-randomness by the Ergodic Theorem and the emerging theory of computable Polish spaces.

Of course, some limits must be set on the content of such a book. For instance, a new line of thought has arisen in recent years over categorical treatments of probability [**199, 200, 395**]. In view of traditional [**350**] and recent [**119, 248, 247**] work on connections between category theory and logic, this work is certainly interesting and relevant, but it is hard, at this stage of the theory, to explain its relationship to the other work.

The book is to be formally self-contained, but realistically anticipates a reader who has completed a first course in logic at the graduate or upper undergraduate level. Such a reader will, after reading the book, be prepared to understand the frontier of the research literature in probability-related areas of computability, model theory, set theory, and logical aspects of artificial intelligence. There is an important place in the world for a reader equipped in this way: A major part of logic in the coming years will involve connections between these fields, and those who understand something of all of them will be well-poised to contribute.

CHAPTER 3

Random Sequences

Of the topics covered by chapters in this book, the subject of the present chapter — often called algorithmic randomness — has perhaps had the most comprehensive and recent treatment in other volumes. The goal here will be to draw attention to some points that seem to have received less attention and to emphasize connections with the other subjects of the present book. The canonical comprehensive treatments of the modern theory of algorithmic randomness are [**158**] and [**387**], while [**193**] provides an update on directions of research that have come to greater prominence in the decade since those two comprehensive treatments were published. The present section will draw heavily [**158**] and [**193**], and all three of these volumes should be the first point of reference for readers interested in pursuing most of these subjects further.

3.1. Normal Sequences

A notable exception to the comprehensiveness of the three books just mentioned is the theory of normal numbers. Indeed, there is good ground for debate on whether they deserve the name of "random" at all, as we shall see. However, there is at least a continuity of thought between normal numbers and the various classes of algorithmically random numbers, narrowly construed, and they have seen recent activity in mathematical logic communities. A good reference on normal numbers from a rather different perspective is [**90**].

3.1.1. Popper and Randomness. Normal numbers were introduced in [**80**], but in some sense the more basic concept is the free sequence. A normal number is a real number whose expansions in various bases constitute a free sequence.

Free sequences are described in Popper's *Logic of Scientific Discovery* as part of his effort to explain the foundations of probability [**407**]. His overall program was to describe the method by which scientific knowledge could be justified.

His view was that a hypothesis would be "put up tentatively," and conclusions deduced from it. It might then be tested empirically by observing evidence for or against the conclusions deduced from the hypothesis. An important aspect of this program is a description of how a statement of probability "can be explicitly tested and corroborated."

Following von Mises' development of the foundations of probability, Popper attempted to describe what it would mean to hypothesize that a given sequence of observations approximates "empirical sequences of a chance-like or random character."

Whatever a sequence of random character might be, we at least sympathize with comic strip character Dilbert. A random number generator is introduced to him as it reports its random sequence "Nine nine nine nine nine nine nine. . . ."

"Are you sure that's random?" Dilbert asks [**12**]. To be sure, whatever random means, a constant sequence can't be it. One way to explain this intuition is to say that the next element of a constant sequence is guaranteed to have a particular value.

Popper suggests that this can be extended. We also recognize

$$(0, 1, 0, 1, 0, 1 \dots)$$

as a non-random sequence. Although $P(x_{n+1} = 1) = \frac{1}{2}$, suggesting a superficial randomness, we can remove this apparent randomness by conditioning on the previous bit. Indeed, $P(x_{n+1} = 1 | x_{n-1})$ is either 1 or 0, depending on the value of x_{n-1}. The same applies to the sequence

$$(0, 0, 1, 1, 0, 0, 1, 1, \dots)$$

with conditioning on the previous two elements. This leads to a definition.

Definition 3.1.1. Let $\mathbb{X} := (x_i : i \in \mathbb{N})$ be a sequence in Σ^ω for some alphabet Σ.

(1) For each $n, k \in \mathbb{N}$ and each $\bar{\ell} \in \Sigma^n$, we define a function $p_k^{\bar{\ell}}(s)$ by

$$p_k^{\bar{\ell}}(s) := \frac{\left|\left\{j \leq k : x_j = s \wedge \bigwedge_{i \leq n} x_{j-i} = \ell_i\right\}\right|}{\left|\left\{j \leq k : \bigwedge_{i \leq n} x_{j-1} = \ell_i\right\}\right|}.$$

(2) We say that $\mathbb{X}$ is *n-free* if and only if for every $\bar{\ell} \in \Sigma^n$ and every $s \in \Sigma$, we the formula

$$\lim_{k \to \infty} p_k^{\bar{\ell}}$$

gives a uniform distribution on Σ.

So in the prior examples, we see that $(0, 1, 0, 1, \dots) \in 2^\omega$ is 0-free but not 1-free, and $(0, 0, 1, 1, 0, 0, 1, 1, \dots)$ is 0-free and 1-free, but not 2-free. In general, an n-free sequence can be constructed by taking the set of all distinct elements $\sigma_1, \dots, \sigma_k \in \Sigma^{n+1}$, concatenating them in any order, and repeating that finite sequence infinitely. Popper points out that any periodic n-free sequence must have period at least 2^{n+1}. The natural limit, then, is a sequence that is n-free for all n.

Definition 3.1.2. We say that a sequence $\mathbb{X}$ is *absolutely free* if and only if it is n-free for every $n \in \mathbb{N}$.

One argument against understanding these sequences as "random" is the fact that there are absolutely free sequences $\mathbb{X}$ such that there is an algorithm to compute, for each n, the value of x_n. Indeed, the kind of "unpredictability" required is quite rigid: no look-back window *of fixed length* should allow a favorable chance of guessing the next element. This does not prevent other methods of guessing, including a prediction of x_n based on $(x_1, \dots, x_{n-1})$, or even a prediction of x_n based solely on n, with no reference to the previous entries.

Theorem 3.1.3 ([**407**]). *There is a computable absolutely free sequence.*

Proof. We have already shown, for any n, how to generate a sequence τ_n of length $n' > n$ which, if repeated infinitely often, is n-free. Moreover, this sequence can be chosen in such a way as to begin with any sequence $\sigma \in \Sigma^n$. There are many ways to do this effectively, but for concreteness, we could list the $n+1$-tuples in lexicographical order.

For some arbitrary $n \in \mathbb{N}$, we generate such a τ_n. Then we pick some $n_1 \geq n'$ and form an n_1-free τ_{n_1} of length n'_1 whose first n' elements are τ_n. For each $i > 1$, we pick some $n_{i+1} > n'_i$, and form an n_{i+1}-free $\tau_{n_{i+1}}$ of length n'_{i+1}. For any fixed t, this process will converge on the tth element of the sequence. □

We will see in the next section methods to control the complexity of this computation. Nevertheless, there are computational limitations.

Definition 3.1.4 ([**18**]). Let Σ be an alphabet.

(1) A *deterministic finite automaton with output* is a quadruple $(Q, \Sigma, \delta, q_0, \tau)$, where Q is a finite set of states q_0 is an initial state, $\delta : Q \times \Sigma \to Q$, and $\tau : Q \to \Sigma$.
(2) We say that a sequence $\mathbb{X} \in \Sigma^\omega$ is *automatic* if and only if there is a deterministic finite automata with output such that if w_n is a representation of n in Σ (say, a base k expansion where $|\Sigma| = k$) and w' is the reverse of w, then $x_n = \tau(\delta(q_0, w'))$.

Clearly every automatic sequence is computable, but the reverse fails.

Theorem 3.1.5. *No automatic sequence is absolutely free.*

Proof. In an automatic sequence, the number of distinct subsequences of length n is bounded by $O(n)$ (see Corollary 10.3.2 of [**18**]). Since an absolutely free sequence must have more distinct subsequences, it cannot be automatic. □

3.1.2. The Problem of Bases, and Normal Numbers. Of course, we can treat a sequence of numbers as a single real number. For instance, a sequence of numbers, each at most k, can be read as a base-k representation. In this sense, we could think of n-free and absolutely free real numbers. There is, however, a problem of robustness in this approach, in that the same real number may be absolutely free in one base and not absolutely free in another base.

Definition 3.1.6 ([**80**]). Let x be a real number and $b \in \mathbb{N}$.

(1) We say that x is *simply normal to base b* if and only if, in the base b representation of x, each symbol from $\{0, 1, \dots, b-1\}$ occurs with limiting probability $\frac{1}{b}$.
(2) We say that x is *normal to base b* if and only if, in the base b representation of x, for each $\ell \in \mathbb{N}$, every sequence in $\{0, 1, \dots, b-1\}^\ell$ occurs with identical limiting probability.

It is straightforward to observe that x is normal to base b if and only if its base b expansion is an absolutely free sequence. In this sense, we have shown that there exist, for each b, computable numbers normal to base b.

Proposition 3.1.7 ([**310**]). *Given $b \in \mathbb{N}$, the set of real numbers that are normal to base b is $\mathbf{\Pi^0_3}$ complete.*

Proof. The $\mathbf{\Pi^0_3}$ definition of normality is straightforward. On the other hand, let $x = \sum_{i=1}^{\infty} \frac{\alpha_i}{b^i}$ be normal to base b, and let $I_0(x)$ be the set of i such that $\alpha_i = 0$.

We now denote by $k(i)$ the position of i in an increasing enumeration of I_0, and we define a function on Cantor space $T : 2^\omega \to \mathbb{R}$ by setting, for each $\sigma \in 2^\omega$,

$$t(\alpha_i) := \begin{cases} 1 & \text{if } i \in I_0(x) \text{ and } \sigma(k(i)) = 1 \\ \alpha_i & \text{otherwise} \end{cases}.$$

We define $T(\sigma) := \sum_{i=1} \infty \frac{t(\alpha_i)}{b^i}$.

Now we consider cerain properties of σ. We define the density of σ, denoted $\delta(\sigma)$ to be

$$\delta(\sigma) := \lim_{i\to\infty} \frac{|\{n \leq i : \sigma(n) = 1\}|}{i}.$$

If $\delta(\sigma) = 0$, then $T(\sigma)$ is normal to base b. Otherwise, $T(\sigma)$ is not even simply nromal to base b. It now suffices to show that the set of σ with density zero is $\mathbf{\Pi^0_3}$ complete, which turns out to be true. □

The property of normality is dependent on the base.

THEOREM 3.1.8 ([**365, 433**]). *Let $b_1 \in \mathbb{N}$ and $x \in \mathbb{R}$. Then x is normal to base b_1 exactly when x is normal to every base $b_2 = b_1^q$ with $q \in \mathbb{Q}$. Moreover, if $B \subseteq \mathbb{N}$ is (relative to $\mathbb{N}$) closed under rational powers, then B can occur as the set of bases to which a real number is normal.*

PROOF. We give a proof of the first assertion of the theorem, for which it suffices to prove that x is normal to base b_1 exactly when x is normal to base b_1^r for some integer r. One side of this implication is straightforward from the definition of normality (as opposed to simple normality).

Suppose x is normal to base b_1^r. Now for any integer t, we have x normal to base b_1^{tr}, so that x is simply normal to base t, as is explained in detail in [**90**]. □

This result can be extended to simple normality. In the following theorem, it will be helpful to recall the definition of Hausdorff dimension. This is described, for instance in [**184**]. It suffices for the present purpose to give the definition for subsets of $\mathbb{R}$.

DEFINITION 3.1.9. Let $p, \delta \geq 0$, and $A \subseteq \mathbb{R}$.

(1) We define $H_{p,\delta}(A)$ to be $\inf \left\{ \sum_{i=1}^{\infty} r_j^p \right\}$, where A is covered by a union of balls $\bigcup_{i=1}^{\infty} B_{r_i}(x_i)$, with all $r_i \leq \delta$, and the infimum is taken over all such collections of balls.
(2) The p-dimensional Hausdorff measure of A, denoted $H_p(A)$, is given by $\lim_{\delta\to 0} H_{p,\delta}(A)$.
(3) The Hausdorff dimension of A is given by the infimum of the set of all nonnegative p such that $H_p(A) = 0$.

If A is a singleton, for instance, or even a discrete set of points in $\mathbb{R}$, then for any $p, \delta \geq 0$, we have $H_{p,\delta}(A) = 0$, so that the Hausdorff dimension of A is 0. On the other hand, the Hausdorff dimension of $\mathbb{R}$ is 1. It is a slightly more involved exercise to show that the Hausdorff dimension of the usual middle-thirds Cantor set is $\log_3 2$. We have the following result on the ubiquity of numbers simply normal to some bases but not to others.

THEOREM 3.1.10 ([**50**]). *Let S be the set of positive integers that are not perfect powers, and let $M : F \to \mathcal{P}(\mathbb{Z}^)+$ such that for every $s \in S$, the following hold:*

(1) *If $m \in M(s)$ and $d|m$, then $d \in M(S)$*
(2) *If $M(s)$ is infinite, then it is equal to $\mathbb{Z}^+$.*

Then there is a nonempty set $X \subset \mathbb{R}$ with the following properties:

(1) *X has Hausdorff dimension 1, and*
(2) *For every $s \in S$ and every $m \in \mathbb{Z}^+$, every $x \in X$ is simply normal to base s^m if and only if $m \in M(s)$.*

PROOF. We first show the nonemptiness of X by constructing $x \in \mathbb{R}$ satisfying the necessary properties. If $M(s)$ is infinite for every $s \in S$, then the result will follow from Theorem 3.1.13, so we assume that there is some s such that $M(s)$ is finite. In that case, we define sequences $(n_j : j \in \mathbb{N})$, $(r_j : j \in \mathbb{N})$, $(s_j : j \in \mathbb{N})$, $(s_j^* : j \in \mathbb{N})$, $(\ell_j : j \in \mathbb{N})$, $(U_j : j \in \mathbb{N})$, $(p_j : j \in \mathbb{N})$, to satisfy several conditions. In many cases the satisfiability of these conditions is nontrivial, but we offer this outline.

We will use the term *balanced*. In particular, for a word σ of length ℓ and a finite set V, with $v \in V$, we define $occ(w, v)$ to be the number of occurrances in w of v. We then define

$$D_-(w, V) = \max\left\{\left|\frac{occ(w,v)}{\ell} - \frac{1}{|V|}\right| : v \in V\right\}.$$

DEFINITION 3.1.11. We define *balanced* in the following senses:

(1) A word $\sigma \in V^\ell$ is said to be balanced for an integer m if
  (a) ℓ is a multiple of m, and
  (b) The sequence τ of blocks of length m whose concatenation is the longest initial segment of σ whose length is a multiple of m has the property that $D_-(\tau, V^\ell) = 0$.
(2) A string σ is said to be balanced for a set M if and only if is balanced for every element of M.
(3) A set $W \subseteq V^\ell$ is said to be balanced for a set M if and only if
  (a) ℓ is a multiple of each element of M, and
  (b) The concatenation of the elements of W is balanced for M.

We construct the sequences to satisfy the following conditions.

(1) s_j is an element of S.
(2) $M(s_j)$ is finite.
(3) $n_j \notin M(s_j)$.
(4) Every pair (n, s) appearing as (n_j, s_j) for some j appears for infinitely many j.
(5) $(r_j : j \in \mathbb{N})$ enumerates $\{s^m : s \in S, m \in M(s)\}$ in increasing order.
(6) $s_j^* = s_j^{\ell_j}$.
(7) U_j is a set of strings of length ℓ_j.
(8) U_j is balanced for $M(s_j)$
(9) U_j is not balanced for n_j
(10) If s_j is odd, then U_j is the set of ℓ_j-tuples of elements from $\{0, \dots, s_{j-1}\}$ except the even singletons.
(11) If s_j is even, then U_j is the set of ℓ_j-tuples of elements form $\{0, \dots, s_{j-1}\}$ except the pairs (a, b) with $a < b$ and a even and b odd.

(12) There is $\epsilon_j > 0$ and a sequence d_j of length n_j from $\{0, \ldots, s_j\}$ such that for any $\delta_j > 0$ there is a positive $\hat{\ell}_j$ such that for $\ell > \hat{\ell}_j$, the number of sequences u of length $\ell\ell_j$ from $\{0, \ldots, s_j\}$ with d_j occurring in $u \restriction_{n_j}$ strictly less than $\frac{1}{s_j^n} - \epsilon_j$ is at least $(1-\delta)|U_j|^\ell$.

(13) p_j is the least positive integer with the property that $r_k^{p_j} \geq 2(j+1)$ for each $k \leq j$.

We now construct a sequence of approximations $(x_t : t \in \mathbb{N})$, and x will be defined by the fact that $x \in \left[x_t, x_t + (s^*_{j_t})^{p_t}\right)$, for appropriate sequences j_t and p_t. We will have a function $\ell : \mathbb{N} \to \mathbb{N}$ that is specified in the paper's full treatment of this proof, but that we do not specify here. Let $z(j, a, y)$ be the least number shuch that there is a sequence of length $\lceil a + \ell(j)/\ln s^*_j \rceil$.

Letting y be the least number $\frac{k}{s_{j_{t+1}}}^\alpha > x_t$, and choose an appropriate a. Then we can define $x_{t+1} = z(j_{t+1}, a, y)$.

The Hausdorff dimension of the set X can be calculated by means of a careful analysis of this construction, in combination with a result of [**167**]. □

It is in some sense a failure of canonicity in the definition of normality that numbers can be normal to one base but not to others, and that such examples are in some sense the norm. To remove this dependency, Borel described another standard of normality.

Definition 3.1.12. We say that a real number x is *absolutely normal* if and only if it is normal to every base.

Borel observed that almost all numbers are absolutely normal, but did not explicitly give a proof. There are several proofs available, but we defer a proof until a later section, in which we prove that almost all real numbers have a stronger property that implies absolute normality.

Theorem 3.1.13 (Borel). *Almost all numbers are absolutely normal.*

Both Borel and Sierpinski proposed a relationship between normality and irrationality. Of course, rational numbers cannot be absolutely normal. The normality of e and π are well-known open questions.

The study of transcendental numbers has a (generally well-deserved) reputation for difficulty, but one class of transcendentals that has been well-explored is the Liouville numbers [**40**].

Definition 3.1.14. Let x be a real number. Then x is said to be Liouville if and only if there is a sequence $\left(\frac{p_n}{q_n} : n \in \mathbb{N}\right)$ of rational numbers and a sequence $(\omega_n : n \in \mathbb{N})$ such that the following hold:

(1) $\limsup \omega_n = \infty$
(2) $\left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{\omega_n}}$.

In particular, a Liouville number witnessed by $\left(\frac{p_n}{b^n} : n \in \mathbb{N}\right)$ will have arbitrarily long sequences of zeroes in its base b representation, making that number very far from being simply normal to base b. We say that such a number is *Liouville to base b*.

Problem 3.1.15 (Slaman). Is there a number which is normal to base 2 but Liouville to base 3?

3.1.3. Computability of Normal Numbers. Writing considerably before the advent of a mathematical approach to the theory of algorithms, Borel observed that in the contemporary state of knowledge, the "effective determination" of an absolutely normal number seemed very difficult — he even proposed that it would be interesting either to do so or to prove that any number that can be "really defined" must fail to be absolutely normal — remarkably prescient in being much more open to algorithmic unsolvability than either Hilbert or Dehn at about the same time. The problem was first solved by Sierpinski in 1917 by demonstrating a computable absolutely normal real number, in the course of giving an elementary proof of Theorem 3.1.13.

There has been considerable work since Sierpinski in the possibilities for effectiveness in normal numbers, and [**430**] provides a recent survey. We have already seem that no absolutely free sequence is automatic. It follows then, that the digits of an absolutely normal number, in whatever base, cannot constitute an automatic sequence. In fact, more is true. We begin by describing a notion of compression.

Definition 3.1.16. We describe the action of a finite-state compressor.

(1) A *finite-state compressor* is a sextuple $\mathcal{C} = (\mathcal{A}, \mathcal{B}, \mathcal{Q}, q_0, \delta, o)$, where $\mathcal{A}$ and $\mathcal{B}$ are alphabets, $\mathcal{Q}$ is a finite set of states, $q_0 \in \mathcal{Q}$ is the initial state, $\delta : \mathcal{Q} \times \mathcal{A} \to \mathcal{Q}$ is the transition function, and $o : \mathcal{Q} \times \mathcal{A} \to \mathcal{B}^*$ generates an output.
(2) Any finite-state compressor, as above, induces functions $\delta^* : \mathcal{Q} \times \mathcal{A}^* \to \mathcal{Q}$ and $o : \mathcal{Q} \times \mathcal{A}^* \to \mathcal{B}^*$ in the obvious way (by composition in δ and by concatenation in o).
(3) $\mathcal{C}$ is said to be *lossless* if and only if the mapping $f : \mathcal{A}^* \to \mathcal{Q} \times \mathcal{B}^*$ given by $f(\sigma) = \langle o^*(q_0, \sigma), \delta^*(q_0, \sigma)\rangle$ is injective.

The name "compressor" arises because of the following calculation.

Definition 3.1.17. We describe the compression of strings by finite-state compressors.

(1) The *compression ratio* for a finite state compressor $\mathcal{C}$ on a finite string $\sigma \in \mathcal{A}^*$, denoted by $\rho_{\mathcal{C}}(\sigma)$, is given by the output length divided by $|\sigma| \log_{|\mathcal{B}|} |\mathcal{A}|$, a standard optimal coding of σ in $\mathcal{B}$.
(2) The compression ratio $\rho_{\mathcal{C}}(\sigma)$ for an infinite string $\sigma \in \mathcal{A}^\omega$ is given by $\liminf_{n\to\infty} \rho_C(\sigma \upharpoonright_n)$.
(3) We say that an infinite string $\sigma \in \mathcal{A}^\omega$ is *compressible* if and only if there is a lossless finite-state compressor $\mathcal{C}$ with $\rho_{\mathcal{C}}(\sigma) < 1$. We say that it is *incompressible* otherwise.

From this perspective, the following result gives limits on the computation by automata of normal numbers.

Theorem 3.1.18 ([**52**]). *A real number is normal to base b if and only if its base b expansion is incompressible.*

Proof. Let x be normal to base b, and let $\mathcal{A} = \{0, \ldots, b-1\}$. Let $\mathcal{C} = (\mathcal{A}, \mathcal{B}, \mathcal{Q}, q_0, \delta, o)$ be a lossless compressor, and $\epsilon > 0$.

For each $\sigma \in \mathcal{A}^*$, we set $a_\sigma = \min_{q\in\mathcal{Q}} |o^*(q, \sigma)|$. For $n \in \mathbb{N}$, we define the set $\mathcal{S}_n := \left\{\sigma \in \mathcal{A}^n : a_\sigma > (1-\epsilon) n \log_{|\mathcal{B}|} |\mathcal{A}|\right\}$, and the proof, in the end, will consist of a lower bound on this set's cardinality. By counting arguments, we can see that

$$|\mathcal{S}_n| > |\mathcal{A}^n| - |\mathcal{B}||\mathcal{Q}|^2|\mathcal{A}|^{(1-\epsilon)n}.$$

We can naturally replace $\mathcal{C}$ by $\mathcal{C}^n$ (with output function $o^{n,*}$, which accepts inputs from $\mathcal{A}^n$, and can view x as a sequence $\hat{x}$ of elements of $\mathcal{A}^n$. By the simle normality of $\hat{x}$, we can show that

$$\rho_{\mathcal{C}^n}(\hat{x}) = \liminf_{k\to\infty} \frac{|o^{n*}(q_0, \hat{x}\restriction_k)|}{k \log_{|\mathcal{B}|} |\mathcal{A}^n|} \geq (1-\epsilon)^3.$$

It follows that $\rho_C(x) \geq (1-\epsilon)^3$. Since this holds for every ϵ, we conclude that x is incompressible.

The proof of the converse is also elementary. □

Further work in [**51**] describes enhancements to a finite state transducers that would allow compression of normal numbers, leaving open the following problem.

PROBLEM 3.1.19. Can a deterministic push-down transducer compress a normal word?

Beyond the world of automata, much of the work on effective normal numbers seems to center on tradeoffs between computational complexity and the speed of convergence to normality. The latter is measured in terms of the *discrepancy.*

DEFINITION 3.1.20. Let $\mathbb{X}$ be a sequence of real numbers.
We define the *discrepancy* of $\mathbb{X}$ to be the quantity

$$D_N(\mathbb{X}) = \sup_{I=(i_1,i_2)} \left| \frac{|\{n \in \{1,\ldots,N\} : x_n \mod 1 \in I\}|}{N} - (i_2 - i_1) \right|$$

where I ranges over all open subintervals of $[0,1)$.
We say that $\mathbb{X}$ is *uniformly distributed modulo 1* if and only if $\lim_{N\to infty} D_N(\mathbb{X}) = 0$.

Now a real number x is normal to base b if and only if $\mathbb{B}x := (b^n x : n \in \mathbb{N})$ is uniformly distributed modulo 1. In that sense, we can consider the discrepancy of this sequence $D_{\mathbb{B}x}(N)$ as a function of N, and use this function as a measure of the speed at which x "converges to normality."

THEOREM 3.1.21 ([**434**]). *Let $\mathbb{X}$ be a sequence of real numbers. Then there is a positive constant c such that for infinitely many N, we have $D_{\mathbb{X}}(N) \geq c\frac{\log N}{N}$.*

The strongest possible result on computable normal numbers, then, would be the construction of a real number x such that $D_{\mathcal{B}x}(N) = O\left(\frac{\log N}{N}\right)$. The best result currently known is the following.

THEOREM 3.1.22 ([**334**]). *There is a computable number which is normal to base b with discrepancy $O\left(\frac{\log\log N}{N}\right)$.*

From the perspective of convergence to normality, Theorem 3.1.13 can be made more specific.

THEOREM 3.1.23 ([**206**]). *For almost every $x \in \mathbb{R}$, for every integer $b > 1$, the sequence $\mathbb{B}x$ has discrepancy $O\left(\sqrt{\frac{\log\log N}{N}}\right)$.*

The problem of an optimal construction remains open. A survey of the recent results can be found in [**430**].

PROBLEM 3.1.24. Does there exist a computable normal number x with $D_{\mathcal{B}x}(N) = O\left(\frac{\log N}{N}\right)$?

3.2. Martin-Löf Randomness

3.2.1. Kolmogorov Complexity. We begin our study of the more traditionally considered forms of algorithmic randomness with the idea of compressibility. This compressibility is in some ways similar to that of the previous section, but uses Turing machines instead of finite state transducers. We define a $\mathbb{N}$-valued function C on strings, and several variants.

DEFINITION 3.2.1. Denote by $2^{<\omega}$ the finite binary strings.

(1) For any $f : 2^{<\omega} \to 2^{<\omega}$, we define $C_f : 2^{<\omega} \to \mathbb{N}$ by
$$C_f(\sigma) := \min\{|\tau| : f(\tau) = \eta(\sigma)\}.$$
(2) Let U be a universal Turing machine. Then $C_U = C_f$ where f is the function computed by U.

We note that for any partial computable function f, the quantity $C_U(\sigma)$ is bounded by $C_f(\sigma) + k_f$, where k_f is a constant depending on f but not on σ. In that sense, a universal quantity $C(\sigma)$ can be defined up to an additive constant by defining $C = C_U$ for some universal machine U. This quantity is sometimes called the *plain Kolmogorov complexity*, to distinguish it from the prefix-free Kolmogorov complexity we describe later. We can similarly define a conditional version of plain Kolmogorov complexity.

DEFINITION 3.2.2. Let $f : (2^{<\omega})^2 \to 2^{<\omega}$. Then we define
$$C_f(\sigma|\eta) = \min\{|\tau| : f(\eta, \tau) = \sigma\}.$$

The definitive reference on Kolmogorov complexity is [**336**]. We will frequently describe an infinite binary sequence (equivalently, a real number), and will consider $C(X \restriction_n)$ as a function of n. Since this quantity is well-defined only up to an additive constant (corresponding to the choice of universal machine), essentially all formulas involving Kolmogorov complexity will include a term of $\pm O(1)$. We can also see that $C(X \restriction_n) \leq n - O(1)$, since it suffices to simply list the elements $X(0), \ldots, X(n-1)$.

In the spirit of the deficiency concept of the previous section, we also have a notion of deficiency here.

DEFINITION 3.2.3. The *randomness deficiency of σ relative to A* is defined by
$$\delta(\sigma|A) = \ell(A) - C(\sigma|A)$$
where $\ell(A)$ is the length of the cardinality of A.

An important property of randomness deficiency is that there are few strings with high deficiency.

THEOREM 3.2.4. *Let $A \subseteq \mathbb{N}$. Then for any $k \in \mathbb{N}$,*
$$|\{x : \delta(x|A) \geq k\}| \leq \frac{|A|}{2^{k-1}}.$$

PROOF. Consider the set $E_1 := \{\tau \in 2^{<\omega} : |\tau| \leq \lambda\}$. Of course, $|E| \leq 2^{\lambda+1}$. Consequently, we can bound, for a fixed A, the number of strings σ with $C(\sigma|A) \leq \lambda$ by $2^{\lambda+1}$.

To have $\ell(A) - C(\sigma|A) \geq k$, then, we must have $C(\sigma|A) \leq \ell(A) - k$. The set of all such strings must have cardinality at most $2^{\ell(A)-k+1} = 2^{\ell(A)}2^{-(k-1)} = \frac{|A|}{2^{k-1}}$. □

Computable sequences $\sigma \in 2^{\mathbb{N}}$ are an important benchmark case. Since the (finite) index for a Turing machine computing σ constitutes a description of $\sigma \upharpoonright_n$ for all $n \in \mathbb{N}$, we have $C(\sigma \upharpoonright_n \lesssim n - O(1)$ for sufficiently large n.

At the level of intuition, Kolmogorov complexity has significant similarity to Shannon's entropy. This measure was introduced in [**441**], as a measure of the information content of a string, and is well-attested in the literature. A good modern reference is [**78**].

DEFINITION 3.2.5. Let σ be a finite string of elements from $\{1, \ldots, m\}$. We define the *entropy of* σ by

$$H_m(\sigma) = \sum_{i=1}^{m} p_i \log_2 p_i,$$

where p_i is the frequency of occurrance in σ of the element i.

A more complete discussion of entropy in its many forms — including a derivation of the form of this definition from reasonable hypotheses — can be found in Section 6.1.3. Indeed, there is a strong quantitative relationship between the two properties.

THEOREM 3.2.6. *Let $\sigma \in 2^{<\omega}$ be a concatenation $\tau_1\tau_2\cdots\tau_k$ where each τ_i is a string of length n, so that σ can be interpreted as a string of natural numbers $\hat{\tau}_i = \sum_{j=0}^{n} \tau_i(j)2^j$. We then have*

$$C(\sigma) \leq k\left(H_m(\sigma) + \frac{2^{n+1}\log_2(m)}{m}\right) + O(1)$$

.

PROOF. Each of the quantities $s_j = p_j m$ imposes a constraint on the set of possible strings σ. Given the list of all strings satisfying these constraints, σ can be uniquely determined by selecting one of them, a choice from among $\binom{m}{s_1,\ldots,s_{2^n}}$ possibilities. Moreover, each of the quantities s_j can be expressed using at most $\log_2(m)$ bits, since $s_j \leq m$. Consequently, we have

$$C(\sigma) \leq 2^{n+1}\log_2(m) + \log_2\binom{m}{s_1,\ldots,s_{2^n}}.$$

The result follows from Stirling's approximation. □

We can, using Kolmogorov complexity, formulate a notion of mutual information parallel to Shannon's. Intuitively, the quantity defined here should represent the information about σ contained in τ.

DEFINITION 3.2.7. The algorithmic information of σ from τ, denoted $I_C(\sigma : \tau)$ is defined by $C(\sigma) - C(\sigma|\tau)$.

An important feature of mutual information in Shannon's theory is its symmetry: $I(\tau : \sigma) = I(\sigma : \tau)$. This is, of course, too much to ask for a notion like I_C, which is defined only up to an additive constant. Indeed, the symmetry of information from a Kolmogorov complexity perspective is weaker even than that.

PROPOSITION 3.2.8. *Let σ, τ be strings. Then*

$$|I_C(\tau : \sigma) - I_C(\sigma : \tau)| = O\left(\log C(\sigma\tau)\right).$$

On the other hand, there are strings σ, τ witnessing that this bound is sharp.

A very strong statement of incompressibility, then, would be to say that real number X has the property that $C(X \upharpoonright_n) \geq n - O(1)$. However, we have the following result.

PROPOSITION 3.2.9 (Martin-Löf). *For any real number X, we have $C(X \upharpoonright_n) \lneq n - O(1)$.*

PROOF. We will show that for n large enough, there is an initial segment σ of $X \upharpoonright_n$ with $C(\sigma) < |\sigma| - k$.

Indeed, let τ be a finite initial segment of X, and let j be the serial number for τ in an enumeration of $2^{<\omega}$. Now let ρ consist of the unique set of j bits such that $\tau\rho$ is an initial segment of X. We set $\sigma = \tau\rho$.

To see that σ has the desired property, we observe that to determine σ, we need only know ρ, since the length of ρ encodes τ. Then $C(\sigma) \leq |\rho| - O(1)$, and if $n = |\sigma|$, then $C(X \upharpoonright_n) \leq |\rho| + O(1) \lneq n - O(1)$. Since τ was of arbitrary length, the result follows. □

For this reason, there can be no real number with the incompressibility property we first stated. Instead, we slightly modify the definition of Kolmogorov Complexity.

DEFINITION 3.2.10. Let C denote plain Kolmogorov complexity.

(1) A partial recursive prefix-free function is a partial recursive function $\phi : 2^{<\omega} \to \mathbb{N}$ such that if $\phi(p)\downarrow$ and $\phi(q)\downarrow$ with $p \neq q$, then p is not a prefix of q.
(2) Let $\hat{\psi}$ be a partial recursive prefix-free function which is universal in the sense that for any recursive prefix-free function f, there is ρ_f such that $\hat{\psi}(\rho_f\sigma) = f(\sigma)$. Then we define the prefix-free Kolmogorov complexity

$$K(\sigma) = C_{\hat{\psi}}(\sigma).$$

This definition raises two technical points. It is not hard to see that a universal prefix-free function exists. Moreover, the defintion of K is given, as usual, only up to an additive constant.

This avoids the difficulty of Proposition 3.2.9, intuitively because a prefix-free machine cannot use both ρ and $|\rho|$. We now have a viable standard of incompressibility. The following definition was sketched in the concluding sentences of [**320**], where the complexity definition was introduced, and later made precise in [**333, 111**].

DEFINITION 3.2.11. We say that $X \in \mathbb{R}$ is 1-random if and only if, for all n, we have $K(X \upharpoonright_n) \geq n - O(1)$.

We can see, by a counting argument, that the number of sequences σ of length n with $K(\sigma) \geq n - c$ is at least $(1 - 2^c)2^n$, so that almost all real numbers are 1-random. We will soon see that every 1-random is absolutely normal, but we defer the proof of this statement until after stating some equivalent definitions and proving their equivalence.

3.2.2. Martin-Löf Characterization by measures. Martin-Löf, in the 1960s, was aware of Kolmogorov's earlier work, and showed that it satisfied certain intuitive properties that should be associated with randomness. His first example was that a random number — whatever that was, should be simply normal to base 2 [**359**].

To show that in the binary expansion of $X = \sum_{i=1}^{\infty} x_i 2^{-i}$, the proportion of ones is $\frac{1}{2}$, we could run a series of tests of increasing precision. For each n, we could calculate

$$E_n = \left| 2\left(\sum_{i=1}^{n} x_i\right) - n \right|.$$

If X is simply normal to base 2, of course, $E_n(X)$ will tend toward zero. We can construct a computable function $f : \mathbb{N} \times \mathbb{N} \times 2^{<\omega} \to \mathbb{N}$ in such a way that the sequence

$$C_n := \{X : E_n \leq f(m, n, X \upharpoonright_n)\}$$

satisfies $\lambda C_n \leq 2^{-m}$, where λ represents Lebesgue measure. Martin-Löf's goal was to show that 1-random reals must pass all such tests — for instance, $E_n \to 0$.

The definition of Martin-Löf randomness is often stated in terms of Lebesgue measure, but Martin-Löf himself suggested the use of arbitrary measures, and the use of measures other than Lebesgue measure has become more important in the recent literature. In any case, it is no addition to the difficulty of the definition to define Martin-Löf randomness in the context of an arbitrary measure. By way of motivation, though, we will see that for Lebesgue measure λ, the λ-Martin Löf random reals are exactly the 1-random reals.

Definition 3.2.12. Let μ be a measure.

(1) A Σ_1^0 class of elements of 2^ω is a set C of the form $\{A : \exists n\ R(A \upharpoonright_n)\}$ for some computable relation R.
(2) A μ-Martin-Löf test is a sequence $(U_n : n \in \mathbb{N})$ of uniformly Σ_1^0 classes such that for any $n \in \mathbb{N}$, we have $\mu U_n \leq 2^{-n}$.
(3) A set $C \subseteq \mathbb{R}$ is said to be *μ-Martin-Löf null* if and only if $C \subseteq \bigcap_{n \in \mathbb{N}} U_n$ for some μ-Martin-Löf test $(U_n : n \in \mathbb{N})$.
(4) A real X is said to be *μ-Martin-Löf random* if and only if the singleton $\{X\}$ is not Martin-Löf null.

Theorem 3.2.13 (Schnorr). *A real X is Martin-Löf random with respect to Lebesgue measure if and only if it is 1-random.*

Proof. Let X be Martin-Löf random, and let $U_n = \{Y : \exists k\ K(Y \upharpoonright_k) \leq k - n\}$. By the discussion following Definition 3.2.11, this set will have measure at most 2^{-n}. Now $X \notin \bigcap_{n \in \mathbb{N}} U_n$, so that X is 1-random.

On the other hand, suppose that X is not Martin-Löf random. Then a test $(U_n : n \in \mathbb{N})$ with $X \in \bigcap_{n\in\mathbb{N}} U_n$ can be transformed into a witness to $K(X \upharpoonright_n) \lneq n + O(1)$. □

An advantage that Martin-Löf pointed out to the approach by measures is the concept of a universal test. A Martin-Löf test $(U_n : n \in \mathbb{N})$ is said to be *universal* if and only if for any other test $(V_n : n \in \mathbb{N})$, we have $\bigcap_{n\in\mathbb{N}} V_n \subseteq \bigcap_{n\in\mathbb{N}} U_n$. The test constructed in the previous proof to show that a Martin-Löf random is 1-random is an example of such a test. This consideration already gives us the following observation.

PROPOSITION 3.2.14 ([**359**]). *The set of 1-random reals in the unit interval has measure 1.*

PROOF. Consider the universal test $(U_n : n \in \mathbb{N})$ already constructed. Since $\bigcap_{n\in\mathbb{N}} U_n$ has measure zero, its complement, consisting of exactly the 1-random reals, has measure 1. □

A much more striking proof of this proposition arises from the consideration of dynamical systems. A complete discussion of ergodicity will come later, in Section 6.1.2, but we observe here that for a probability space $(\Omega, \mathcal{M}, P)$ and a measure-preserving function $f : \Omega \to \Omega$, we say that the system (f, Ω, P) is *ergodic* if and only if for any $S \in \mathcal{M}$, if $T^{-1}(S) = S$, then $P(S) \in \{0, 1\}$.

THEOREM 3.2.15 (Birkhoff Ergodic Theorem). *Let $(\Omega, \mathcal{M}, P)$ be a probability space, and let $f : \Omega \to \Omega$ be a measure-preserving transformation. Let $A \in \mathcal{M}$. Then for almost every $x \in \Omega$, then the limit*

$$\lim_{n\to\infty} \frac{1}{n} \left(\sum_{j=1}^{n} \chi_A \left(f^j(x) \right) \right)$$

exists. Moreover, if (f, Ω, P) is ergodic, then the limit is equal to $P(A)$.

The claim to be advanced is that the full-measure set of x for which the conclusion of the theorem holds coincides exactly (under approriate provisos) with the P-Martin-Löf random elements of Ω. We say that a point $x \in \Omega$ is *Birkhoff* for a set B and a measure-preserving function f if and only if

$$\lim_{n\to\infty} \frac{1}{n} \left(\sum_{j=1}^{n} \chi_A \left(f^{j-1}(x) \right) \right) = P(B).$$

THEOREM 3.2.16 ([**70, 191**]). *Let $(\Omega, \mathcal{M}, P)$ be a probability space, and $x \in \Omega$. Then the following are equivalent.*

(1) *x is P-Martin-Löf random.*
(2) *x is Birkhoff for every Σ^0_1 class $U \subseteq \Omega$ for every measure-preserving, ergodic, computable $f : \Omega \to \Omega$.*

PROOF. We first suppose that x is P-Martin-Löf random. Let U be a Σ^0_1 class of Ω. We define

$$g_n(x) = \frac{1}{n} \left(\sum_{j=1}^{n} \chi_A \left(f^{j-1}(x) \right) \right),$$

and will show that $\limsup g_n(x) \leq P(U)$ and $\liminf g_n(x) \geq P(U)$. For each rational $r > P(U)$, we set

$$G_N = \{x : \exists(n \geq N)\ g_n(x) > r\}.$$

This set G_N is a Σ_1^0 class of Ω, and form a decreasing sequence, whose intersection has measure zero (by the Ergodic Theorem). Fix N such that $P(G_N) < 1$. We can show that there is i such that $f^i(x) \notin G_N$. Thus, $\limsup g_n\left(f^i(x)\right) \leq r$. On the other hand, we can approximate U from below by closed sets and apply the argument of the previous paragraph to the complement of these approximations, proving $\liminf g_n(x) \geq P(U)$. Similar arguments establish the converse. □

Work continues on the connection of randomness to ergodic theory. There is a recent survey of this work in [**487**]. However, Theorem 3.2.16 is a natural effectivization of Birkhoff's theorem, and in conjuction with Theorem 3.2.13 and results of the following section it does provide significant evidence that the 1-randoms are a natural notion of randomness.

3.2.3. Characterization by Games.

We return again to one of the insights around normal numbers. In a random sequence, it should not be possible to bet profitably on the next entry in the sequence. Of course, this must be contextualized.

Recall from Section 2.4 that a *martingale* is a stochastic process $x(t)$ with σ-algebras $\mathcal{F}_t$ such that for all $2 < t$ we have $\mathcal{F}_s \subseteq \mathcal{F}_t$, such that $x(t)$ is $\mathcal{F}_t$-measurable, such that $x(t)$ has finite expected value at any time, and also such that for any time t and any $s < t$, we have $E\left(x(t)|\mathcal{F}_s\right) = x(s)$. This can be applied either in discrete ($t \in \mathbb{N}$), or continuous ($t \in \mathbb{R}$) time, using one and the same definition. In the probability literature, the term "martingale" seems to unambiguously name this concept, and this concept is central to modern probability [**284, 285, 332**]. We consider a related concept, which is often called a "martingale" in the algorithmic randomness literature.

DEFINITION 3.2.17. A *martingale indicator* is a function $d : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ such that, for all $\sigma \in 2^{<\omega}$, we have

$$d(\sigma) = \frac{d(\sigma 0) + d(\sigma 1)}{2}.$$

The relation of these functions to martingales is explored in detail in [**259**]. We can, of course, take a martingale indicator to represent a discrete-time stochastic process in $\mathbb{R}$ by defining $X_{d,n}(\mathbb{X}) = d(\mathbb{X}\restriction_n)$ for each $\mathbb{X} \in 2^\omega$. Now certainly $X_{d,n}$ has finite expectation for each n, and if we take $\mathcal{F}_n$ to be the family generated by the basic open sets in 2^ω defined by initial sequences of length n, we have $E\left(X_{d,n+1}|\mathcal{F}_n\right) = X_{d,n}$. On the other hand, not every discrete-time martingale occurs in this form: the σ-algebras $\mathcal{F}_n$ need not be this particular sequence. Hitchcock and Lutz describe this as a distinction between the "bit history" (the case for martingale indicators) and the "capital history" (the case for general martingales) [**259**].

This distinction of "bit history" and "capital history" refers to a specific interpretation to which we will need to refer again. We can view a martingale indicator as a betting system. It prescribes a payoff for the next bit, allowing the gambler to condition on knowledge of the previous bets. Then if $|\sigma| = n$, the quantity $d(\sigma 1)$ represents the total capital held by the gambler after the $(n+1)$st bit is revealed to be a 1. At issue in randomness is whether there is a computable martingale indicator d that can reliably win — that is, can reliably increase the capital. This should not be possible for a random sequence. The Dutch book condition, $d(\sigma) = \frac{d(\sigma 0)+d(\sigma 1)}{2}$, says, in this case, that the bet placed on the next bit is a fair one.

We say formally that a martingale indicator d succeeds on a sequence $\mathbb{X} \in 2^\omega$ if and only if $\limsup_{n\to\infty} d(\mathbb{X}\restriction_n) = \infty$. We define the *success set* of a martingale indicator to be the set of sequences $\mathbb{X}$ such that d succeeds on $\mathbb{X}$.

The following concept was proposed by Hitchcock and Lutz to explain the boundary between martingales and martingale indicators.

Definition 3.2.18. Let $d : 2^{<\omega} \to \mathbb{R}$. We define a discrete stochastic process $X_{d,n} : 2^\omega \to \mathbb{R}$ by $X_{d,n}(\mathbb{X} \in 2^\omega) := d(\mathbb{X}\restriction_n)$, as before.

(1) For each n, we denote by $c_{\mathbb{X},n}$ the value $d(\mathbb{X}\restriction_n)$.
(2) For each n, let $\mathcal{F}_{\mathbb{X},n}$ be the σ-algebra generated by the set of all strings σ such that $X_{d,n}(\mathbb{X} \in 2^\omega) = c_{\mathbb{X},n}$.
(3) We say that d is a *martingale process* if and only if for every n, we have $E(X_{d,n}|\mathcal{F}_{\mathbb{X},n-1}) = X_{d,n}$

For resource-bounded notions of randomness, this distinction is important. Hitchcock and Lutz showed that for every computable nonnegative martingale process, there is a polynomial-time exactly computable nonnegative martingale process with the same success set. In the definition of randomness that is equivalent to 1-randomness, though, there is no difference, as we will see.

Theorem 3.2.19 ([**259, 367, 435, 436**]). *Let $\mathbb{X} \in 2^\omega$. The following are equivalent:*

(1) *$\mathbb{X}$ is 1-random.*
(2) *No computably enumerable martingale process succeeds on $\mathbb{X}$.*
(3) *No computably enumerable martingale indicator succeeds on $\mathbb{X}$.*

Proof. Let d be a martingale indicator. Then we can determine a Martin-Löf test by $U_n = \{\sigma : d(\sigma) \geq 2^n\}$. Thus, if $\mathbb{X}$ is in the success set of d, there is a Martin-Löf test containing $\mathbb{X}$, so that if a computably enumerable martingale indicator succeeds on X, then it is not 1-random. On the other hand, for any Martin-Löf test $(V_n : n \in \mathbb{N})$, we can define a computably enumerable martingale indicator d such that d succeeds on $\mathbb{X}$ if and only if $\mathbb{X} \in \bigcap_{n\in\mathbb{N}} V_n$.

Now let d be a computably enumerable martingale process, and assume without loss of generality that $d(\emptyset) = 1$. For each natural number k, we define

$$A_k = \left\{\sigma \in 2^{<\omega} : \max_i d(\sigma\restriction_i) < 2^k \leq d(\sigma)\right\}.$$

Each of these sets includes has the property that if $\sigma \in A_k$ and for all i we have $d(\tau\restriction_i) = d(\sigma\restriction_i)$, then $\tau \in A_k$. We can show that $\sum_{\sigma\in A_k} \leq 2^{-k}$. We can then define, for each k, the function

$$d'_k(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ extends an element of } A_k \\ \sum_\tau 2^{|\sigma|-|\tau|} & \text{otherwise} \end{cases},$$

where the sum is taken over all extensions τ of σ with $\tau \in A_k$. Setting $d' = \sum_{k\in\mathbb{N}} d'_k$, we have a martingale indicator d'. All of this may be done effectively. The success set of d' will match that of d. In this way, it follows that if a computably enumerable martingale process succeeds on $\mathbb{X}$, then a computably enumerable martingale indicator succeeds on $\mathbb{X}$.

We have now established that conditions 1 and 3 are equivalent, and that 3 implies 2. We conclude by showing that 2 implies 1.

Let $(U_n : n \in \mathbb{N})$ be a Martin-Löf test. We will construct a computably enumerable martingale process that succeeds on $\bigcap_{n\in\mathbb{N}} U_n$.

To this end, we first identify, for each string σ, the set V_σ, the set of τ extending σ whose extensions are all in $U_{|\sigma|+1}$. We also identify the set $V_\sigma^- = \{\tau : \sigma\tau \in V_\sigma\}$. Using the effectiveness of these sets, we can define, for each k and each σ, a martingale process $d_{\sigma,k}$ that, with $d_{\sigma,k}(\emptyset) = 2^k$ achieves $d_{\sigma,k}(\tau) = 2^{k+1}$ for all $\tau \in V_\sigma^-$. We refine this process by setting

$$d'_{\sigma,k}(\tau) = \begin{cases} d_{\sigma,k}(\eta) & \text{if } \tau = \sigma\eta \\ 2^k \text{if } \tau \text{ does not extend } \sigma \end{cases} .$$

We now define a martingale process δ, not depending on σ or k, that will succeed on $\bigcap_{n\in\mathbb{N}} U_n$. We start with $\sigma_0 = \emptyset$, and $\delta_0(\emptyset) = d'_{\emptyset,0}$. At stage $s+1$, we find σ such that $\delta_s(\sigma) = 2\delta_{s-1}(\sigma_{s-1})$, and set $\sigma_{s+1} = \sigma$ and $\delta_{s+1} = d'_{\sigma_{s+1},s+1}$. The limit $\delta = \lim_{s\to\infty} \delta_s$ is a computably enumerable martingale process, and succeeds on $\bigcap_{n\in\mathbb{N}} U_n$. □

The approach to Martin-Löf randomness by martingales gives us easy access to a result generalizing some of the initial properties sought in normality.

Proposition 3.2.20. *Let $\sigma \in 2^\omega$ be Martin-Löf random, and let $f : \omega \to \omega^k$ be a computable injection. Then for any $w \in 2^k$ there are infinitely many i such that $\sigma \restriction_{\{i,i+1,\ldots,i+k-1\}} = w$.*

Proof. If there were only finitely many i such that $\sigma \restriction_{\{i,i+1,\ldots,i+k-1\}} = w$, then we coud create a computable martingale betting that the sequence would never be completed, which would almost always win. □

It may be objected that the form of betting strategy represented by martingales is too rigid. Real gamblers certainly randomize their bets, in addition to facing the probabilistic nature of the environment. They also sometimes decline to take a bet in any direction at some points. Buss and Minnes explored the randomness notions that arise from weaker forms of betting strategies.

Definition 3.2.21. Denote by D_n the set of pairs of binary strings of length n, and by D the disjoint union $\bigcup_{n\in\mathbb{N}} D_n$. A probabilistic strategy $A = (p_A, q_A)$, with $p_A : D \to \mathbb{Q} \cap [0,1]$ and $q_A : D \to \mathbb{Q} \cap [0,2]$.

We define the action of a probabilistic strategy in the following way. At each stage, the strategy will determine, based on its own history and the current initial segment of the sequence, whether to bet on the next bit, and, if so, how much. For both p_A and q_A, the first input coordinate reflects the history of decisions on whether to bet or not, and the second input reflects the current initial segment.

In particular, at stage n, having placed bets on the values of $\{x_i : i \in I_n\}$, where $I_n \subseteq \{0, \ldots, n-1\}$, and having seen $\mathbb{X}_n = (x_0, \ldots, x_{n-1})$, the gambler will, with probability $p_A(\chi_{I_n} \upharpoonright_n, \mathbb{X}_n)$ place a bet of size $q_A(\chi_{I_n} \upharpoonright_n, \mathbb{X}_n)$ that x_n will have the value 1, and with probability $1 - p_A(\chi_{I_n} \upharpoonright_n, \mathbb{X}_n)$ does nothing.

To formally compute the winnings of such a strategy, we first compute the probabilities of achieving particular sets I_n of bets. We set $P_A(\pi, \sigma)$ to be the probability that, running on a string $\mathbb{X}$ with initial segment σ, we have $\chi_{I_n} \upharpoonright_n = \pi$. We can compute this quantity inductively. We can also compute inductively the cumulative winnings $C_A(\pi, \sigma)$ if A is running on a string $\mathbb{X}$ with initial segment σ and $\chi_{I_n} \upharpoonright_n = \pi$. We can then define the *expected winnings after n rounds*, $E_A(\mathbb{X}, n)$ as follows:

$$E_A(\mathbb{X}, n) = \sum \pi \in 2^n P_A(\pi, \mathbb{X} \upharpoonright_n) C_A(\pi, \mathbb{X} \upharpoonright_n).$$

Finally, we give a success criterion.

Definition 3.2.22. We say that a probabilisitc strategy A *succeeds on $\mathbb{X}$ in expectation* if and only if

$$\lim_{n \to \infty} E_A^{\mathbb{X}}(n) = \infty.$$

A sequence $\mathbb{X} \in 2^\omega$ is said to be Ex-random if and only if no computable probabilistic strategy succeeds on $\mathbb{X}$ in expectation.

The sequences which are random with respect to expected winnings of probabilistic betting strategies are precisely those which are random with respect to martingales.

Theorem 3.2.23 ([**91**]). *The following properties of a sequence $\mathbb{X}$ are equivalent:*

(1) *$\mathbb{X}$ is 1-random*
(2) *There is no computable probabilistic strategy that succeeds on $\mathbb{X}$ in expectation.*

Proof. If A is a probabilistic strategy with $\lim_n E_A^{\mathbb{X}}(n) = \infty$, then we define a Martin-Löf test $(U_n : n \in \mathbb{N})$ such that $\mathbb{X} \in \bigcap_{n \in \mathbb{N}} U_n$. In particular, we define U_n to be the sequences $\mathbb{Y}$ such that $\exists i \left(E_A^{\mathbb{Y}}(i) \geq 2^n \right)$. Consequently, $\mathbb{X}$ is not 1-random.

Now suppose that there is a Martin-Löf test $(U_n : n \in \mathbb{N})$ such that $\mathbb{X} \in \bigcap_{n \in \mathbb{N}} U_n$. We define a probabilistic strategy A that will succeed on $\mathbb{X}$ in expectation. As each string $\sigma_{n,j}$ is enumerated into U_n, we set $b_{n,j} = 2^{n - |\sigma_{n,j}|}$. We define p_A and q_A inductively. Let π be the least string from which $p_A(\pi, \sigma), q_A(\pi, \sigma)$ are not yet defined, and let $n = \sum_i \pi(i)$. We then define p_A by, for each j, the condition

$$p_A\left(\pi 0^j, \sigma \prod_{i=0}^{j} \left(1 - p_A(\pi 0^i, \sigma)\right)\right) = b_{n+1,j}.$$

Moreover, for each j and for each k with $1 \leq k < |\sigma_{n+1,j}| - n$, we set $p_A(\pi 0^j 1^k, \sigma) = 1$. Finally, we set

$$q_A(\pi 0^j 1^k, \sigma) = \begin{cases} 0 & \text{if } \sigma_{n+1,j}(n+k) = 1 \\ 2 & \text{if } \sigma_{n+1,j}(n+k) = 0 \end{cases}.$$

If $\mathbb{X} \in \bigcap_{n\in\mathbb{N}} U_n$, then $\limsup_n E^{\mathbb{X}}_A(n) = \infty$. This probabilistic strategy A can be modified into a probabilistic strategy A' that succeeds on $\mathbb{X}$ in expectation. □

3.3. Other Notions of Algorithmic Randomness

At this point, it is natural to summarize our results as follows.

THEOREM 3.3.1. *Let $\mathbb{X} \in 2^\omega$. The following conditions are equivalent.*

(1) $K(\mathbb{X}\restriction_n) \geq n - O(1)$
(2) *For any Martin-Löf test $(U_n : n \in \mathbb{N})$, we have $\mathbb{X} \notin \bigcap_{n\in\mathbb{N}} U_n$.*
(3) *$\mathbb{X}$ is Birkhoff for every Σ^0_1 class $U \subseteq \Omega$ in every measure-preserving, ergodic, computable $f : \Omega \to \Omega$.*
(4) *No computably enumerable martingale process succeeds on $\mathbb{X}$.*
(5) *No computably enumerable martingale indicator succeeds on $\mathbb{X}$.*
(6) *No computable probabilistic strategy succeeds on $\mathbb{X}$ in expectation.*

Even without reference to any of the other known equivalent statements, this list might suggest to us a situation not unlike Church's Thesis: If we have defined the same thing in six ways and come up with equivalent definitions, we must have the right definition. Indeed, there is much to recommend this philosophy. However, it suffers significantly from selection bias. There are other definitions that we could have made that are *not* equivalent to 1-randomness.

A full survey of all randomness conditions now in the literature is a major study in itself, and far beyond the scope of this book. A starting place to appreciate the complexity of such a study would be the "Randomness Zoo" of Antoine Taveneaux, which describes the relative implications of thirty-two pairwise non-equivalent definitions of randomness, plus at least one proper infinite chain of randomness notions [**477**]. We point here to a few definitions not equivalent to 1-randomness, mostly to avoid the impression that 1-randomness (perhaps on account of being frequently studied) is unequivocally the correct definition, or even unequivocally the most interesting.

3.3.1. n-Randomness.

One natural way to modify the definition of 1-randomness is to weaken the effectiveness conditions on Martin-Löf tests or the notion of Kolmogorov complexity. There are several ways in which this could be done.

THEOREM 3.3.2 ([**295**]). *The following conditions on a sequence $\mathbb{X}$ are equivalent:*

(1) *Let $K^A(\sigma)$ be defined by analogy to Kolmogorov complexity, but giving the universal machine access to an oracle for A. Then*
$$K^{\emptyset^{(n-1)}}(\mathbb{X}\restriction_n) \geq n - O(1).$$
(2) *For any sequence $(U_n : n \in \mathbb{N})$ of uniformly Σ^0_n classes with $\lambda U_n \leq 2^{-n}$, we have $\mathbb{X} \notin \bigcap_{n\in\mathbb{N}} U_n$.*
(3) *For any sequence $(U_n : n \in \mathbb{N})$ of* open *uniformly Σ^0_n classes $\lambda U_n \leq 2^{-n}$, we have $\mathbb{X} \notin \bigcap_{n\in\mathbb{N}} U_n$.*
(4) *No Σ^0_n martingale indicator succeeds on $\mathbb{X}$.*

PROOF. Certainly 2 implies 3. Moreover, since relativized Kolmogorov complexity corresponds to relativized Martin-Löf tests (all of which are open), we have 3 implies 1.

Suppose there is a sequence $(U_n : n \in \mathbb{N})$ of uniformly Σ^0_n classes with $\lambda U_n \leq 2^{-n}$. We can use this sequence to construct a Martin-Löf test relative to $\emptyset^{(n-1)}$, which, in turn, gives rise to a computation that $K^{\emptyset^{(n-1)}}(\mathbb{X}\restriction_n) \geq n - O(1)$.

The equivalence of the martingale indicator definition is analogous to the 1-random case. □

This equivalence gives rise to a definition.

DEFINITION 3.3.3. A sequence $\mathbb{X}$ is said to be n-random if and only if it satisfies any of the equivalent conditions of Theorem 3.3.2.

Obviously, for $n = 1$ this definition matches the earlier definition of 1-randomness. The first natural question is whether this hierarchy is strict.

PROPOSITION 3.3.4. *For any $n \in \mathbb{N}$, if $\mathbb{X}$ is $(n+1)$-random, then $\mathbb{X}$ is n-random. On the other hand, for any n, there are n-random sequences which are not $(n+1)$-random.*

PROOF. For any n, every Δ^0_n set is non-n-random. On the other hand, there are $\Delta^0_n + 1$ sets which are n-random. □

We sometimes speak of *weak n-randomness*, as well.

DEFINITION 3.3.5. We say that $\mathbb{X}$ is *weakly n-random* if and only if it is a member of all Σ^0_n classes of measure 1.

Weak n-randomness does *not* coincide with weak 1-randomness relative to $\emptyset^{(n-1)}$.

3.3.2. Shnorr Randomness. Another possible direction in which the notion of 1-randomness can be strengthened is to require additional effectiveness in the tests.

THEOREM 3.3.6 ([**436, 157**]). *The following conditions on a sequence $\mathbb{X} \in 2^\omega$ are equivalent.*

(1) *Let $(U_n : n \in \mathbb{N})$ be a Martin-Löf test with $\mu(U_n)$ computable, uniformly in n. Then $\mathbb{X} \notin \bigcap_{n\in\mathbb{N}} U_n$.*
(2) *For any nondecreasing function $h : \mathbb{N} \to \mathbb{N}$ and any computable martingale indicator d, we have*
$$\limsup_n \frac{d(\mathbb{X}\restriction_n)}{h(n)} < \infty.$$
(3) *For any prefix-free machine M such that the measure of the domain of M is computable, we have $K_M(\mathbb{X}\restriction_n) \geq n - O(1)$.*

PROOF. To show 1 implies 2, let $(U_n : n \in \mathbb{N})$ be as described in condition 1. For each n, let R_n be a uniformly computable prefix-free set of strings such that U_n is the set of paths through R_n, and $R_{n,k} \subseteq R_n$ the set of elements of R_n of length at least k, and $V_{n,k}$ the set of paths through $R_{n,k}$. We then find a computable non-decreasing function $h_0 : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n\in\mathbb{N}} \mu(V_n) \leq 2^{-2h_0(k)}$.

There is, then, a computable non-decreasing function $h_1 : \mathbb{N} \to \mathbb{N}$ with

$$\sum_{n \in \mathbb{N}} \left(\sum_{\sigma \in R_{n,h_1(k)}} 2^{h_0(k)-|\sigma|} \right) < 2^{-k}.$$

We define $d_{n,k}$ in the following way. If $\sigma \in R_{n,h_1(k)}$ and τ extends σ, add $2^{h_0(k)}$ to the value of $d_{n,k}(\tau)$. Further, for each $i \leq |\sigma|$, add $2^{h_0(k)-|\sigma|+i}$ to the value of $d_{n,k}(\sigma \restriction_i)$. Finally, we set $d(\tau) = \sum_{n,k \in \mathbb{N}} d_{n,k}(tau)$.

Now if $\mathbb{X} \in \bigcap_{n \in \mathbb{N}} U_n$, then we can show that

$$\limsup_n \frac{d(\mathbb{X} \restriction_n)}{h(n)} < \infty.$$

To see that 2 implies 1, suppose that d is a computable martingale indicator and $h : \mathbb{N} \to \mathbb{N}$ nondecreasing. Letting $[\![\sigma]\!]$ represent the set of paths through σ, we define

$$U_i = \bigcup_{d(\sigma) > h(\sigma) > 2^i} [\![\sigma]\!].$$

This constitutes a Martin-Löf test with uniformly computable measures, as required, so that $\limsup_n \frac{d(\mathbb{X}\restriction_n)}{h(n)} = \infty$ if and only if $\mathbb{X} \in \bigcap_{n \in \mathbb{N}} U_n$.

To show that 3 implies 1, we similarly assume that $(U_n : n \in \mathbb{N})$ is a Martin-Löf test as in condition 1, with $\mathbb{X} \in \bigcap_{n \in \mathbb{N}} U_n$. Again, we take a uniformly computable sequence of prefix-free sets R_n as before, and consider the set of pairs $(|\sigma - k|, \sigma)$ where $k \geq 1$ and $\sigma \in R_{2k}$. A technical result known as the Kraft-Chaitin Theorem allows us to produce from this a machine M whose domain has computable measure witnessing $K_M(\sigma) \leq |\sigma| - k$, so that $K(\mathbb{X} \restriction_n) < n - O(1)$.

The proof that 3 implies 1 is similar to the proof for the analogous Martin-Löf random situation. □

This result gives rise to a definition.

Definition 3.3.7. We say that a sequence $\mathbb{X}$ is *Schnorr random* if and only if it satisfies either of the equivalent conditions of Theorem 3.3.6.

It is immediate from the definition that every 1-random is Schnorr random. The implication is strict.

While the combinatorics of Schnorr randoms are less accommodating than the situation with 1-randoms (e.g. there is no universal Schnorr test), Jason Rute has made a case that Schnorr randomness is the appropriate notion of randomness for applications involving analysis. In support of this thesis, he offers the following data.

Theorem 3.3.8 (Various authors; see [**426**]). *For a real number $x \in [0,1]$, the following are equivalent.*

(1) *x is Schnorr random.*
(2) *For every increasing computable sequence of continuous functions $g_n : [0,1] \to [0,\infty)$, if there is some computable probability measure μ such that for any Borel set A, we have $\int_A g_n(x)dx \leq \mu(A)$, then $\sup_n g_n(x)$ is finite.*

(3) *If $f : [0,1] \to \mathbb{R}$ is a function of bounded variation with effectively integrable derivative f, then f is differentiable at x.*
(4) *For every effectively integrable function f, the averages $\frac{1}{2r}\int_{x-r}^{x+r} f(y)dy$ converge as $r \to 0$.*
(5) *For every computable martingale X_n with $\sup_n ||f_n||_{L^1} < \infty$, the sequence $f_n(x)$ converges.*

3.3.3. Stochasticity. Neither martingale indicators nor martingale processes nor martingales is the most intuitive way to formalize the idea of computably betting on future bits of a binary sequence. Perhaps the most intuitive approach is something closer to the idea of a normal sequence: to have a computable function that tries to predict places at which the sequence will take value 1. Two notions termed *stochasticity* formalize this.

DEFINITION 3.3.9. For any function $f : 2^{<\omega} \to \{0,1\}$, we define $\tilde{f} : 2^\omega \times \mathbb{N} \to \mathbb{N}$ by letting

$$\tilde{f}(\sigma, n) = |\{i < n : \sigma(i) = f(\sigma \upharpoonright_i) = 1\}|.$$

(1) We say that $\mathbb{X}$ is *von Mises-Wald-Church stochastic* if and only if for any partial computable function $f : 2^{<\omega} \to \{0,1\}$ with $f(i) = 1$ on infinitely many i, we have
$$\lim_{n\to\infty} \frac{\tilde{f}(\mathbb{X}\upharpoonright_n, n)}{n} = \frac{1}{2}.$$
(2) We say that $\mathbb{X}$ is *Church stochastic* if and only if for any total computable function $f : 2^{<\omega} \to \{0,1\}$ with $f(i) = 1$ on infinitely many i, we have
$$\lim_{n\to\infty} \frac{\tilde{f}(\mathbb{X}\upharpoonright_n, n)}{n} = \frac{1}{2}.$$

It is clear from the definition that if $\mathbb{X}$ is von Mises-Wald-Church stochastic, then $\mathbb{X}$ is Church stochastic. The following result relates these concepts to 1-randomness.

THEOREM 3.3.10. *If $\mathbb{X}$ is 1-random then $\mathbb{X}$ is von Mises-Wald-Church stochastic.*

PROOF. Suppose $\mathbb{X}$ is not von Mises-Wald-Church stochastic. In particular, suppose that f is a partial computable function with $f(i) = 1$ on infinitely many i and

$$\lim_{n\to\infty} \frac{\tilde{f}(\mathbb{X}\upharpoonright_n, n)}{n} = \frac{1}{2}.$$

We produce a computably enumerable martingale indicator that succeeds on $\mathbb{X}$ by reading f's predictions about the next bit at each stage. We begin by defining a pair of sequences of martingale indicators $(d_{i,k} : i \in \{0,1\}, k \in \mathbb{N})$. We define these inductively, with $d_{i,k}(\emptyset) = 1$ for all (i,k).

If $d_{i,k}(\sigma)$ has been defined, we consider $f(\sigma)$, and define $d_{i,k}(\sigma j)$ for each $j \in \{0,1\}$. It is helpful to remember that $d_{i,k}$ need only be computably enumerable. If $f(\sigma)\uparrow$, then we set $d_{i,k}(\sigma j) = 0$. If $f(\sigma) = 1$, then we "bet on i" by setting $d_{i,k}(\sigma i) = \left(1 + 2^{-k}\right) d_{i,k}(\sigma)$. If $f(\sigma) = 0$, then we decline to place a new bet by setting $d_{i,k}(\sigma 0) = d_{i,k}(\sigma 1) = d_{i,k}(\sigma)$. We then compose d by

$$\sum_{k\in\omega} 2^{-k}\left(d_{0,k} + d_{1,k}\right).$$

Now d will succeed on $\mathbb{X}$. □

In each case, the implication is proper. There are Church stochastic sequences that are not von Mises-Wald-Church stochastic, and there are von Mises-Wald-Church stochastics which are not 1-random.

An additional notion of stochasticity was introduced by Loveland. He pointed out that in many real models of probability, observations need not be sequential. For instance, in a quality inspection problem, the testing of items produced later might be used to predict the quality of items produced earlier [**340**].

To formalize this, we indicate by the term *non-monotonic selection rule* a partial function $f : (\mathbb{N} \times \{0,1\})^{<\omega} \to \mathbb{N}$ with $f(\sigma) \notin \pi_1(\sigma)$. We interpret such a function intuitively as examining a sequence $i_0, \ldots, i_k$ of places in a sequence $\mathbb{X}$, and if $\mathbb{X}(i_j) = \ell_j$ for each j, then f predicts that $\mathbb{X}\left(f\left((i_0, \ell_0), \ldots, (i_k, \ell_k)\right)\right) = 1$. In particular, by analogy with the previous case, we define $\tilde{f} : 2^{<\omega} \to \mathbb{N}$ by

$$\tilde{f}(\sigma) = \left|\{i < |\sigma| : \exists \tau \left[(\pi_1(\tau) \subseteq |\sigma) \wedge (\sigma\left(f(\tau)\right) = 1)\right]\}\right|.$$

Definition 3.3.11. We say that $\mathbb{X} \in 2^\omega$ is *Kolmogorov-Loveland stochastic* if and only if for any computable non-monotonic selection rule f, we have

$$\lim_{n\to\infty} \frac{\tilde{f}(\mathbb{X}\restriction_n)}{n} = \frac{1}{2}.$$

Certainly any Kolmogorov-Loveland stochastic sequence is von Mises-Wald-Church stochastic, but Loveland showed that the reverse is not true. A separate proof that Kolmogorov-Loveland stochasticity does not imply 1-randomness is given in [**447**]. A nonmonotonic construction analogous to martingale indicators was introduced in [**383**] and used to describe yet another notion of randomness, called Kolmogorov-Loveland randomness. Its equivalence to 1-randomness appears to be unknown at the time of this writing. These results are further described in [**367**].

Bienvenu, contextualizing a result of Shen, describes a family of measures that he calls the *generalized Bernoulli* measures. Each has a sequence parameter $\bar{p} = (p_i : i \in \mathbb{N})$, where each $p_i \in [0,1]$. The $\bar{p}$-generalized Bernoulli measure on 2^ω is the measure corresponding to the ith bit having value 1 independently with probability p_i, so that the standard Bernoulli measure with parameter p is a generalized Bernoulli measure with $p_i = p$ for all i. In particular, Lebesgue measure on 2^ω is a generalized Bernoulli measure with $p_i = \frac{1}{2}$ for all i.

Theorem 3.3.12 ([**447**]). *Let $\bar{p} = (p_i : i \in \mathbb{N})$ such that $\lim_{i\to\infty} p_i = \frac{1}{2}$. Let μ be a strongly positive generalized $\bar{p}$-Bernoulli measure. Then every μ-Martin-Löf random is Kolmogorov-Loveland stochastic.*

Proof. The exposition of this proof owes much to [**69**]. Let μ be as hypothesized, and let f be a computable non-monotonic selection rule. We pick a sequence $\mathbb{X}$ at random with distribution given by μ, and let $I = \{i_0, i_1, \ldots\}$ be the positions of $\mathbb{X}$ selected by f.

We define random variables $(Y_n : n \in \mathbb{N})$ by setting Z_n equal to the number of zeros in the subsequence $(x_{i_0}, \ldots, x_{i_n})$ and then $Y_n = Z_n - \sum_{k=0}^{n} p_{i_k}$. These random variables are determined by μ through their dependence on $\mathbb{X}$. If $\frac{Y_n}{n}$ does not tend to zero, then we can create a μ-Martin-Löf test demonstrating that $\mathbb{X}$ is not μ-Martin-Löf random.

If $\mathbb{X}$ is μ-Martin-Löf random, then, since $p_i \to \frac{1}{2}$, we must have

$$\lim_{n\to\infty} \frac{Z_n}{n} = \frac{1}{2},$$

so that

$$\lim_{n\to\infty} \frac{\tilde{f}(\mathbb{X}\restriction_n)}{n} = \frac{1}{2}.$$

□

3.4. Effective Dimension

3.4.1. Effective Hausdorff Dimension. Smooth manifolds admit induced measures and integration theory through their charts, which gives an intrinsic way to measure the size, for instance, of certain n-dimensional subsets of $(n+k)$-dimensional Euclidean spaces. The unit 2-sphere, for instance, when considered from the perspective of the measure induced by a set of charts, has positive finite measure from the perspective of 2-dimensional Lebesgue measure, infinite measure from the perspective of 1-dimensional Lebesgue measure, and measure zero from the perspective of 3-dimensional Lebesgue measure. In this sense, dimension 2 is really the natural place in which to measure it, independent of any prejudice we might have from its charts.

To extend this way of approaching dimensionality to contexts that do not admit such a geometric structure, Hausdorff proposed a parameterized set of measures, derived from Carathéodory's definition of measures [**251, 105**].

Definition 3.4.1. Let Ω be a metric space. We define the *r-dimensional Hausdorff outer measure* on subsets of Ω by, for any set E, setting $\mathcal{H}_r(E) = \lim_{\epsilon\to 0} S_{r,\epsilon}(E)$, where $S_{r,\epsilon}(E)$ is the infimum of $\sum_{i=1}^{\infty} \delta(E_i)^r$ over all collections $(E_i : i \in I)$ where $E \subseteq \bigcup_{i=1}^{\infty} E_i$ and $\delta(E_i)$ is the diameter of E_i.

It is a standard result that for any set E, there exists a critical value r_0 such that for $r > r_0$, we have $\mathcal{H}_r(E) = 0$ and for $r < r_0$ we have $\mathcal{H}_r(E) = \infty$. We define the *Hausdorff dimension of* E, denoted $\dim_H(E)$ to be this value r_0.

Example 3.4.2. Let $\mathcal{C} = \bigcap_{k\in\mathbb{N}} C_k$ be the standard middle-thirds Cantor set in $\mathbb{R}$, with $C_0 = [0,1]$ and C_{n+1} the result of removing the middle third from each interval in C_n. We compute $dim_H(\mathcal{C})$ as follows. In this case, open intervals are as good as any other choice of E_i. Certainly C_k (hence also $\mathcal{C}$ can be covered by 2^k open intervals, each of diameter 3^{-k}. Consequently, for each r, we have $\mathcal{H}_r(\mathcal{C}) \leq \frac{2^k}{3^{rk}}$. As k increases, this quantity approaches zero if and only if $r > \frac{\log 2}{\log 3}$, so $\dim_H(\mathcal{C}) = r > \frac{\log 2}{\log 3}$, strictly between the dimension of a discrete set of points and that of an interval.

Example 3.4.3. Let $\mathcal{S}$ be the Sierpiński carpet, the result of starting with the unit square and then, at each stage, dividing each component square into a 3×3 grid and removing the middle of the nine resulting squares. By an argument similar to the previous example, $dim_H(\mathcal{S}) = \frac{\log 8}{\log 3}$, strictly between that of a line segment and that of the unit square.

EXAMPLE 3.4.4. We next consider a much more complicated example from [**346**], one in the spirit of Chapter 6. We let $(L, A_1, \ldots, A_L)$ be a random vector where L is a $\mathbb{N}$-valued random variable, and for each i we have $A_i \in (0, 1]$. We let $(L_{i,n} : i, n \in \mathbb{N})$ be independent copies of L. Then we construct a tree T in the following way. At level 0, we have a root. At level n, the ith vertex at level n will have $L_{i,n}$ offspring at level $(n + 1)$.

We then add edge capacities according to the A_i, in the following way. For each vertex σ in the tree, we generate independent identically distributed random variables $\left(L_\sigma, A_{\tau_1}, \ldots, A_{\tau_{L_\sigma}}\right)$, where the τ_i are the offspring of σ. We now, for each σ in the tree, set the capacity of the edge joining x to its parent equal to $\prod_{\eta \preceq \sigma} A_\eta$. Such a structure is called a *Galton-Watson process*.

We now assign a subset I_T of $\mathbb{R}^n$ to T in the following way. For a set S we let $cl(S)$ and $int(S)$ denote the closure and interior of S, respectively, and $\delta(S)$ the diameter. We also denote, for each $\sigma \in T$, the predecessor of σ by $p(\sigma)$. For each $\sigma \in T$, we assign a compact nonempty set I_σ satisfying the following conditions.

(1) $I_\sigma = cl\,(int(I_\sigma))$
(2) For any non-root σ, we have $I_\sigma \subseteq I_{p(\sigma)}$
(3) If σ_1, σ_2 are distinct vertices with the same predecessor, then $int(\sigma_1)$ and $int(\sigma_2)$ have empty intersection.
(4) $\inf_{\sigma \in T} \frac{\mu(I_\sigma)}{\delta(I_\sigma)^n} > 0$, where μ is the n-dimensional Lebesgue measure.
(5) $\frac{\delta(I_\sigma)}{\delta(I_\emptyset)}$ is the capacity of the edge linking σ to its predecessor.

Denote by T' the subnetwork of T in which every vertex extends to an infinite path, and set $I_{T'} = \bigcup_P \bigcap_{\sigma \in P} I_\sigma$, where P ranges over all infinite paths through T'. Then

$$dim_H(I_{T'}) = \min \left\{ r : E\left(\sum_{i=1}^{L} A_i^r \right) \leq 1 \right\}.$$

This result is nontrivial, and a proof may be found in [**346**].

Introductions to this classical notion of Hausdorff dimension can be found in [**166**] and [**184**]. Deeper treatments, including applications of this dimension to dynamical systems and connections to classical box dimension, can be found in [**47, 400**].

Lutz gave an alternate characterization of Hausdorff dimension, which provides the gateway to effective Hausdorff dimension and its connection to algorithmic randomness.

THEOREM 3.4.5 ([**344**]). *For any $X \subseteq 2^\omega$, we define $G(X, s)$ to be the set of martingale indicators d such that $\limsup_n \frac{d(X\restriction_n)}{2^{(1-s)n}} = \infty$. Then*

$$dim_H(X) = \inf \{s \in \mathbb{Q} : \exists d \in G(X, s)\}.$$

PROOF. Let $s > dim_H(X)$. Now for each $k \in \mathbb{N}$, there exists a prefix-free set $U_k \in 2^{<\omega}$ such that X is contained in the set of paths through U_k and $\sum_{\sigma \in U_k} 2^{s|\sigma|} \leq$

2^{-k}. Now for each $\sigma \in 2^{<\omega}$, we set $U_{k,\sigma} = \{\tau \in U_k : \sigma \preceq \tau\}$, and then

$$d_k(\sigma) = \begin{cases} 2^{|\sigma|} \sum_{\tau \in U_{k,\sigma}} 2^{-s|\tau|} & \text{if } U_{k,\sigma} \neq \emptyset \\ 2^{(1-s)m} & \text{if } \sigma \restriction_m \in U_k \\ 0 & \text{otherwise} \end{cases}$$

We compose d by adding the d_k, as usual, and note that it is a martingale indicator with the necessary growth properties.

On the other hand, if $d \in G(X, s)$, we can define a cover of $\mathbb{X}$ by arbitrarily small sets as follows. We set

$$V_k = \left\{ \sigma : \frac{d(\sigma)}{2^{(1-s)|\sigma|}} \geq 2^k \right\},$$

and U_k a refinement of V_k to a prefix-free set. Then we have $\mathcal{H}_s(U_k) \leq 2^{-k}$, so that $\mathcal{H}_r(X) = 0$. □

While it is not obvious how to effectivize the standard definition of Hausdorff dimension, Theorem 3.4.5 gives a version that can be effectivized in a more straightforward way.

Definition 3.4.6. Let $X \subseteq 2^\omega$. The *effective Hausdorff dimension of* X, denoted $dim_e(X)$, is given by

$$\inf \left\{ s \in \mathbb{Q} : \exists d \in G(X, s) \cap \Sigma^0_1 \right\}.$$

It is common to refer to the effective Hausdorff dimension of a single element $\mathbb{X} \in 2^\omega$, meaning the effective Hausdorff dimension of the singleton.

Proposition 3.4.7. *Every 1-random has effective Hausdorff dimension 1.*

Proof. Let $\mathbb{X}$ be 1-random, and let $s < 1$, with $d \in G(\mathbb{X}, s)$ computably enumerable. Then d succeeds on $\mathbb{X}$, a contradiction. □

The converse is false — Kolmogorov-Loveland stochastics have effective Hausdorff dimension 1, by work of [**367**] — but we will see a partial converse in a later chapter as Theorem 4.5.14.

Effective Hausdorff dimension also stands in close relationship with the other quantitative measures of sequence complexity.

Theorem 3.4.8 ([**366**]). *For any* $\mathbb{X} \in 2^\omega$*, we have*

$$\dim_e(\mathbb{X}) = \liminf_{n \to \infty} \frac{K(\mathbb{X} \restriction_n)}{n}.$$

Proof. The difficult side is to show that

$$\dim_e(\mathbb{X}) \leq \liminf_{n \to \infty} \frac{K(\mathbb{X} \restriction_n)}{n}.$$

To this end, we take

$$s > t > \liminf_{n \to \infty} \frac{K(\mathbb{X} \restriction_n)}{n},$$

with $s, t \in \mathbb{Q}$, and consider the (computably enumerable) set B of all strings σ with $K(\sigma) \leq t|\sigma|$. Write $B = \bigcup_{n \in \mathbb{N}} B_n$, where all elements of B_n have length n, so that for some constant c, a standard counting argument gives us $|B_n| \leq 2^{tn-K(n)+c}$. In that case, from

$$\tilde{d}(\sigma) = 2^{(s-t)|\sigma|}\left(\sum_{\sigma\tau\in B} 2^{-t|\tau|} + \sum_{\substack{\tau\in B\\ \tau\preceq\sigma}} 2^{(t-1)(|\sigma|-|\tau|)}\right)$$

we can find $d \in G(\mathbb{X}, s)$. □

From the perspective of entropy, Staiger notes that a formulation naturally arising in effective fractal dimensions closely matches Shannon's formulation for entropy.

Definition 3.4.9 ([**461, 462**]). Let $C \subseteq 2^{<\omega}$. Then the *entropy rate* of C is given by

$$H_C = \limsup_{n\to\infty} \frac{\log_2 |C \cap 2^n|}{n}.$$

Staiger proved in [**461**] that

$$H_C = \inf\left\{s : \sum_{\sigma} \in C 2^{-s|\sigma|} < \infty\right\}.$$

This led Hitchcock to effectivize the notion of entropy rate.

Definition 3.4.10 ([**258**]). Let $C \subseteq 2^{<\omega}$, and $X \subseteq 2^{\omega}$.

(1) We define
$$C^{\delta} := \{\mathbb{X} \in 2^{\omega} : (\exists^{\infty} n)\, \mathbb{X} \upharpoonright_n \in C\}.$$
(2) We define the computably enumerable entropy rate (sometimes called the constructive entropy rate) by
$$\inf\left\{H_C : \left(C \in \Sigma_1^0\right) \wedge X \subseteq C^{\delta}\right\}.$$

This leads to Hitchcock's key result, which has relevance for other approaches to effective dimension.

Theorem 3.4.11 ([**258**]). *For any $X \subseteq 2^{\omega}$, we have*

$$\dim_e(X) = \inf\left\{H_C : \left(C \in \Sigma_1^0\right) \wedge X \subseteq A^{\delta}\right\}.$$

Proof. For any $C \in \Sigma_1^0$, and for any $t > s > H_C$, we can, in a way that is by now familiar, construct a computably enumerable martingale indicator of appropriate s-growth on C^{δ} to show that

$$\dim_e(X) \leq \inf\left\{H_C : \left(C \in \Sigma_1^0\right) \wedge X \subseteq A^{\delta}\right\}.$$

On the other hand, if d is a martingale indicator of appropriate s-growth on X, we set $C = \{\sigma : d(\sigma) > 1\}$. This set is computably enumerable, and $H_C < s$, so that

$$\inf\left\{H_C : \left(C \in \Sigma_1^0\right) \wedge X \subseteq A^{\delta}\right\} \leq dim_e(X).$$

□

We mention an observation that will be straightforward for the reader at this point, although it may not feel so obvious when we use it later in the proof of Theorem 4.5.14.

PROPOSITION 3.4.12 (Lemma 1.5 of [**231**]). *Let* $\mathbb{X} \in 2^\omega$ *and suppose that* $dim_e(\mathbb{X}) = 1$. *Then*

$$\lim_{m\to\infty} \frac{K\left(\mathbb{X}\restriction_{[2^m,2^{m+1})} |\mathbb{X}\restriction_{2^m}\right)}{2^m} = 1.$$

Recent work has suggested an equivalent formulation of effective Hausdorff dimension that seems in some ways more natural and is certainly more transparently a strengthening of the condition of normality from Section 3.1.2.

THEOREM 3.4.13 ([**99**]). *The following conditions on* $\mathbb{X} \in 2^\omega$ *are equivalent:*

(1) $\dim_e \mathbb{X} = 1$
(2) *For every total computable function* $f : \{0, \ldots, b\}^* \to \mathbb{Q}$, *for every* $\epsilon > 0$, *and for sufficiently large* n, *we have*

$$\min\left(\left\{|\sigma| : f(\sigma) - x| < b^{-n}\right\} \cup \{n-1\}\right) \geq n(1-\epsilon).$$

If we were only to require Condition 2 to hold where f is the function encoding the usual b-ary representation of rationals, we would have exactly a characterization of normality to base b found in [**437**], and (of course) if we require Condition 2 to hold of the b-ary representation functions for all b, we have a characterization of absolute normality.

One feature of effective Hausdorff dimension that may have struck the reader is how much of the theory of this dimension is carried out in the world of singletons. Another exciting recent area of advance in the theory of effective Hausdorff dimension is the *point-to-set principle* demonstrated by Lutz and Lutz. This principle relates the effective dimension of a single point in a set E to the classical Hausdorff dimension of E.

THEOREM 3.4.14 ([**345**]). *For every set* $E \subseteq \mathbb{R}^n$, *we have*

$$\dim_H(E) = \min_{A\subseteq\mathbb{N}} \left(\sup_{x\in E} \dim_e^A(x)\right),$$

where $\dim_e^A$ *denotes the effective dimension relative to an oracle for* A.

Lutz and Lutz initially used this result to give a novel proof of the previously known two-dimensional case of the Kakeya conjecture. Further applications are emerging rapidly.

3.4.2. Packing Dimension. An alternative approach to classical fractal dimensions arises by replacing coverings with packings in the definition of Hausdorff measure. To be precise, we obtain the packing measure $\pi_{\Delta,s}(X)$ by taking the supremum of $\sum_{i\in\mathbb{N}} \delta(B_i)$, where $(B_i : i \in \mathbb{N})$ is a disjoint sequence of closed balls with centers in X and diameters bounded by Δ. We then set $\pi_s(X) = \inf_{\Delta>0} \pi_{\Delta,s}(X)$. Since π_s is not subadditive, we instead construct a measure by

$$\Pi_s(X) = \inf\left\{\sum_{i\in\mathbb{N}} \pi_s(A_i) : X = \bigcup_{i\in\mathbb{N}} A_i\right\}.$$

We can again define a dimension $\dim_p(X)$ by the infimum of the set of dimensions s for which $\Pi_s(X) = 0$. This dimension is bounded from below by the Hausdorff dimension, and coincides with it on both the Cantor set and the Sierpinski carpet. More on the classical packing dimension can be found in [**166, 362**].

This packing dimension has also been effectivized, along a similar argument to that for Hausdorff dimension.

THEOREM 3.4.15 ([**35**]). *Let $X \subseteq 2^\omega$. We define $g(X,s)$ to be the set of all martingale indicators d such that for all $\mathbb{X} \in X$, we have*

$$\liminf_n \frac{d(\mathbb{X} \restriction_n)}{2^{(1-s)n}} = \infty.$$

Then $\dim_p(X)$ is the infimum of all s such that $g(X,s)$ is nonempty.

We defer the proof of this result to the next section when we consider box dimension, but it readily lends itself to effectivization. The *effective packing dimension* of X, denoted $dim_{ep}(X)$, is the infimum of all s such that $g(X,s) \cap \Sigma_1^0 \neq \emptyset$.

THEOREM 3.4.16 ([**35**]). *For any $C \subseteq 2^{<\omega}$, we denote by C^η the set*

$$\{\mathbb{X} \in 2^\omega : \mathbb{X} \restriction_n \in A \textit{ for all but finitely many } n\}.$$

Then for $X \subseteq 2^\omega$, we have

$$dim_{ep}(X) = \inf\left\{H_C : (X \subseteq A^\eta) \wedge \left(A \in \Sigma_1^0\right)\right\}.$$

The proof of this theorem is analogous to the proof of Theorem 3.4.11.

3.4.3. Box (a.k.a. "Box Counting," or "Minkowski" Dimension). A second straightforward modification of the Hausdorff dimension is to replace arbitrary coverings with small sets by using only coverings of small sets all of the same size.

DEFINITION 3.4.17. For $X \subseteq 2^\omega$ and $\epsilon > 0$, we define

$$N(X,\epsilon) = \min\left\{k : \exists(x_1,\ldots,x_k)\ X \subseteq \bigcup_{i=1}^k B_\epsilon(x_i)\right\}.$$

Let $B_r(x)$ denote the ball about x of radius r.

(1) The upper box dimension (also called the upper box counting dimension or upper Minkowski dimension) is defined by

$$\overline{\dim}_B(X) = \inf\left\{s : \limsup_{\epsilon\to 0} N(X,\epsilon)\epsilon^s = 0\right\}.$$

(2) The lower box dimension (also called the lower box counting dimension or lower Minkowski dimension) is defined by

$$\underline{\dim}_B(X) = \inf\left\{s : \liminf_{\epsilon\to 0} N(X,\epsilon)\epsilon^s = 0\right\}.$$

The "boxes" in the terminology arise from an equivalent definition in which the balls of fixed radius are replaced with axis-aligned boxes (squares, cubes, etc.). Since 2^ω does not have an obvious and intrinsic analogue to this approach, we use balls instead. Some authors call these dimensions "box counting" dimensions because of the central role played by the number of boxes (or balls) needed to cover X. A more detailed account of the different approaches can be found in [**166, 181, 362**].

One challenge with box dimension is that it represents some sets to be larger than seems intuitive, based only on countable phenomena. Consider, for instance, the set $F = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$. For any positive $\epsilon < \frac{1}{2}$, we find k such that

$$\frac{1}{(k+1)k} \leq \epsilon < \frac{1}{(k-1)k}.$$

Now to cover F, we need k balls of diameter ϵ, and we can calculate $\overline{\dim}_B(F) = \underline{\dim}_B(F) = \frac{1}{2}$. However, with the exception of the point 0, the set F is discrete, suggesting an intuitive dimension of 0. The usual solution of this concern, at cost of ruining the simplicity of box dimension, is to take the infimum over all possible countable covers $X \subseteq \bigcup_{i \in \mathbb{N}} X_i$ of the supremum over all i of the box dimension (upper or lower) of X_i. This gives, respectively, the upper or lower *modified box dimension*, denoted $\overline{dim}_{MB}(X)$ or $\underline{dim}_{MB}(X)$, respectively.

Definition 3.4.18 ([**413**]). Let $X \subseteq 2^\omega$.

(1) We say that $C \subseteq 2^{<\omega}$ is an effective cover iff it is computably enumerable and all elements of X restrict to an element of C.
(2) For any set $C \subseteq 2^{<\omega}$, we define $\nu(C, n) = \frac{\log |C \cap 2^n|}{n}$.
(3) We define the effective upper box dimension of $X \subseteq 2^\omega$, denoted $\overline{\dim}^1_B(X)$, as the infimum over all effective covers C of X of the quantity $\limsup_{n\to\infty} \nu(C, n)$.
(4) We define the effective lower box dimension of $X \subseteq 2^\omega$, denoted $\underline{\dim}^1_B(X)$, as the infimum over all effective covers C of X of the quantity $\liminf_{n\to\infty} \nu(C, n)$.

In consideration of singletons, there is no advantage in using modified Box dimension, so it is not standard to do so.

Proof of Theorem 3.4.15. It is a standard result (see, for instance, [**181**]), that $dim_P(X) = \overline{dim}_{MB}(X)$. Let $d \in g(X, s)$. We let B_n denote the set of strings σ of length n such that $d(\sigma) > d(\emptyset)$, and $\overline{B}_n$ the set of sequences in 2^ω that restrict to elements of B_n. Now we have

$$X \subseteq \bigcup_{i \in \mathbb{N}} \bigcap_{n=i}^{\infty} \overline{B}_n.$$

Further, $\overline{dim}_B \left(\bigcap_{n=i}^{\infty} \overline{B}_n \right) \leq s$, so that $\overline{dim}_{MB}(X) \leq s$.

On the other hand, if $s > t > \overline{dim}_{MB}(X)$, there is some cover $X = \bigcup_{i\in\mathbb{N}} X_i$ such that for all i we have $\overline{dim}_B(X_i) < t$. We let $B_{n,i}$ be the set of restrictions to length n of elements of X_i, and can then define $d \in g(X_i, s)$, which suffices to establish the theorem. □

3.5. An Example: Brownian Motion

Most probabilists, if asked to construct a random sequence, might first think of Brownian motion. In this standard process, we generalize the symmetric random walk by taking steps independently at random in either positive or negative direction over vanishing time intervals.

DEFINITION 3.5.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $w : [0,1]\times\Omega \to \mathbb{R}^n$ a stochastic process. We then say that w is a *Brownian motion* (also called a *Wiener process*) if and only if

(1) For each $s \in [0,1]$, the random variable $w(s,x) : \Omega \to \mathbb{R}^n$ is Gaussian with mean zero,
(2) For all finite partitions $(t_i)_{i\leq m}$ of I, the random variables
$$w(t_0,x), w(t_1,x) - w(t_0,x), \dots w(t_m,x) - w(t_{m-1},x)$$
are independent, and
(3) There is some constant σ such that for all $t, r \in [0,1]$, the random variable $w(t,x) - w(r,x)$ is Gaussian with mean zero and variance $\sigma^2|t-r|$.

Further, we say that $f \in C[0,1]$ is a *realization of a Brownian motion* w if and only if there is some $a \in \Omega$ such that $f(x) = w(x,a)$.

From its early observation by botanist Robert Brown down to its present application in modeling financial securities, such a process has a sound claim on status as a "standard" random process. It is worth considering what the theory developed in the present chapter tells us about Brownian motion.

Constructing a stochastic process with these properties was historically a challenge. One approach is the following one, due to Donsker, summarized in [**187**].

THEOREM 3.5.2 ([**155**]). *Let $C[0,1]$ denote the space of continuous functions on the unit interval. Let $(y_n : n \in \mathbb{N})$ be independent identically distributed variables with mean 0 and variance 1, and let $S_n = \sum_{i=1}^{n} y_n$. Define*
$$X_n(t) = \frac{S_{\lfloor nt\rfloor} + \big(nt - \lfloor nt\rfloor\big)y_{\lfloor nt\rfloor+1}\big)}{\sqrt{n}}.$$
Then there is a probability measure W on $C[0,1]$ such that for any Borel set A with W-null boundary, we have
$$\lim_{n\to\infty} P(X_n \in A) = W(A).$$
Moreover, W-almost every continuous function is a realization of a Brownian motion $\sigma = 1$.

This measure W is called *Wiener measure*. The perspective of Wiener measure allows us to connect Brownian motion to Kolmogorov complexity [**31**]. We denote by C_n the set of continuous functions f on the unit interval such that

- $f(0) = 0$
- On the interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, the function f is linear with slope $\pm\sqrt{n}$.

Algorithmically random sequences of these functions will be seen to approximate realizaitons of Brownian motions. We define a mapping $\alpha_n : C_n \to 2^n$ as follows.
$$\alpha_n(f)(i) := \begin{cases} 0\text{if } f(x) \text{ is increasing on } \left[\frac{i-1}{n}, \frac{i}{n}\right] \\ 1\text{otherwise} \end{cases}$$
This mapping allows treatment of the Kolmogorov complexity of functions in $\bigcup_{n\in\mathbb{N}} C_n$. We now have the approximation result.

THEOREM 3.5.3 ([**31**]). *For W-almost every function $f \in C[0,1]$, there is a sequence $(f_n : n \in \mathbb{N}) \subseteq C[0,1]$ with the following properties:*

(1) $f_n \in C_n$
(2) $K(\alpha_n(f_n)) \geq n - O(1)$.
(3) $\sup\limits_{x\in[0,1]} |f_n(x) - f(x)| \leq \frac{1}{n^{10}}$

We say that f is a *complex oscillation* if there is a sequence $(f_n : n \in \mathbb{N}) \subseteq C[0,1]$ satisfying the first two properties in the conclusion of the Theorem, but with the weaker convergence criterion that there is some total recursive function $\nu : \mathbb{N} \to \mathbb{N}$ such that if $n > f(m)$, we have

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| \leq \frac{1}{m+1}.$$

All complex oscillations are, in a certain sense, random: the Kolmogorov complexity of $\alpha_n(f_n)$ is maximal.

Theorem 3.5.4 ([**31, 187**]). *Let W denote the Wiener measure. Then if f is a complex oscillation, then for any W-Martin-Löf test $(U_n : n \in \mathbb{N}$, we have $f \notin \bigcap\limits_{n\in\mathbb{N}} U_n$.*

In this sense, the W-almost sure set of realizations of Brownian motions and the W-almost sure set of complex oscillations intersect in a W-almost sure set, so that we can regard these two sets (probabilistically) as identical. While the literature on complex oscillations and effective features of Brownian motion is still expanding, we state here a few representative results.

Theorem 3.5.5 ([**17**]). *Let f be a complex oscillation.*

(1) *The set of positive zeros of f does not contain a computable real.*
(2) *If $x > 0$ with $f(x) = 0$, then $dim_e(x) \geq \frac{1}{2}$.*
(3) *Given any computable real $\alpha > \frac{1}{2}$, there is $x > 0$ with $f(x) = 0$ and $dim_e(x) = \alpha$.*

Theorem 3.5.6 ([**409**]). *Let $(X_i : i \in \mathbb{N})$ and $(Y_i : i \in \mathbb{N})$ be a sequence of independent normal random variables of mean 0 and variance 1. We then construct a stochastic process f by*

$$f(t) = X_0 t + \frac{1}{\sqrt{2}\pi} \sum_{n\in\mathbb{N}} \frac{1}{n} \left(X_n \sin(2\pi n t) + Y_n(1 - \cos(2\pi n t))\right).$$

Then for any 1-random real $\mathbb{X}$, the realization of $f(t)$ given by evaluating all X_i and Y_i at $\mathbb{X}$ is a complex oscillation.

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