

Mathematical Logic and Probability

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Preface

In the late 19th and early 20th centuries, logic and probability were frequently treated as closely related disciplines. Each has, in an important sense, gone its own way, so that neither, in its modern form, is in any proper sense a systematization of the “Laws of Thought,” as Boole called them.

However, the last four decades have seen a remarkable rapprochement. On the most obvious level, the various probability logics have developed as formal systems of reasoning in the modern sense of logic.

At a deeper level, though, attempts have been made to formulate logics in which model theory of random variables, stochastic processes, and randomized structures can be explored from the perspective of model theory. Continuous first-order logic as a context for stability theory on metric structures is perhaps the most conspicuous example, but others exist.

At the same time, algorithmic randomness in its various forms has come to play a core role in computability theory, while probabilistic computation of various kinds (randomized computation, interactive proofs, and others) has come to dominate major parts of computational complexity. The older recursion-theoretic program of machine learning, initiated by Gold in the 1960s, has become much more important thanks to Valiant’s reformulation in probabilistic terms to allow for reasonable errors.

The model theory of random objects, Fraïssé limits, and pseudofinite structures, each of which embodies some important aspect of 0-1 laws, has been important for longer, but advances in stability, simplicity, and the transition from finite to infinite model theory have enriched this subject.

In set theory, too, the study of dynamics that respect probability measures has played a central role in the study of equivalence relations. Probability is frequently at the center of modern descriptive set theory.

Nor have these developments been independent. The PAC learning theory of Valiant is inextricably linked to the model theory of NIP theories. The dynamics of computable Polish spaces have become an important emerging area in computability. Randomized computation is the natural computation on metric structures. Notions of random structures have become intertwined with algorithmic randomness, and are naturally described in continuous first order logic.

Many of these developments have been adequately treated in isolation by various books. Probability logic has been discussed at length from various perspectives in [10, 239, 242, 394, 419]. Bayesian networks are well-covered, for instance, in [219, 397, 398], and a monograph on adapted distributions also exists [180]. Randomized computation has a detailed treatment in [30]. Algorithmic randomness is the subject of three relatively recent books, [158, 387, 194]. Zero-one laws are treated at length in [165, 230], and other places, and [266] includes an

extended treatment of Fraïssé limits. Random graphs are extensively covered in [74, 125, 338]. The definitive reference on PAC learning is [297]. In the field of set-theoretic dynamics, there have been several treatments at several levels of detail, among which [55, 260, 298, 303] merit special mention. There is no shortage of book-length treatments of subjects within the range of this book.

However, a reader in a well-stocked library might well pass all these separate books without knowing that they had anything substantive in common. Indeed, one could read most of them in detail — in addition to the long papers that give strong expositions on many related subjects (the seminal paper [61] on continuous first-order logic comes to mind) — without finding a commonality.

It is true that [238] describes connections between probability logics and Bayesian networks. However, it is silent on the rest of these issues.

The present book, then, attempts to take a unified — or, at least, unifying — approach to this subject. The expanding literature in each of these fields has seen more interaction between them, so that a model theorist might well want to know more about the frontier of probabilistic work in set theory, or a computability theorist more about the relevant work in model theory.

We focus here on *mathematical* logic and probability. Probability logic and its relatives seem frequently to arise as works of *philosophical* logic, and this has implications for the questions that are asked about it. Frequently it is seen in connection with the theory of rational decision, as in [242]. Mathematical logic, by contrast, asks about computability and undecidability; about theories and their models; about reducibilities and regularity of sets. Alternate logics are of interest to mathematical logic inasmuch as they provide the necessary infrastructure for carrying out this program in interesting settings. Applications of logic to artificial intelligence and other modeling contexts are important, but they arise as applications of the theory, not as its defining elements.

Chapter 1 begins to lay out the central thesis of the book: that all the other chapters have something to say to one another. This is done by identifying several important cross-cutting themes that come up in several of the other chapters.

In the next chapter, we begin the technical section of the book by describing the various logics useful for probability. Continuous first-order logic has a central role, not least because it generalizes many others. Probability logic is extensively studied, and is explored here as well, as are some other approaches.

In a third chapter, we will consider the theory of algorithmic randomness, with special attention to normal numbers, Martin-Löf randomness, and their relation to computation. This treatment will not be complete, of course — the subject is well-covered elsewhere. Rather, the focus will be on those aspects of algorithmic randomness that interact with other areas of advance in the logic and probability community.

The chapter on randomized computation involves the leap of reasoning that computability and complexity still have something to say to one another. Recent work on generic and coarse computability, as well as that on derandomization, descriptive complexity, and continuous first-order logic support this hypothesis.

The following two chapters will take up the various approaches to random structures. The investigation of random structures seems to have arisen historically from the study of random graphs, which invited generalization to 0-1 laws, and which

connected with the earlier beginnings of Fraïssé limits. More recent approaches consider the “random” structure as a single structure that somehow embodies the possible variation — graphons, Keisler randomizations, invariant random subgroups, and the like. Others use algorithmic randomness to define the structure.

In taking up the problem of learning theory, there is a fair viewpoint from which learning after the tradition of Gold, probably approximately correct (PAC) learning after the tradition of Valiant, and the model theory of NIP structures are wildly different fields. The chapter devoted to these topics takes the opposite view. Valiant’s definition is a natural extension of Gold’s framework, and the theory of Vapnik-Chervonenkis dimension governs both PAC learning and NIP theories.

The final chapter surveys the general area of dynamics. An introduction to orbit equivalence relations and Borel cardinality is given, and several topics on the relation of measure to equivalence relations are considered, including the implications of ergodicity and Hjorth’s notion of turbulence. Recent model-theoretic approaches to Szemerédi Regularity and Furstenberg Correspondence belong here, too, as does the characterization of 1-randomness by the Ergodic Theorem and the emerging theory of computable Polish spaces.

Of course, some limits must be set on the content of such a book. For instance, a new line of thought has arisen in recent years over categorical treatments of probability [199, 200, 395]. In view of traditional [350] and recent [119, 248, 247] work on connections between category theory and logic, this work is certainly interesting and relevant, but it is hard, at this stage of the theory, to explain its relationship to the other work.

The book is to be formally self-contained, but realistically anticipates a reader who has completed a first course in logic at the graduate or upper undergraduate level. Such a reader will, after reading the book, be prepared to understand the frontier of the research literature in probability-related areas of computability, model theory, set theory, and logical aspects of artificial intelligence. There is an important place in the world for a reader equipped in this way: A major part of logic in the coming years will involve connections between these fields, and those who understand something of all of them will be well-poised to contribute.

CHAPTER 2

Formulating Probability

2.1. Axioms for Probability

2.1.1. The Prehistory of Probability. While there have certainly been individual utterances related to probability from quite early on in recorded thought, as well as hints of at least potential organized knowledge of the field, it is difficult to avoid the impression that logic came first. Certainly mathematical logic came before mathematical probability.

In its medieval usage, “probability” was not a mathematical or quantitative concept at all. Some things, collectively called *scientia*, could be known certainly by “demonstration,” a precursor of modern concepts of proof. Other kinds of propositions, collectively called *opino* — and it was thought a genuinely different category — were the subject of belief, including belief based on sense perception. To say that an opinion was *probable* meant exactly that some trustworthy person believed it. There was no paradox, then, when Gibbon said of a certain opinion, “Such a fact is probable but undoubtedly false.”

This led to the common accusation, not unfamiliar in our times, that one’s opponent had first settled on an opinion, and then found some learned worthy whose words could be taken as acceptance. Such an approach, sometimes called “probabilism,” was one that conscientious scholars would presumably take pains to avoid.

Although various problems on the fair division of the stakes in an uncompleted game of chance circulated in the years before it, the first accounts of probability in anything like the modern sense appear to have arisen in the context of the Jansenist controversy in theological circles. Pascal was a collaborator and correspondent with several on the Jansenist side, including Antoine Arnauld and Pierre Nicole, authors of the 1662 *La Logique ou l’art de penser*, better known as the *Port Royal Logic*.

The *Port Royal Logic* started in a relatively conventional way, treating in its first part the problems around forming concepts; in the second part, deductive reasoning generally; in the third, syllogistic reasoning specifically. The fourth part begins as an exposition of analysis and synthesis, the two traditional parts of geometric reasoning, but the final four chapters take up a very different kind of reasoning. The matter under examination there is the application of reason to events “which may or may not be, when we inquire about the *future*, but which cannot be otherwise when we inquire about the *past*” [29]. It would be foolish to think, in probabilist fashion, that something is false in general merely because it fails in a particular case, or to think it true merely because it is possible. These uncertain beliefs, though, are explicitly compared to games of chance, and the frequency of the various outcomes of these games are used to reckon the degree of probability

of the proposition — for the first time a quantity. This analysis ends with the first published form of Pascal’s wager.

The history of probability, or the course by which Pascal, Fermat, Leibniz, Huygens, and finally Bernoulli extended these observations into what we now recognize as mathematical probability is not the subject of the present book. However, it is important to note two features regarding the first emergence of mathematical probability. First, it is important to note that it emerged as a branch of logic intended to deal with uncertain events. Second, we note that the genesis of modern probability happens exactly with the merger of what are later called the Bayesian and frequentist interpretations of probability.

In the frequentist interpretation, probability represents the proportion of one outcome in a (perhaps large) class of indistinguishable trials. In the Bayesian interpretation, the probability of a hypothesis reflects our degree of belief in a hypothesis. The authors of the *Port Royal Logic* used the frequency of truth to address the problem of assigning a degree of belief. Of course, the philosophy of probability, including the Bayesian/frequentist debate, has a large literature of its own. The explanation here of the early history of the subject was heavily influenced by [236].

The *Port Royal Logic* was widely circulated, but by 1847 Augustus De Morgan still felt a need to explain the inclusion of probability in his *Formal Logic*:

The old doctrine of modals is made to give place to the numerical theory of probability. Many will object to this theory as extralogical. But I cannot see on what definition, founded on real distinction, the exclusion of it can be maintained. When I am told that logic considers the validity of the inference, independently of the truth or falsehood of the matter, or supplies the conditions under which the hypothetical truth of the matter of the premises gives hypothetical truth to the matter of the conclusion, I see a real definition, which propounds for consideration the forms and laws of inferential thought. But when it is further added that the only hypothetical truth shall be absolute truth, certain knowledge, I begin to see arbitrary distinction, wanting the reality of that which preceded.... Not however to dispute upon names, I mean that I should maintain, against those who would exclude the theory of probability from logic, that, call it by what name they like, it should accompany logic as a study.[381]

In any case, De Morgan described probability in the context of logic, including arithmetical calculations and the assignment of a scale of probability, offering the Fahrenheit temperature scale as an example, but exploring in more detail the interval $[-1, 1]$.

When Boole undertook a general mathematicization of logic, he, too, included probability in its scope. Significantly for our investigation of algorithmic randomness in Chapter 3 and of machine learning in Chapter 7, one of his first motivating examples is the use of the fact that “the same characters and successions of characters recur with determinate frequency” to interpret archaeological texts from Ireland and the Near East. Boole sets out an axiomatic calculus of probabilities that would be familiar today either to students of elementary probability or to the

axiomatizers of the modern probability logics discussed in Section 2.2.2, as well as a “general method for the solution of problems in the theory of probabilities” [75].

Keynes described probability in much more strictly logical terms, axiomatizing an algebraic structure capable of serving as the possible values of probability [306]. For Keynes, too, probability was a kind of logic.

The tradition of interpreting probability as a generalization of logic continued for some time afterward in philosophical circles in the hands of Popper, Carnap, and others, and some of this work (in particular that of Popper) will be relevant for algorithmic randomness. However, it is not primarily this philosophical tradition that concerns us, but the communication of modern *mathematical* logic with modern probability.

2.1.2. The Kolmogorov Axiomatization. I have broken some with the standard scholarly delineation of the history. Most would place everything after Pascal — and certainly everything after Bernoulli — in the *history* of probability, not the prehistory. There is certainly fairness in this. However, the point I want to emphasize is that all of the probability discussed in the previous section belongs firmly to an age of the field at least as distant from the practice of modern probability as De Morgan’s classification of syllogistic forms is from that of modern logic.

A fair point of delineation in logic might be the Zermelo-Frankel formulation of set theory, or, more likely, the quick succession in the 1930s of Gödel’s completeness and incompleteness theorems, Tarski’s undefinability theorem, and Turing’s negative solution to the *Entscheidungsproblem*. That which came before is certainly not modern logic. That which came afterward will generally be accepted by modern logicians as being in continuity with modern research in the field. The analogous demarcation point in probability is Kolmogorov’s measure-theoretic formulation of probability.

It is true that Kolmogorov’s measure-theoretic approach opened up significant new vistas for the field, especially toward non-finite probability spaces, but his motivation, it seems germane to state, was

... to give an axiomatic foundation for the theory of probability.
The author set himself the task of putting in their natural place, among the general notions of modern mathematics, the basic concepts of probability theory — concepts which until recently were considered to be quite peculiar.[319]

Kolmogorov developed probability as a special case of Lebesgue’s measure theory.

DEFINITION 2.1.1. A probability space is a triple (Ω, \mathcal{B}, P) with the following properties:

- (1) Ω is a set.
- (2) \mathcal{B} is a σ -algebra of subsets of Ω .
- (3) $\Omega \in \mathcal{B}$
- (4) $P : \mathcal{B} \rightarrow \mathbb{R}^{\geq 0}$
- (5) $P(\Omega) = 1$
- (6) If $A, B \in \mathcal{B}$ are disjoint, then $P(A \cup B) = P(A) + P(B)$
- (7) If $A_1 \supseteq A_2 \supseteq \dots$ is a decreasing sequence of elements of \mathcal{B} with empty intersection, we have $\lim_{n \in \mathbb{N}} P(A_n) = 0$.

It is routine to see that this last condition is equivalent to the assertion that P is countably additive.

For any set S , an S -valued random variable is a function $X : \Omega \rightarrow S$. Typically the set S has some measure of its own and random variables are restricted to be the measurable functions. From this perspective, for a measurable subset $T \subseteq S$, we write $P(X \in T) = P(X^{-1}(T))$.

Using this formalism, it is possible to generalize the probability known to Bernoulli and Boole to contacts with what might be called a “completed infinity” of possible outcomes, those involving infinite-dimensional distributions. We express the expectation $E(X) = \int X \, dP$.

In this world, a large range of applications opens up. For instance, let \mathcal{G} be a graph, and equip \mathcal{G} with a sequence of functions $\eta_t : V \rightarrow \{0, 1\}$ on its vertices, as t ranges over \mathbb{N} .

EXAMPLE 2.1.2. Each vertex represents an individual in a population susceptible in principle to an infectious disease. Those vertices v with $\eta_t(v) = 1$ are said to be *infected* at time t and those with $\eta_t(v) = 0$ are said to be healthy. Of course, many models could describe the dynamics of this system over time, perhaps prescribing $\eta_{t+1}(v)$ in terms of the values of η_t on the neighborhood of v . This is called a *contact process*. Under specific models of the evolution of η_t , we could ask questions about the probability that, for instance, there is some \hat{t} such that for $t > \hat{t}$ there is some v such that $\eta_t(v) = 1$.

There arises, of course, the question of the necessity of this approach. In a sense, this necessity is established by a theorem of de Finetti [143, 145, 144]. Given a measurable space (Ω, \mathcal{B}) *betting system* is a function $b : \mathcal{B}' \times \Omega \rightarrow \mathbb{R}$, where $\mathcal{B}' \subseteq \mathcal{B}$ is countable, such that for each $S \in \mathcal{B}'$ there is a $q_S \in \mathbb{R}$ such that

$$b(S, x) := \begin{cases} 1 - q_S & \text{if } x \in S \\ -q_S & \text{otherwise} \end{cases}$$

That is, the gambler pays q_S for a potential payoff of 1. In this sense, we can interpret any function $\mu : \mathcal{B} \rightarrow \mathbb{R}$ as a betting system b_μ , by setting $q_S = \mu(S)$. A rational actor who, for each $S \in \mathcal{B}'$, believes with strength $\mu(S)$ that $x \in S$ will expect value 0 from $b_\mu(S, \cdot)$, and will accept any bet giving μ -expectation larger than that.

If b is a betting system, a Dutch book on b is a set $\mathcal{C} \subseteq \mathcal{B}'$ such that for any $x \in \Omega$, we have

$$\sum_{S \in \mathcal{C}} b(S, x) < 0.$$

A Dutch book, then, is a combination bet guaranteeing a loss for the agent who adopts betting system b . The following result, variously attributed to [412] and [144], shows that the axioms of probability are necessary, in the sense that a Dutch book can be made on any betting system that violates them.

THEOREM 2.1.3. *Let (Ω, \mathcal{B}) be a measurable space, and let μ be a function from \mathcal{B} to $\mathbb{R}^{\geq 0}$. If $(\Omega, \mathcal{B}, \mu)$ is not a probability space, then there is a Dutch book on b_μ .*

The proof consists of an elementary truth table construction, and is described in detail in [495].

A second result, which seems unequivocally attributed to de Finetti, suggests a kind of uniqueness for the probability measure on particular events. It seems difficult to locate in de Finetti's works an explicit statement of the theorem attributed to him — or, indeed, any proofs or many theorems — but the sentiments expressed by the theorem certainly arise in his work. A standard modern expression of the theorem can be found in [285]. For the statement of this theorem, we define a sequence $(X_i : i \in \mathbb{N})$ of random variables to be *exchangeable* if and only if for any permutation $\sigma \in S_\infty$ of the natural numbers and for any $j \in \mathbb{N}$, the joint distribution of $(X_{\sigma(1)}, \dots, X_{\sigma(j)})$ is equal to that of (X_1, \dots, X_j) .

THEOREM 2.1.4 (de Finetti's Theorem). *Let $(X_i : i \in \mathbb{N})$ be a sequence of random variables over a Borel measure space, taking values in S , with probability measure P . Then the following are equivalent:*

- (1) $(X_i : i \in \mathbb{N})$ is exchangeable.
- (2) There is some random variable ν taking values in the set of probability measures on S so that P is the product measure ν^∞ almost surely.

This “uniqueness” aspect of this theorem is explained, for instance, by Jeffrey in [279]: de Finetti did not believe in objective values of probability. However, he was able to give meaning to probability apart from any objective value by proving that any two observers would eventually agree upon a subjective probability measure. Dawid explains the impact of this theorem in some detail, including the observation that a person believing only in the exchangeability of a sequence of coin tosses would converge on a subjective probability distribution exactly matching that given by the usual (objective) axiomatization of Bernoulli trials [142].

If we consider the sequence $(X_i : i \in \mathbb{N})$ to be exchangeable, then as we observe X_i for more and more i , the subjective probability measure we would all eventually agree on is ν^∞ . In particular, if we write $P(T) = \nu^\infty(T)$ and take expected values, then we obtain

$$P(T) = E\nu^\infty = \int m^\infty P(\nu \in dm),$$

where m in the integral ranges over all probability measures on S .

PROOF. The implication from 2 to 1 is obvious from the definitions. To show the opposite direction, let $\mathbb{X} = (X_i : i \in \mathbb{N})$ be an exchangeable sequence. Let \mathcal{I} be the shift-invariant σ -algebra in the natural probability space of sequences on which \mathbb{X} is a random variable, and let $\mathcal{I}_{\mathbb{X}}$ be the pre-image of this σ -algebra under \mathbb{X} .

For a probability measure P , a random variable A and a σ -algebra \mathcal{C} , we define the conditional probability $P(A|\mathcal{C})$ in the standard way [284]. That is, we first define the conditional *expectation* first for the case where $A \in L^2$, so that $E(A|\mathcal{C})$ is the orthogonal projection of A onto the subspace of \mathcal{C} -measurable functions. This can be extended, by continuity, to $A \in L^1$. We then define $P(A|\mathcal{C}) := E(\chi_A|\mathcal{C})$. Thus, $P(A|\mathcal{C})$ is not a constant, but a random element of the space of probability measures. This specializes to the elementary conditioning on a single event when \mathcal{C} is the smallest σ -algebra containing that event. We now define $\nu = P(X_1|\mathcal{I}_{\mathbb{X}})$.

To verify that this ν has the properties required, we take $I \in \mathcal{I}$ and let f_1, \dots, f_m be bounded measurable functions, and with the dominated convergence theorem (recalling that we have assumed \mathbb{X} is exchangeable) we let $n \in \mathbb{N}$ and

calculate

$$\begin{aligned} E(\chi_I(\mathbb{X})) \prod_{i=1}^m f_i(X_i) &= \lim_{n \rightarrow \infty} \frac{1}{n^m} \sum_{j_1, \dots, j_m \leq n} E(\chi_I(\mathbb{X})) \prod_{k=1}^m f_k(X_{kn+j_k}) \\ &= \lim_{n \rightarrow \infty} E(\chi_I(\mathbb{X})) \prod_{k=1}^m \frac{\sum_{j=1}^n f_k(X_{kn+j})}{n}. \end{aligned}$$

We now use the definition of conditional probability, combined with the mean ergodic theorem, to find that this quantity converges to $E(\chi_I(\mathbb{X})) \prod_{k=1}^m \nu f_k$. Now for any product-measurable set B , we have $P(\mathbb{X} \in B | \mathcal{I}_{\mathbb{X}}) = \nu^\infty(B)$ almost surely, as required. \square

It is natural to ask about the effectiveness of the preceding proof: given a sequence of random variables that is, in some appropriate sense, effective, can we find a computable measure ν ? This question has added significance in light of Jeffrey's interpretation. Following the reasoning of algorithmic game theory, if the agents can't effectively find the Nash equilibrium, then its power to predict their behavior is limited. Similarly, if the so-called *de Finetti measure* is not computable, we could not expect computable agents to agree on it.

THEOREM 2.1.5 ([195]). *Let $F_{\mathbb{X}}$ be the distribution of a real-valued exchangeable sequence \mathbb{X} and F_ν be the distribution of the measure whose existence is guaranteed by Theorem 2.1.4. Then $F_{\mathbb{X}} \equiv_T F_\nu$ in the sense of type-two functions.*

PROOF. Let \mathbb{X} be an exchangeable sequence of real variables, and B a Borel set. We hope to define a random variable Y_B by $Y_B = F_\nu(B)$, and it suffices to define it where B is an interval (a, b) with rational endpoints.

Now if $\beta : \mathbb{N} \rightarrow \mathcal{B}$ is an enumeration of some Borel sets, by de Finetti's theorem, we have $P\left(\bigcap_{i=1}^k \{X_i \in \beta(i)\}\right) = E\left(\prod_{i=1}^k Y_{\beta(i)}\right)$.

It can be shown that if $R : \mathbb{N} \rightarrow \mathcal{B}$ is an enumeration of finite unions of open rectangles in \mathbb{Q}^k , then from a degree computing $F_{\mathbb{X}}$, we can enumerate all rational lower bounds on $P\left(\bigcap_{i=1}^k \{X_i \in R(i)\}\right)$. By noting that almost surely F_ν places no mass on boundaries, an enumeration of upper bounds can at least be simulated.

LEMMA 2.1.6. *Let \mathbb{Y} be a sequence of $[0, 1]$ random variables with distribution F_η . Then F_η is Turing equivalent to the sequence $E\left(\prod_{i=1}^k Y_{j(i)}\right)$ as k ranges over \mathbb{N} and j ranges over \mathbb{N}^k .*

PROOF. That the mixed moments $E\left(\prod_{i=1}^k Y_{j(i)}\right)$ should be computable relative to F_η is straightforward from the computability of polynomials and of integrals.

On the other hand, let B be a k -tuple of open intervals with rational endpoints, and consider that $F_\eta = E(\chi_B(Y))$. We can find a sequence of polynomials converging pointwise to χ_B from below, completing the proof of the Lemma and the Theorem. \square

\square

In considering Kolmogorov’s axiomatization of probability, it is also interesting to point out the recent formalization of a proof of the Central Limit Theorem in [37] in the Isabelle proof assistant. Much of the work in this formalization included formalization of the appropriate objects in measure theory.

2.2. Reasoning With Probability

2.2.1. The Primacy of Probability. If you want to compute the probability of a conjunction, you could think of

$$P(A \wedge B) = P(A|B) \cdot P(B).$$

Even under some additional hypotheses, you could reckon

$$P(A \wedge B|H) = P(A|B \wedge H) \cdot P(B|H).$$

Similarly, you could compute the probability of a disjunction by

$$P(A \vee B|H) = P(A|H) + P(B|H) - P(A \wedge B|H),$$

and that of a negation by $P(\neg A|H) = 1 - P(A|H)$, and from there you’re off to all propositional logic.

Probability is a natural enough way for us, as mathematicians to think about events “which may or may not be,” and in fact it’s hard to think of a reason we would do anything else. In fact, there’s a proof, due to Cox [139] that probability is enough. There’s a bit of setup for it.

Let’s use $L(A|B)$ to denote some \mathbb{R} -valued “likelihood,” some credibility that we assign to A under the assumption of B . Probability is certainly the key example, but you could certainly think of others. One could certainly take any function $f : \mathbb{R} \rightarrow \mathbb{R}$ and take $L(A|B) = f(P(A|B))$. In common language, one often sees $f(x) = 100x$, or a “rareness” function like

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x > 10^{-100} \\ 10^{100} & \text{otherwise.} \end{cases}$$

A good start is to restrict what sorts of factors can be considered. One reasonable restriction arises from an accounting of the things that “should be” relevant, and restricting L to only considering those. In a US presidential election, the likelihood that a given candidate will win the votes of both Ohio and Georgia should depend heavily on the likelihood of winning Georgia and the likelihood of winning Ohio given the candidate wins Georgia. There may be other ways to calculate it, but if you know those two, then everything else has either been accounted for or is irrelevant. So we have the first axiom: There is some function F such that

$$(1) \quad L(C \wedge B|A) = F(L(C|B \wedge A), L(B|A)).$$

If L is probability, then we even know what this function is: it’s multiplication.

If we also believe that \wedge is associative, we get some restriction on F . If we start with $L(D \wedge C \wedge B|A)$, then by our axiom, it must equal

$$F(F(L(D|C \wedge B \wedge A), L(C|B \wedge A)), L(B|A)).$$

On the other hand, it must also equal

$$F(L(D|C \wedge B \wedge A), F(L(C|B \wedge A), L(B|A))).$$

Consequently, we have

$$F(F(x, y), z) = F(x, F(y, z)).$$

LEMMA 2.2.1. *For any solution F to the functional equation above, having second derivatives, there must exist a unary function f and a constant C such that $Cf(x, y) = f(x)f(y)$.*

PROOF. We start with the given equation,

$$F[F(x, y), z] = F[x, F(y, z)],$$

and let $F(x, y) = u$ and $F(y, z) = v$. Then our equation is

$$F(u, z) = F(x, v).$$

We differentiate with respect to x and get

$$F_1(u, z) \frac{\partial u}{\partial x} = F_1(x, v).$$

Differentiating this with respect to x gives

$$F_{11}(u, z) \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + F_1(u, z) \frac{\partial^2 u}{\partial x \partial y} = F_{12}(x, v) \frac{\partial v}{\partial y}.$$

If we differentiate each time with each of the three variables, we get

$$\frac{\partial^2 u / \partial x \partial y}{\partial u / \partial x} - \frac{\partial^2 u / \partial y^2}{\partial u / \partial y} = \frac{\partial^2 v / \partial y \partial z}{\partial v} \partial z - \frac{\partial^2 v / \partial y^2}{\partial v / \partial y}.$$

Using logarithmic differentiation, we get

$$\frac{\partial}{\partial y} \ln \left(\frac{\partial u / \partial x}{\partial u / \partial y} \right) = - \frac{\partial}{\partial y} \ln \left(\frac{\partial v / \partial y}{\partial v / \partial z} \right).$$

By the definitions of u and v , we have

$$\frac{\partial}{\partial y} \ln \left(\frac{F_1(x, y)}{F_2(x, y)} \right) = - \frac{\partial}{\partial y} \ln \left(\frac{F_1(y, z)}{F_2(y, z)} \right).$$

Since the x 's only appear on one side and the z 's on the other, each side of the equation must be determined entirely by y , so we set the left side equal to

$$\frac{d}{dy} \ln \Phi(y),$$

and the right hand side equal to its negative. By renaming variables in the equation for the right hand side, we can have

$$\frac{\partial}{\partial x} \ln \left(\frac{F_1(x, y)}{F_2(x, y)} \right) = - \frac{d}{dx} \ln \Phi(x).$$

We can now combine these to get

$$\frac{\partial}{\partial x} \ln \left(\frac{F_1(x, y)}{F_2(x, y)} \right) dx + \frac{\partial}{\partial y} \ln \left(\frac{F_1(x, y)}{F_2(x, y)} \right) dy = -d \ln \Phi(x) + d \ln \Phi(y).$$

We can integrate this, and get

$$\frac{F_1(x, y)}{F_2(x, y)} = C_0 \frac{\Phi(y)}{\Phi(x)}.$$

When we combine this with all the first derivatives, we get

$$\frac{\partial F(x, y)}{\partial \cdot} = \frac{\Phi(F(x, y))}{\Phi(\cdot)}.$$

Now the equation comes up to be

$$\frac{dF}{\Phi(F)} = \frac{dx}{\Phi(x)} + \frac{dy}{\Phi(y)},$$

and we can integrate that to get $C_1 f(F) = f(x)f(y)$, as was to be shown. \square

As we already said, any f will do, so we'll choose the identity. Then we have

$$C_1 F(p, q) = pq,$$

so that

$$F(p, q) = \frac{1}{C_1} pq,$$

and, in particular,

$$L(C \wedge B|A) = \frac{1}{C_1} (L(C|B \wedge A)) \cdot (L(B|A)).$$

We can also specify C_1 based on convention and first principles. Let $C = B$. Then this last equation is equivalent to

$$L(B|A) = \frac{1}{C_1} (L(B|B \wedge A)) \cdot (L(B|A)).$$

If we clear the denominator and divide both sides by $L(B|A)$, we get

$$C = L(B|B \wedge A).$$

So we have to pick what the likelihood of certainty is.

We're ready for the second assumption: There is some function S such that

$$(2) \quad L(\neg B|A) = S(L(B|A)).$$

Intuitively, the likelihood of B failing is determined by that of B holding. Since we probably want $L(\neg\neg B|A) = L(B|A)$, we have $S(S(x)) = x$. Then by De Morgan's law (which we presumably also want),

$$S(L(C \vee B|A)) = L(\neg C \wedge \neg B|A).$$

But then using the previous part, we have

$$S(L(C \vee B|A)) = S(C|\neg B \wedge A)S(B|A).$$

With another application of S and a little more work, we get

$$\frac{S(L(B|C \wedge A))L(C|A)}{S(L(B|A))} = S\left(\frac{S(L(C \vee B|A))}{S(L(B|A))}\right).$$

We can rewrite this to have the same hypothesis in all cases.

$$S\left(\frac{L(C \wedge B|A)}{L(C|A)}\right) L(C|A) = S\left(\frac{S(L(C \vee B|A))}{S(L(B|A))}\right) S(L(B|A)).$$

Now let $B = C \wedge D$, so that

$$S\left(\frac{L(C \wedge D|A)}{L(C|A)}\right) L(C|A) = S\left(\frac{S(L(C|A))}{S(L(C \wedge D|A))}\right) S(L(C \wedge D|A)).$$

Now we have another functional equation, this time for S :

$$xS\left(\frac{S(y)}{x}\right) = yS\left(\frac{S(x)}{y}\right).$$

LEMMA 2.2.2. *Let S be twice differentiable, and let it satisfy the previous equation. Then there is some m such that $S(p) = (1 - p^m)^{1/m}$.*

PROOF. Write $u := \frac{S(y)}{x}$ and $v := \frac{S(x)}{y}$. Then the equation is

$$xS(u) = yS(v).$$

If we differentiate by x and then y , we get

$$\frac{uS''(u)S'(y)}{x} = \frac{vS''(v)S'(x)}{y}.$$

Multiplying by the previous version, we get

$$uS''(u)S(u)S'(y) = vS''(v)S(v)S'(x).$$

With the first derivatives, we can get

$$\frac{uS''(u)S(u)}{(uS'(u) - S(u))S'(u)} = \frac{vS''(v)S(v)}{(vS'(v) - S(v))S'(v)}.$$

Again, we have separated variables, so that the right hand side is equal to a constant k . Then

$$uS''(u)S(u) = k(uS'(u) - S(u))S'(u),$$

so that

$$\frac{dS'}{S'} = k \left(\frac{dS}{S} - \frac{du}{u} \right).$$

We can integrate to get

$$S' = A_0 \left(\frac{S}{u} \right)^k.$$

We can integrate again to get

$$S^{1-k} = A_0 u^{1-k} + A_1.$$

For this to actually satisfy the equation, $A_0 = -1$ and $A_1 = 1$. The result follows, with $m = 1 - k$. \square

Now, in particular, $L(B|A)^m + L(\neg B|A)^m = 1$. We make another conventional assumption and assume $m = 1$. If $B = A$, then we find that the likelihood of impossibility is 0.

With these assumptions in place, and using the foregoing calculations, we also can get the usual inclusion-exclusion formula.

LEMMA 2.2.3. *The following hold:*

- (1) $L(C \wedge B|A) + L(\neg C \wedge B|A) = L(B|A)$
- (2) $L(C \vee B|A) = L(C|A) + L(B|A) - L(C \wedge B|A)$.

DEFINITION 2.2.4 ([242]). A conditional probability measure is a binary function $\mu : \mathcal{E} \times \mathcal{E} \rightarrow [0, 1]$ such that

- (1) $\mu(U|U) = 1$ for any U
- (2) If V_1 and V_2 are disjoint, then $\mu(V_1 \cup V_2) = \mu(V_1) + \mu(V_2)$
- (3) $\mu(V|U) = \mu(V \cap U|U)$
- (4) $\mu(U_1|U_3) = \mu(U_1|U_2) \times \mu(U_2|U_3)$ whenever $U_i \subseteq U_{i+1}$.

For finite probability spaces, this agrees exactly with Kolmogorov's definition in the previous section, and for infinite spaces we need only extend the additivity condition. We have shown that any function L , as described in the foregoing discussion, is a conditional probability measure.

THEOREM 2.2.5 (Cox, 1946). *Let $L(\cdot|\cdot)$ be a binary \mathbb{R} -valued function on events such that*

- (1) \wedge *is associative in the second argument,*
- (2) \neg *is a reflection in the first argument,*
- (3) *For all B, C , De Morgan's law $\neg(C \vee B) \Leftrightarrow \neg C \wedge \neg B$ holds in the first argument.*
- (4) *There is a twice-differentiable function F such that for all A, B, C , we have*

$$L(C \wedge B|A) = F(L(C|B \wedge A), L(B|A)),$$

- (5) *There is a twice-differentiable function S such that for all A, B , we have*

$$L(\neg B|A) = S(L(B|A)).$$

Then L is a conditional probability measure.

Halpern noted some issues with this proof. The big one is that the functional equations are assumed to hold for all possible values. We could use continuity and deal with only a dense set of values. This means, among other things, that for any U and any $x \in [0, 1]$, there is some V such that $L(V|U) = x$.

Also, it assumes that L is real-valued. This means that every event is comparable in likelihood to every other event — even under any (different) conditions on either of them.

THEOREM 2.2.6 (Halpern 1999 [240]). *There is a function L_0 , a finite domain W , and functions S , F , and G such that*

- (1) $L_0(\neg V|U) = S(L_0(V|U))$
- (2) $L_0(V_1 \wedge V_2|U) = F(L_0(V_2|V_1 \wedge U), L_0(V_1|U))$
- (3) $L_0(V|U) \in [0, 1]$
- (4) $S(x) = 1 - x$
- (5) $G(x, y) = x + y$
- (6) F *is infinitely differentiable, strictly increasing in each argument, commutative, $F(x, 0) = 0$ and $F(x, 1) = x$.*
- (7) $L_0(V_1 \vee V_2|U) = G(L_0(V_1|U), L_0(V_2|U))$.
- (8) *There is no function f making L a conditional probability measure.*

PROOF. Let $W = \{w_1, \dots, w_{12}\}$, and weight each point as follows:

$$\begin{aligned} f(w_1) &= 3 \\ f(w_2) &= 2 \\ f(w_3) &= 6 \\ f(w_4) &= 5 \times 10^4 \\ f(w_5) &= 6 \times 10^4 \\ f(w_6) &= 8 \times 10^4 \\ f(w_7) &= 3 \times 10^8 \\ f(w_8) &= 8 \times 10^8 \\ f(w_9) &= 8 \times 10^8 \\ f(w_{10}) &= 3 \times 10^{18} \\ f(w_{11}) &= 2 \times 10^{18} \\ f(w_{12}) &= 14 \times 10^{18} \end{aligned}$$

Define $f(U) = \sum_{w \in U} f(w)$, and define $P(U) = f(U)/f(W)$.

We will pick a certain δ , and define

$$f'(w) = \begin{cases} (3 - \delta) \times 10^{18} & \text{if } w = w_{10} \\ (2 + \delta) \times 10^{18} & \text{if } w = w_{11} \\ f(w) & \text{otherwise} \end{cases} \quad ll,$$

and set $W' = \{w_{10}, w_{11}, w_{12}\}$. For any nonempty U , define

$$L_0(V|U) = \begin{cases} f'(V \cap U)/f(U) & \text{if } W' \subseteq U \\ f(V \cap U)/f(U) & \text{otherwise} \end{cases}.$$

Now pick $\Delta > 0$ so that whenever $P(V_1|U_1) > P(V_2|U_2)$ we have $L_0(V_1|U_1) > L_0(V_2|U|2)$

The remainder of the proof of the counterexample can be found in [240]. Further discussion of the conditions under which Cox's theorem does hold is found in [241]. \square

Note that given an unconditional real-valued likelihood measure L , we can always define a conditional version with

$$L(A|B) = \frac{L(A \wedge \neg B) - L(\neg B)}{1 - L(\neg B)}.$$

Similarly, given a conditional likelihood measure, we can always define an unconditional version by conditioning on the entire sample space.

2.2.2. Probability Logic. In the most straightforward connection of probability to logic, and the one suggested by the preceding sections, we understand classical logic to have truth values at the two ends of the spectrum: certainty and impossibility. We then extend the possible truth values by assigning probabilities to sentences. There are several systems of syntax and semantics for this task in the literature. Although this interpretation goes back, as we have seen, deep in the history of probability, the root of current work in the area seems to be in the 1960's. At that time, Gaifman proposed assigning a measure to sentences.

Throughout, we let L be a first-order signature, and $L(U)$ be an expansion of L by constants U .

DEFINITION 2.2.7. If G is a set of sentences of $L(U)$, then we say that $\mu : G \rightarrow \mathbb{R}^{\geq 0}$, not everywhere zero, is a syntactic measure if and only if its domain is closed under boolean combinations and for all $\varphi, \psi \in G$, we have

- (1) If $\vdash \varphi$ and $\vdash \psi$ then $\mu(\varphi) = \mu(\psi)$, and
- (2) If $\vdash \neg(\varphi \wedge \psi)$ then $\mu(\varphi \vee \psi) = \mu(\varphi) + \mu(\psi)$.

We say that μ is a syntactic probability measure if and only if $\mu(\varphi) = 1$ whenever $\vdash \varphi$.

Gaifmain notes that a first-order theory gives a syntactic measure mapping the true sentences to 1 and the false ones to 0. It is natural to ask about the connection to Kolmogorov's continuity axiom (axiom 7 of the definition).

LEMMA 2.2.8. *Let μ be a syntactic measure on the quantifier-free sentences of $L(U)$. Let $(\varphi_i : i \in \mathbb{N})$ be a sequence of quantifier-free sentences such that $\varphi_{i+1} \rightarrow \varphi_i$ for every i , with $\{\varphi_i : i \in \mathbb{N}\}$ inconsistent. Then $\lim_{i \rightarrow \infty} \mu(\varphi_i) = 0$.*

PROOF. By compactness, there must either be an n for which $\{\varphi_i : i \leq n\}$ is inconsistent, or else $\{\varphi_i : i \in \mathbb{N}\}$ is consistent. Consequently, there must be some n such that for $m > n$ we have $\mu(\varphi_m) = 0$. \square

For the semantics of this approach, Gaifman proposed *probability models* of the following form.

DEFINITION 2.2.9. A probability model for L is a pair (U, μ) where U is a set of constants and μ is a syntactic probability measure on the quantifier-free formulas of $L(U)$.

There is, of course the problem of extending μ to sentences which are not quantifier-free. Gaifman gave the following characterization

THEOREM 2.2.10 ([207]). *Let (U, μ) be a probability model. Then there is a unique syntactic probability measure μ^* on the full set of sentences of $L(U)$ which extends μ and satisfies*

$$\mu^*(\exists x \varphi(x)) = \sup_{\bar{a} \subseteq U} \left\{ \mu^* \left(\bigvee_{i=1}^n \varphi(a_i) \right) \right\}.$$

PROOF. We show how to construct μ^* in the countable case. Let \mathcal{K} be the set of all L -structures with universe U . We take \mathcal{K}_0 to be the Boolean algebra of subsets B_φ of \mathcal{K} defined by their satisfaction of a quantifier-free formula φ . We use μ to define a measure μ_1 on \mathcal{K}_0 .

Since μ_1 is continuous (as per Lemma 2.2.8), it can be extended to a countably additive probability measure on the σ -algebra generated by \mathcal{K}_0 . Since, for any quantifier-free φ , the set of structures satisfying $\exists x \varphi(x)$ is a countable union of elements of \mathcal{K}_0 , we can define μ^* by induction on the quantifier depth. \square

An important class of probability measures that we will explore at length in Chapter 6 consists of the exchangeable measures.

THEOREM 2.2.11 ([207]). *Let μ be a syntactic probability measure on $L(U)$. Then there is a set $V \supseteq U$ and a probability model (V, m) agreeing with μ which is invariant under permutations of $V - U$.*

PROOF. Let V be a proper superset of U , and S_D the set of permutations of $V - U$. We denote by $S_{D,0}$ the finitely supported elements of S_D . For each $\sigma \in S_{D,0}$ we define $[\sigma]$ to be the set of functions $\tau \in S_D$ such that $\tau \upharpoonright_{\text{dom}(\sigma)} = \sigma$.

We now construct a measure on S_D under which the σ -algebra generated by the sets $[\sigma]$ is measurable. Let $\{a_i : i \in \mathbb{N}\}$ be the distinct elements of $V - U$, and let ν be a countably additive probability measure on $V - U$ that gives each element positive measure. We can now interpret the product measure on $(V - U)^\omega$ as a measure on S_D by identifying $\sigma \in F$ by $\sigma(\vec{a})$.

Let m be a syntactic probability measure on $L(V)$. We define a syntactic measure m_ν on a sentence $\varphi(\vec{b})$, with \vec{b} representing all members of $V - U$ occurring in it, by

$$m_\nu(\varphi(\vec{b})) = \sum_{\sigma \in S_{D,0}} (m(\varphi(\sigma(\vec{b}))) \nu[\sigma]).$$

This “averaging” method achieves the necessary invariance, and also represents a strategy that we will see several times in Chapter 6. \square

Since Gaifman, many others have approached this topic. Indeed, Williamson complains, “The problem is that the literature contains a plethora of new axiomatisations of probability on sentences, few of which bear a close resemblance [to] the mathematical formulation of probability” [501]. One approach that does follow mathematical probability closely is that of Adams, who requires the following for all sentences:

- (1) $0 \leq P(\varphi) \leq 1$
- (2) If φ is logically true, then $P(\varphi) = 1$.
- (3) If φ logically implies ψ , then $P(\varphi) \leq P(\psi)$.
- (4) If φ and ψ are logically inconsistent, then $P(\varphi \vee \psi) = P(\varphi) + P(\psi)$.

For obvious reasons, although carefully distinguishing his instantiation of the axioms by sentences, instead of sets, Adams calls these the Kolmogorov axioms. Frequently, Adams treats the *uncertainty* of a sentence φ , defined as $1 - P(\varphi)$, as the more fundamental concept, and demonstrates the following

THEOREM 2.2.12 ([10]). Let $\{\varphi_1, \dots, \varphi_n\} \vdash \psi$.

- (1) $1 - P(\psi) \leq \sum_{i=1}^n (1 - P(\varphi_i))$
- (2) Let u_1, \dots, u_n be nonnegative numbers with $\sum_{i=1}^n u_i \leq 1$. Assume also
 - (a) $\{\varphi_1, \dots, \varphi_n\}$ is consistent, and
 - (b) For any $T \subsetneq \{\varphi_1, \dots, \varphi_n\}$ we have $T \not\vdash \psi$.
 Then there is a probability measure P such that for all i we have

$$1 - P(\varphi_i) = u_i$$

and

$$1 - P(\psi) = \sum_{i=1}^n u_i.$$

Adams develops the theory in some detail, including a theory to recognize a change in probability assigned in the face of new information. Hailperin proposed another probability logic similar in some ways to Gaifman’s, and others have given a definition of *probabilistic logic* sufficient to interpret many of these attempts. In the

framework of [238], we consider premises and conclusions of the form $P(\varphi) \in X$, where $X \subseteq [0, 1]$. Then in the logic of Gaifman, for instance, or in the logic proposed in [239], if S is a set of sentences and ψ is a sentence, then the entailment relation $S \models P(\psi) \in Y$ holds if every probability measure P satisfying S , we have $P(\psi) \in Y$. The key enhancement in Hailperin's treatment is that the probabilities need not have sharp values.

DEFINITION 2.2.13 ([238]). A *probabilistic argumentation system* is a triple $\mathcal{A} = (V, W, \Phi, P)$, where

- (1) V is a set of propositional atoms, with $W \subseteq V$
- (2) Φ is a set of sentences in V
- (3) P is a probability measure defined on subsets of the space 2^W of joint values for the elements of W .

In this context, for each propositional formula ψ , we define the set $A_{\mathcal{A}}(\psi)$ to be the set of $\sigma \in 2^V$ such that if the values of σ are instantiated for the atoms V , then the formula ψ is satisfied. We further define the *degree of support* for ψ by

$$d_{s,\mathcal{A}}(\psi) = \frac{P(A_{\mathcal{A}}(\psi)) - P(A_{\mathcal{A}}(\perp))}{1 - P(A_{\mathcal{A}}(\perp))},$$

and the *degree of possibility* for ψ by

$$d_{p,\mathcal{A}}(\psi) = \frac{1 - P(A_{\mathcal{A}}(\neg\psi))}{1 - P(A_{\mathcal{A}}(\perp))}.$$

This approach is equivalent to the belief functions of Dempster and Shafer.

DEFINITION 2.2.14 ([149, 439]). Let Ω be a set. Consider a partial function $B : \mathcal{P}(\Omega) \rightarrow [0, 1]$. We say that B is a *belief function* if and only if the following conditions hold:

- (1) $B(\emptyset) = 0$
- (2) $B(\Omega) = 1$
- (3) $B\left(\bigcup_{i=1}^n U_i\right) \geq \sum_{i=1}^n (-1)^{i+1} \sum_{|I|=i} B\left(\bigcap_{j \in I} U_j\right).$

The connection arises from the following result.

THEOREM 2.2.15 ([237, 238]). For any probabilistic argumentation system \mathcal{A} there exists an equivalent Dempster-Shafer belief function B such that $d_{s,\mathcal{A}} = B$, and conversely.

It is worth repeating: there are many probability logics in the literature, many of them with similar names and many with very similar (but seldom identical) formulations. Much of the work around them appears to happen in the literature of philosophical logic and artificial intelligence, and somewhat less, to date, in the literature of mathematical logic. While these boundaries are, of course, imprecise, a notable exception in recent years has been Paris and Vencovská's recent volume merging the more recent probability logics with the work on first-order predicate logic by Gaifman, with which this section started [394].

2.3. Axiomatizing and Computing Conditionality

2.3.1. Independence and Bayesian Networks. Kolmogorov identified independence as the central feature to distinguish probability in particular from measure theory in general, pointing out that the major theorems of probability known at his time generally either assumed independence as a hypothesis or introduced some weaker condition similar to it. In practical terms, too, dependence and independence have a special place. Much of the applicability of probability turns on the question of predicting one event with greater certainty than would otherwise be possible, using information about another event. For this reason, there has been substantial work in the artificial intelligence community to formalize this notion.

A major direction of this work has been Pearl's concept of Bayesian networks, which are intended to simplify conditional probability computations by representing the dependence relations efficiently [397]. A full table of the joint distribution even of n binary variables would be rather large for computation, and even with that distribution represented, the computation of even a simple conditional probability would be slow.

The first thing to typically do, then, is to compute whether things are independent, and ignore irrelevant information. Since computing the full joint distribution, and then computing from it the conditional and unconditional probabilities, as in the traditional definitions of independence, is computationally unfeasible (for instance, the full joint distribution table for n Boolean variables would have 2^n entries), there must, Pearl reasoned, be some other approach.

DEFINITION 2.3.1. We say that a ternary relation $I(X, Y, Z)$ on sets is an independence relation if and only if the following conditions hold:

- (1) $I(X, Z, Y)$ if and only if $I(Y, Z, X)$.
- (2) If $I(X, Z, Y \cup W)$, then $I(X, Z, Y)$ and $I(X, Z, W)$.
- (3) If $I(X, Z \cup W, Y)$ and $I(X, Z \cup Y, W)$, then $I(X, Z, Y \cup W)$.
- (4) If $I(X, Z, Y \cup W)$, then $I(X, Z \cup W, Y)$.
- (5) If $I(X, Z, Y)$ and $I(X, Z \cup Y, W)$, then $I(X, Z, Y \cup W)$.

It is routine to verify that for any probability measure P , the relation $I_P(X, Y, Z)$ defined by $P(X|Y, Z) = P(X|Z)$ is an independence relation. It should be noted that this differs from the definition of independence relation in model theory; in addition to the concepts like invariance that would require additional context even to formulate, transitivity is lacking. The first critical issue is the question of whether this characterization precisely captures the notion of conditional independence.

The following ‘‘Completeness Conjecture’’ was initially put forward, and would settle the matter were it true.

CONJECTURE 2.3.2 ([399]). *Let I be an independence relation. Then there is a probability space (Ω, \mathcal{B}, P) such that I is a relation on \mathcal{B}^3 and coincides exactly with probabilistic independence on (Ω, \mathcal{B}, P) .*

As we will see, this conjecture is not quite true, but some important variants of it are true. A basic Horn formula is a first-order formula of the form $\left(\bigwedge_{i=1}^n \theta_i\right) \rightarrow \psi$, where each θ_i is atomic and ψ is atomic. A Horn formula is a basic Horn formula, perhaps bound by some global quantifiers. All of the clauses in Definition 2.3.1 are (or can be expressed by) universal Horn formulas in the language with one ternary relation I .

THEOREM 2.3.3 ([469]). *There is no finite Horn axiomatization satisfied exactly by the ternary relations arising as the conditional independence relation of some probability space.*

PROOF. Let $N \in \mathbb{N}$, and $3 \leq k < N$. Let

$$S(j) := \begin{cases} j+1 & \text{if } j \in \{1, \dots, n-1\} \\ n & \text{if } j = n \end{cases},$$

and I_μ the conditional independence relation given by probability measure μ on $\{0, \dots, N\}$. Then we can show that the following conditions are equivalent:

- (1) For all $j \in \{1, \dots, k\}$, we have $I_\mu(\{0\}, \{j\}, \{S(j)\})$.
- (2) For all $j \in \{1, \dots, k\}$, we have $I_\mu(\{0\}, \{S(j)\}, \{j\})$.

Now let $\varphi_1, \dots, \varphi_\ell$ be Horn formulas. Then we will find some independence relation I_* satisfying φ but not arising as I_μ for any μ . Pick n large enough to exceed the number of θ_i conjuncts in the antecedents of all φ_j . We define $I_*(A, B, C)$ to hold of a triple (A, B, C) of subsets of $\{1, \dots, n\}$ in exactly the following cases:

- (1) $(A, B, C) = (\{0\}, \{S(j)\}, \{j\})$ for some $j \in \{1, \dots, n\}$.
- (2) $(A, B, C) = (\{j\}, \{S(j)\}, \{0\})$ for some $j \in \{1, \dots, n\}$.
- (3) A, B , and C are pairwise disjoint and nonempty.

This I_* does not satisfy the conditions above, but most satisfy $\varphi_1, \dots, \varphi_\ell$. \square

There are some partial recoveries from this obstruction, however. The following two results constitute a sort of completeness theorem with respect to a slightly exotic logic based on graphs.

DEFINITION 2.3.4. Let $G = (V, E)$ be an undirected graph, and $I : \mathcal{B}^3 \rightarrow \{0, 1\}$ an independence relation on a measurable space (Ω, \mathcal{B}) .

- (1) We say that G is a dependency map of I if and only if there is a bijection $f : \Omega \rightarrow V$ such that if $I(X, Z, Y)$ then all paths in G from elements of $f(X)$ to elements of $f(Y)$ pass through elements of $f(Z)$.
- (2) We say that G is an independency map of I if and only if there is a bijection $f : \Omega \rightarrow V$ such that if all paths in G from elements of $f(X)$ to elements of $f(Y)$ pass through elements of $f(Z)$, then $I(X, Y, Z)$.
- (3) We say that G is a perfect map of I if and only if it is both a dependency map and an independency map.

In this sense, a graph which is a perfect map of I constitutes a syntactic representation of the independence relation I . The completeness result for this logic is as follows.

THEOREM 2.3.5 ([399]). *Let (Ω, \mathcal{B}) be a measurable space, and let $I : \mathcal{B}^3 \rightarrow \{0, 1\}$. Then the following are equivalent:*

- (1) *There is an undirected graph that is a perfect map of I .*
- (2) *I satisfies the following conditions:*
 - (a) *$I(X, Z, Y)$ if and only if $I(Y, Z, X)$*
 - (b) *If $I(X, Z, Y \cup W)$, then $I(X, Z, Y)$ and $I(X, Z, W)$.*
 - (c) *If $I(X, Z \cup W, Y)$ and $I(X, Z \cup Y, W)$, then $I(X, Z, Y \cup W)$.*
 - (d) *If $I(X, Z, Y)$, then $I(X, Z \cup W, Y)$.*
 - (e) *If $I(X, Z, Y)$ then either $I(X, Z, W)$ or $I(W, Z, Y)$*

PROOF. Vertex separation in any graph satisfies the stated condition, so one direction of implication is clear. In the other direction, we construct a graph. To do this, we start with a complete graph with a vertex for each element of \mathcal{B} , and delete each edge (X, Y) where there is some $Z \in \mathcal{B}$ with $I(X, Z, Y)$. \square

This gives rise to a representation of independence relations as graphs, and even a notion of maximal parsimony — avoiding the computational complexity explosion inherent in working with a full joint distribution table.

DEFINITION 2.3.6. A Markov Network for I is an independency map G of I with the property that deleting any edge would result in a graph which is not an independency map.

THEOREM 2.3.7 ([399]). *Every function I satisfying the equivalent conditions of Theorem 2.3.5 has a unique Markov network.*

PROOF. For each element $x \in \Omega$, a Markov blanket of x is a subset $S \subseteq \Omega$ such that $I(\{x\}, S, U - (S \cup \{x\}))$. A Markov boundary of x is a minimal Markov blanket. We can show that the Markov boundary of an element is unique. We then set exactly the elements of the Markov boundary of x adjacent to x . \square

The difficulty with Markov networks is that the transitivity condition is too strong. Of course, Theorem 2.3.7 does not assume transitivity, but a form of it comes back in when a graph is a perfect map of I . If A occurs exactly when unconditionally independent events B_1 and B_2 occur, then an independency map will require an edge between B_1 and B_2 , introducing a spurious dependency. This issue can be addressed by replacing undirected graphs by directed acyclic graphs.

DEFINITION 2.3.8. Let $G = (V, E)$ be a directed acyclic graph.

- (1) We say that G to be an independency map of an independence relation I if and only if there is a bijection $f : \Omega \rightarrow V$ such that for any disjoint sets X, Y, Z , if all paths in G from elements of $f(X)$ to elements of $f(Y)$ pass through elements of $f(Z)$, then $I(X, Y, Z)$.
- (2) We say that G to be a dependency map of an independence relation I if and only if there is a bijection $f : \Omega \rightarrow V$ such that for any disjoint sets X, Y, Z , if $I(X, Y, Z)$ then all paths in G from elements of $f(X)$ to elements of $f(Y)$ pass through elements of $f(Z)$.
- (3) We say that G is a perfect map of I , if and only if it is both a dependency map and an independency map.
- (4) We say that G is a minimal independency map of I if and only if it is an independency map and no edge can be deleted without losing this property.
- (5) We say that G is a Bayesian network of a probability measure P if and only if it is a minimal independency map of the conditional independence relation I_P .

THEOREM 2.3.9. *Let (Ω, \mathcal{B}, P) be a probability space. Then the following conditions on a directed acyclic graph G are equivalent:*

- (1) G is a Bayesian network of P .
- (2) Each measurable set $X \in \mathcal{B}$ is conditionally independent of its non-descendants given its parents, and no proper subset of the parent relation satisfies this property.

We now arrive at the canonical completeness result for Bayesian networks.

THEOREM 2.3.10 ([212]). *For any directed acyclic graph $G = (V, E)$ there exists a probability space (Ω, \mathcal{B}, P) such that \mathcal{D} is a perfect map of P .*

PROOF. For any particular sets X, Y, Z of vertices of G , with some path from X to Y not passing through Z . Then there is a probability measure on the vertices of G such that $I_P(X, Z, Y)$ fails. We describe how to do this in the case of a finite graph. We associate with each element of V a binary random variable, and partition V as $V_1 \cup V_2$ where V_1 contains exactly the n_1 vertices on the path from X to Y avoiding Z and V_2 has cardinality n_2 . Preliminary to describing the full joint distribution we construct, set

$$f(x, y) = \begin{cases} \frac{1}{2} & \text{if } x = y = 0 \\ \frac{3}{4} & \text{if } x = y = 1 \\ \frac{1}{4} & \text{if } x = 1 \text{ and } y = 0 \\ \frac{1}{4} & \text{if } x = 0 \text{ and } y = 1 \end{cases}.$$

We now set

$$P(\bar{x}) = \left(\frac{1}{2}\right)^{(n_2+1)} \left(\prod_{i=1}^{n_1-1} f(x_i, x_{i+1})\right).$$

We take the product of all such probability spaces (V, \mathcal{F}, P) to obtain a single probability space satisfying exactly the desired independence relation. \square

A Bayesian network does not require computation or storage of the full joint distribution, but only of the conditional distribution of each vertex conditioned on its parents. There are standard algorithms for constructing Bayesian networks, but their results depend on the ordering of the variables. Informally, it appears that the most compact (i.e. sparse) representations are achieved when causes precede effects.

To generate a Bayesian network for a finite system of variables, we first order the variables X_1, \dots, X_n . Then at each stage s , choose, from X_1, \dots, X_{s-1} a minimal set Π_s of variables such that X_s is conditionally independent of X_1, \dots, X_{s-1} given Π_s . For each j with $X_j \in \Pi_s$, we make an edge from X_j to X_s , and no other edges to X_s . While the traditional applications involve finite sets of variables [425], this algorithm would still produce a computable Bayesian network on a countably infinite set of variables, provided the necessary conditional probability computations could be made — which, as we will see, is not always the case, even for finite sets of variables.

Pearl observed that the method of Bayesian networks could be extended to systems involving continuous random variables. His system, treated only in outline in [397] does not seem to have attracted as much further study as the similar models arising from contemporary work on conditional Gaussian distributions [330]. Instead of Boolean-valued random variables with distribution conditioned on the parent vertices, each vertex has a continuous (in Pearl's early examples, normal) distribution conditioned on a linear combination of the parent vertices.

There is, in fact, a difficulty in computing conditional probabilities. To express this difficulty, we introduce some terminology. If Ω and X are computable Polish spaces (in the sense that there is a computably enumerable dense subset, called the *ideal points*, on which the distance function produces uniformly computable real numbers), then $f : \Omega \rightarrow X$ is said to be computable on a set $S \subseteq \Omega$ exactly when

there is a computable sequence $(U_n : n \in \mathbb{N})$ of computably enumerable open sets in X such that for any ideal point q of X , any rational $r > 0$, and any $n \in \mathbb{N}$, we have $f^{-1}(B_r(q)) \cap S = U_n \cap S$. We say that a function is *almost computable* if and only if it is computable on a set of full measure in Ω , and *almost continuous* if and only if it is continuous on such a set. In that sense, the following result demonstrates in a fairly strong sense that having all information we might wish about random variables does not allow us to compute the conditional distribution.

THEOREM 2.3.11 ([8]). *There are P -almost computable $[0, 1]$ -random variables X and Z such that the conditional distribution map $z \mapsto P(X|Z = z)$ is P_Z -almost continuous but not P_Z -almost computable.*

PROOF. Let N, C, X, U , and V be independent computable random variables with the following distributions:

- N : geometric distribution with parameter $p = \frac{1}{5}$
- C : Bernoulli distribution with parameter $p = \frac{1}{3}$
- X, U, V : each with uniform distribution taking values in $[0, 1]$.

We produce the random variable Y as follows. Define

$$Y_k = \frac{2\lfloor 2^k V \rfloor + C + U}{2^{k+1}},$$

and note that $\lim_{k \rightarrow \infty} Y_k = V$ almost surely. Call this limit Y_∞ . We compose the variable Y as a random mixture (depending on the random variable N) of the variables $\{Y_i : i \in \mathbb{N} \cup \{\infty\}\}$ by setting h_n to be the time by which the n th Turing machine halts on input 0, or ∞ if it does not halt, and setting $Y = Y_{h_N}$, so that Y is equal to V almost surely if φ_N diverges, and an approximation to V (whose closeness depends on t , if $\varphi_{N,t} \downarrow$). We can show that this Y is almost continuous.

As usual, we can simulate a geometric random variable M with parameter $\frac{1}{2}$ by

$$M := \left\lfloor \frac{\log X}{\log(1 - \frac{1}{2})} \right\rfloor.$$

We define Z by $P(Z|Y)(x) = P(X|N = M(x))$.

Now $P(X \in (2^{-n-1}, 2^{-n})|Z = x) = P(N = n|Y = x)$. From computing this, even on a set of full measure, we could compute \emptyset' . \square

Finally, we observe a connection of Bayesian networks with the probabilistic argumentation systems of Section 2.2.2. To determine $P(\psi)$ from assumptions on the values of $P(\varphi_i)$ for each i , we construct a Bayesian network whose vertices are the propositional atoms V of the probabilistic argumentation system. It is customary to assume that the independence relation on the propositional atoms is known *a priori*.

To compute the conditional probabilities represented by the network from the assumptions on $P(\varphi_i)$, we rewrite each φ_i in disjunctive normal form as $\varphi_i = \bigvee_m \bigwedge_n A_{s_i(n)}^\pm$, where $s_i(n)$ enumerates propositional atoms of increasing rank (taking, as usual, the vertices with no edges in to have rank zero, and each other vertex to have the least rank strictly greater than that of any of its parents).

Now we have

$$P(\varphi_i) = \sum_m \prod_n \gamma_{n,m},$$

where $\gamma_{n,m}$ is the probability of $A_{s_i(n)}^{\pm}$ given its parents in the networks. These equations, as i ranges over the set of assumptions, give constraints on the variables $\gamma_{n,m}$. By reversing the procedure, these constraints can be used to give constraints on $P(\psi)$. The full computation can be laborious, and is explained in full detail in [238].

2.3.2. Nonmonotonic Logic and Conditionality. It is perhaps not surprising that conditional probability can interpret certain nonmonotonic reasoning, since the standard examples generally refer to probabilistic judgments. Most people are willing, in general, given that x is a bird, to deduce that x can fly. On the other hand, if afterward informed that x is one of the critically endangered kakapo, they would (perhaps after looking the matter up) not make that deduction. Such reasoning is called “nonmonotonic” because an increase in the information available may cause a recision of deductions that were valid from a smaller set of information.

There are well-documented logics to describe such reasoning. The subject in general is beyond the scope of the present book, but is treated in some detail in [354]. What will concern us here is Pearl’s development of default logic through a semantics of conditional probability, and some exposition of the background theory of nonmonotonic reasoning and default logic will be given in service of that goal.

The syntax of (propositional) default logic has two types. One type consists of the well-formed formulas of the propositional language. The other consists of defaults. A default is a triple $\langle \alpha, J, \gamma \rangle$, where α and γ are formulas and J is a finite set of formulas. We then define the appropriate proof system.

DEFINITION 2.3.12. Let T be a classical propositional theory, W a set of propositional sentences, and D be a set of defaults. A proof of φ from W relative to T consists of a finite sequence $\varphi_1, \dots, \varphi_n$ such that $\varphi_n = \varphi$ and such that for every i , one of the following holds:

- (1) $\varphi_i \in W$
- (2) φ_i follows from $\{\varphi_1, \dots, \varphi_{i-1}\}$ by the usual rules of propositional logic
- (3) There is a default $\langle \alpha, J, \varphi_i \rangle \in D$ such that $\alpha \in \{\varphi_1, \dots, \varphi_{i-1}\}$ and $J \cap T = \emptyset$.

Of course, the semantics for such a system will be a good deal more subtle than for classical propositional logic itself. In the example of the bird, we might express the “default” by saying that “Typically, birds are able to fly, unless they are either of a flightless species or wounded.” (The problems of expressing these assertions exactly in propositional, as opposed to predicate, logic are well-understood, and need not detain us here; in any case, the example is standard.) This expresses the matter well enough in the sense that as the contextual theory T changes we might go from concluding that x can fly to not drawing that conclusion if T grows to include the statement “ x is a kakapo.” But to put semantics on this deductive system, we have to put a more concrete meaning to “Typically.”

Since the goal for Marek and Truszczyński was logic programming, semantics were not an essential feature of their treatment. In Reiter’s seminal paper on default logic, he also devotes primary place to the syntactical realm [414], although in a later survey he does explicitly warn against an exclusively “statistical” reading of defaults [415]. Indeed, it is not fair to interpret “Typically A ” unequivocally as a mere statement that the probability of A is high.

Pearl nevertheless gives a semantics for defaults in terms of limiting conditional probabilities. We introduce formulas of the form $P(A|B) \geq r$, where A, B are sentences and $r \in [0, 1]$. For default set D whose defaults are all of the form $\langle A, \emptyset, B \rangle$, we compose the set

$$D_\epsilon := \{P(B|A) \geq 1 - \epsilon : \langle A, \emptyset, B \rangle \in D\}.$$

The semantic notion of entailment is then given by writing $(T, D) \models \varphi$ if and only if for every probability measure P satisfying all conditions in D_ϵ also satisfies $P(\varphi|T) \geq 1 - O(\epsilon)$. We likewise say that a probability measure P is a model of (T, D) if and only if $P(\varphi) = 1$ for all $\varphi \in T$ and P satisfies D_ϵ for every $\epsilon \geq 0$.

The calculation including our kakapo now goes in the following way. We will make reference to a bird named “Lisa” by researchers, who has played a pivotal role in the conservation of her species. We set

$$D := \{\langle \text{bird}, \emptyset, \text{fly} \rangle, \langle \text{kakapo}, \emptyset, \neg \text{fly} \rangle\},$$

and

$$T := \{\text{kakapo} \rightarrow \text{bird}\}.$$

Now from the axiom “Lisa is a bird,” we can certainly write a proof in default logic of “Lisa can fly.” We will denote “Lisa can fly” by “fly,” and similarly for other properties. For any model $P \models (T, D)$, we have $P(\text{fly}|\text{bird}) \geq 1 - \epsilon$ for each $\epsilon > 0$. On the other hand,

$$P(\text{fly}|\text{bird}, \text{kakapo}) = \frac{P(\text{fly}|\text{kakapo})}{P(\text{bird}|\text{kakapo})} \leq \epsilon.$$

The system of derivations proposed here by Geffner and Pearl is not transparently identical to that of Reiter, Marek, and Truszczyński. Nevertheless, he does state a derivation system which is sound and complete with respect to this semantics, and which has some similarities to theirs.

THEOREM 2.3.13 ([211]). *Let (T, D) be a default theory, where every element of D is of the form $\langle a, \emptyset, b, \rangle$. Then for any sentence φ , we have $(T, D) \models \varphi$ if and only if φ is derivable from (T, D) using the following rules of inference:*

- (1) *If $p \rightarrow q$, then from p we can derive q .*
- (2) *For each default $\langle p, \emptyset, q \rangle$, from p , we can derive q .*
- (3) *If from p we can derive each of q, r , then from $p \wedge q$ we can derive r .*
- (4) *If from p we can derive q and from $p \wedge q$ we can derive r , then from p we can derive r .*
- (5) *If from each of p, q we can derive r , then from $p \vee q$ we can derive r .*

The default rule of this system certainly matches the appropriate special case of that from Definition 2.3.12, and the remaining rules are all sound derivations of propositional calculus. Conversely, by clause 1, all propositional calculus derivations are valid here. Thus, the Geffner-Pearl system coincides with the special case of Definition 2.3.12 in which D consists of elements of the form $\langle a, \emptyset, b, \rangle$. The connection to Pearl’s graph semantics for conditional probability is made by the following result.

THEOREM 2.3.14 ([9, 396]). *Let D be a set of defaults of the form $\langle a, \emptyset, b \rangle$ with a and b atomic or negation atomic propositions. Then the following are equivalent.*

- (1) *For each ϵ , the set of P such that $P \models D$ is non-empty.*

- (2) For every nonempty subset $D_0 \subseteq D$, there is a truth assignment σ such that for some $\langle a, \emptyset, b, \rangle \in D_0$ we have $\sigma \models a \rightarrow b$, and for no such element do we have $\sigma \models a \wedge \neg b$.
- (3) For every pair $\langle a_1, \emptyset, b, \rangle, \langle a_2, \emptyset, \neg b, \rangle$, the following hold:
 - (a) $a_1 \neq a_2$
 - (b) There is no cycle of the form $\langle k_1, \emptyset, k_2, \rangle, \langle k_2, \emptyset, k_3, \rangle \cdots \langle k_n, \emptyset, k_1, \rangle$ in which both a_1 and a_2 are among the k_i .

The final clause of this last result justifies the representation of D as a directed graph that forbids certain kinds of cycles.

2.4. Adapted Spaces and Distributions

We now begin to consider a different logical approach to probability. It represents a significant departure from classical syntax, in exchange for modeling random variables and stochastic processes in a way that is much more tightly connected to the modern view of probability. In addition to being interesting in its own right, it is also an important precursor to the continuous first-order logic of Section 2.6. Fajardo and Keisler considered stochastic processes, with a focus on results that can be established in nonstandard analysis. Their process considers stochastic processes in an appropriate structure, and then passes to a saturated structure. Much of this work is accumulated in [180].

Exactly what sort of structure is appropriate is perhaps a natural logical question, but depends, of course, on the objects under study.

DEFINITION 2.4.1. Let (Ω, \mathcal{F}, P) be a probability space.

- (1) An X -valued stochastic process is a function $x : \Omega \times [0, 1] \rightarrow X$, where for each t , the function $x(\cdot, t) : \Omega \rightarrow X$ is measurable.
- (2) We say that a stochastic process x is *measurable* if it is measurable with respect to the product σ -algebra (using the Borel σ -algebra on $[0, 1]$).

It is also standard in modern investigations of stochastic processes to consider the probability space with a *sequence* of σ -algebras, $(\mathcal{F}_t : t \in [0, 1])$ with $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$ whenever $t_1 < t_2$. This is often said to capture the notion of added information over time, in the sense that the conditional distribution $P(x|\mathcal{F}_{t_2})$ carries more information than $P(x|\mathcal{F}_{t_1})$.

DEFINITION 2.4.2. An *adapted space* is a triple $(\Omega, (\mathcal{F}_t : t \in [0, 1]), P)$, where

- (1) For each $t < 1$, we have $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$,
- (2) \mathcal{F}_0 contains all subsets of measure-zero sets.

It is these structures that constitute the structures of the logic to be described. The next step is to describe the formulas of the appropriate logic. The definition arises from the observation that two random variables x, y are equidistributed if and only if for every bounded continuous function φ , we have $E(\varphi(x)) = E(\varphi(y))$, where E denotes the expectation.

DEFINITION 2.4.3. Let x be a stochastic process defined on an adapted space $(\Omega, (\mathcal{F}_t : t \in [0, 1]), P)$. We define the set of *adapted functions* by induction as follows.

- (1) Every bounded continuous function φ from \mathbb{R}^n to \mathbb{R} , in conjunction with an n -tuple of check-points $\bar{t} \in [0, 1]^n$, gives rise to an adapted function $\varphi_{\bar{t}}$. We evaluate $\varphi_{\bar{t}}$ on x by the random variable $\varphi_{\bar{t}}(x)$ which arises from evaluating x at each of the check-points, and then evaluating φ on that n -tuple.
- (2) If ψ is a bounded continuous function from \mathbb{R}^n to \mathbb{R} , and f_1, \dots, f_n are adapted functions, then $\psi(\bar{f})$ is an adapted function, evaluated in the obvious way.
- (3) If f is an adapted function and $t \in [0, 1]$, then the expression $E(f|t)$ is an adapted function, defined (up to almost everywhere equivalence) by

$$E(f|t)(x)(\omega) = E(f(x)|\mathcal{F}_t)(\omega).$$

Now we define a fundamental equivalence relation.

DEFINITION 2.4.4. Let x_i be a stochastic process on an adapted space Ω_i . We say that $(\Omega_1, x_1) \equiv (\Omega_2, x_2)$ if and only if for each adapted function f , we have $E(f(x_1)) = E(f(x_2))$. Where the context is obvious, we sometimes write $x_1 \equiv x_2$.

As the notation suggests, this relation will serve in the place of elementary equivalence. It does, indeed, preserve many important properties.

A *martingale*, which we will see in a few contexts over the course of the present volume, is a stochastic process x with the following two additional properties:

- (1) For any fixed $t \in [0, 1]$, we have $E(x(t)) < \infty$, and
- (2) For any $t \in [0, 1]$ and any $s < t$, we have $E(x(t)|\mathcal{F}_s) = x(s)$.

PROPOSITION 2.4.5. *If x is a martingale and $x \equiv y$, then y is a martingale.*

PROOF. Note that the truncated absolute value function

$$|x|_n = \begin{cases} |x| & \text{if } |x| \leq n \\ n & \text{otherwise} \end{cases}$$

gives rise to an adapted function $|\cdot|_{n,t}$. Thus,

$$\lim_{n \rightarrow \infty} \int |x|_{n,t} dP = \lim_{n \rightarrow \infty} \int |y|_{n,t} dP$$

since $x \equiv y$, making the individual terms in the limit equal. By the dominated convergence theorem, $E(y(t)) < \infty$ for every t .

Using a similar truncation strategy, we can reason from

$$\int |E(x(t)|\mathcal{F}_s) - x(s)| dP = 0$$

to the corresponding equation for y . □

This shows that \equiv preserves at least some of the important properties. Similarly, we have the following.

PROPOSITION 2.4.6. *If, for all t and almost all ω , we have $x(\omega, t) = y(\omega, t)$, then $x \equiv y$.*

The definition of equivalence can be adapted to consider pairs of stochastic processes. If x_1, x_2 both take values in X , then we can consider the pair (x_1, x_2) as a stochastic process taking values in X^2 . If X is a Polish space, then X^2 is, as well, so that the formulation of adapted functions in terms of continuous bounded functions still makes sense.

DEFINITION 2.4.7. An adapted space Ω is said to be saturated if for any adapted space Λ and any pair of stochastic processes x_1, x_2 on Λ , the following holds:

- For every y_1 on Ω with $y_1 \equiv x_1$, there is a process y_2 on Ω such that $(x_1, x_2) \equiv (y_1, y_2)$.

A notion of homogeneity can be defined in a similar way (with almost sure bijections which preserve measure and \mathcal{F}_t for all t taking the role of automorphisms), as well as a back-and-forth relation and an appropriate version of Ehrenfeucht-Fraïssé games. Saturated adapted spaces exist, but the most obvious adapted spaces are not saturated.

PROPOSITION 2.4.8. *Let $(\Omega, (\mathcal{F}_t : t \in [0, 1]), P)$ be an adapted space with Ω a Polish space and P the completion of a probability measure on the family of Borel sets. Then Ω is not saturated.*

It will be noticed that the set of adapted functions is unusually large for the set of formulas of a logic. Fortunately, it is not always necessary to consider the full set of such functions.

DEFINITION 2.4.9. We say that a set F of adapted functions is *dense* if for any stochastic processes x and sequence $(x_n : n \in \mathbb{N})$ of stochastic processes, then if $E(f(x_n)) \rightarrow E(f(x))$ for every $f \in F$, then the convergence holds for every adapted function f .

We can always take a countable set of bounded continuous functions that generate a dense set of adapted functions.

In considerations of definability, the base notion in classical model is the definable set. It is probably less natural to think of definable subsets of an adapted space, and, in any event, farther from the motivating examples of stochastic processes. Instead, we say what it means for one stochastic process to be definable from another.

DEFINITION 2.4.10. Let $(\Omega, (\mathcal{F}_t : t \in [0, 1]), P)$ be an adapted space and x a stochastic process on Ω .

- (1) The *intrinsic σ -algebra* of x on Ω , denoted \mathcal{I}_x is the σ -algebra generated by the null sets and the sets of the form $f(x)$ where f ranges over all adapted functions.
- (2) We say that y is definable from x if and only if $\mathcal{I}_y \subseteq \mathcal{I}_x$.

An important property of definable stochastic processes is the following argument that facilitates some back-and-forth arguments.

THEOREM 2.4.11. *Let x_i be a stochastic process on Ω_i . If $x_1 \equiv x_2$ and y_1 is a stochastic process on Ω_1 definable from x_1 , then there is a stochastic process y_2 on Ω_2 , definable from x_2 , such that $(x_1, y_1) \equiv (x_2, y_2)$.*

2.5. Approximate Measure Logic

Motivated by several disparate attempts to incorporate measure and probability into model theory, including the work of [305, 270], Goldbring and Towsner proposed a logic that extended first-order logic to accommodate measure as a logical primitive.

DEFINITION 2.5.1 ([223]). Let L be a first-order signature. Then the AML L -formulas are defined by induction:

- (1) Every atomic L -formula is an AML L -formula.
- (2) If φ, ψ are AML L -formulas, then the following are AML L -formulas:
 - (a) $\neg\varphi$
 - (b) $\varphi \wedge \psi$
 - (c) $\varphi \vee \psi$
 - (d) $\exists x\varphi$
 - (e) $\forall x\varphi$
- (3) If \bar{x} is a sequence of distinct variables, q is a non-negative rational, and φ is an AML L -formula, then the following are AML L -formulas:
 - (a) $m_{\bar{x}} \leq q \cdot \varphi$
 - (b) $m_{\bar{x}} < q \cdot \varphi$

It is clear how to interpret the formulas generated by most clauses of this definition, but the final clause requires some comment. First, a variable of \bar{x} that is free in φ is bound in $m_{\bar{x}} \leq q \cdot \varphi$ or $m_{\bar{x}} < q \cdot \varphi$, making this clause something like a quantifier. We interpret $m_{\bar{x}} < q \cdot \varphi$ (for instance) as the statement that the set defined by φ has measure less than q .

Toward formalizing this semantic ideology, we inductively define a rank on the formulas. A classical L -formula has rank 0, and for any φ , we have

$$rk(\varphi) = rk(\neg\varphi) = rk(\forall\varphi) = rk(\exists\varphi).$$

Further, we set

$$rk(\varphi \wedge \psi) = rk(\varphi \vee \psi) = \max(rk(\varphi), rk(\psi)).$$

On the other hand, we let

$$rk(m_{\bar{x}} \leq q \cdot \varphi) = rk(m_{\bar{x}} < q \cdot \varphi) = rk(\varphi) + 1.$$

DEFINITION 2.5.2. Let L be a first-order signature.

- (1) An AML L -quasistructure is a first-order L -structure \mathcal{M} , with the following additional data:
 - (a) For each $n \geq 1$, an algebra \mathcal{B}_n of subsets of M^n , and a finitely additive measure μ_n on \mathcal{B}_n , with the property that for any n, m , the measure space $(M^{n+m}, \mathcal{B}_{n+m}, \mu_{n+m})$ extends the product space $(M^n, \mathcal{B}_n, \mu_n) \times (M^m, \mathcal{B}_m, \mu_m)$.
 - (b) For each $n \geq 1$, a function $v_n : \mathcal{B}_n \rightarrow \{\oplus, \ominus, \odot\}$ such that $v_n(X) = \odot$ whenever $\mu_n(X) \notin \mathbb{Q}$.
- (2) An AML L -structure is an AML L -quasistructure satisfying, for each $n \geq 1$, the condition that for any formula $\varphi(\bar{x}, \bar{y})$ of rank n , and for all \bar{y} , we have $\varphi(M, \bar{b}) \in \mathcal{B}_{|\bar{x}|}$.

Again, most clauses are intuitive. The assignments v_n will designate when the structure determines the measure of a set exactly. In particular, we will say that $\mathcal{M} \models m_{\bar{x}} < r \cdot \varphi(\bar{x}, \bar{b})$ if and only if *either* $\mu(\varphi(M), \bar{b}) < r$ *or* $\mu(\varphi(M), \bar{b}) = r$ and $v(\varphi(M), \bar{b}) = \ominus$, and we say that $\mathcal{M} \models m_{\bar{x}} \leq r \cdot \varphi(\bar{x}, \bar{b})$ if and only if *either* $\mu(\varphi(M), \bar{b}) \leq r$ *or* $\mu(\varphi(M), \bar{b}) = r$ and $v(\varphi(M), \bar{b}) = \oplus$.

We can define much of the usual model-theoretic apparatus (e.g. substructures, definable sets, types) in the usual way. Indeed, Goldbring and Towsner were able to prove the following compactness theorem for AML.

THEOREM 2.5.3 ([223]). *Let T be an AML theory. If every finite subset of T has a model, then T has a model.*

PROOF. Let \mathcal{I} be the set of all finite subsets of T , and for each $T_i \in \mathcal{I}$, let $\mathcal{M}_i \models T_i$. We will construct an ultrafilter D over \mathcal{I} such that $\prod_D \mathcal{M}_i \models T$.

To this end, for each $\varphi \in T$, we define $\mathcal{J}_\varphi \subseteq \mathcal{I}$ to be the set of all members of \mathcal{I} containing φ . The set $\{\mathcal{J}_\varphi : \varphi \in T\}$ is closed under finite intersection. Thus, it can be extended to a set $D \notin \{\emptyset, \mathcal{I}\}$ which is closed upwards, closed under finite intersection, and cannot be properly extended to another set having these properties (that is, D is an ultrafilter).

We now construct a new structure in the following way. We let $\mathcal{M}_0 = \prod_{i \in \mathcal{I}} \mathcal{M}_i$, and let \mathcal{M} be the quotient of \mathcal{M}_0 by the relation identifying two functions $f, g \in \mathcal{M}_0$ if and only if the set on which they agree is an element of D , and define a measure $\mu_n^{\mathcal{M}}$ on \mathcal{M} as the product measure of the measures on the \mathcal{M}_i , modulo D . We claim that $\mathcal{M} \models T$.

Observe that, by definition, for each $\varphi \in T$, the set \mathcal{J}_φ is in D . We will show that $\mathcal{M} \models \varphi$ if and only if $\mathcal{J}_\varphi \in D$, which would complete the proof. This would be quite true in classical first-order logic by the so-called *Łoś Theorem*, a standard result, whose proof proceeds by induction on the formation of formulas. If the result holds for $\varphi(\bar{x}, \bar{y})$, we consider $m_{\bar{x}} < q \cdot \varphi(\bar{x}, \bar{y})$.

If $\mu^{\mathcal{M}_i}(\varphi(M_i, \bar{y})) < r$ on an element of D , then it follows that $\mu^{\mathcal{M}}(\varphi(M_i, \bar{y})) < r$, so that $\mathcal{M} \models m_{\bar{x}} < q \cdot \varphi(\bar{x}, \bar{y})$. The other cases are similar. \square

It is important to note that this result includes a proof of the Łoś theorem for AML: a sentence is true in the ultraproduct if it is true on an element of the corresponding ultrafilter. Significant results from classical first-order model theory are possible here; the Tarski-Vaught test and the downward Löwenheim-Skolem theorem both hold. In the context of approximate measure logic, a substructure should, in addition to preserving the first-order signature, also preserves the algebras \mathcal{B}_n . A substructure is said to be elementary in the usual case, except that satisfaction of all AML formulas must be preserved.

THEOREM 2.5.4 ([223]). *Suppose that \mathcal{M} is an AML substructure of \mathcal{N} . Then \mathcal{M} is an AML-elementary substructure if and only if the following conditions hold:*

- (1) *For all formulas $\varphi(\bar{x}, y)$ and all $\bar{a} \in \mathcal{M}$, if $\mathcal{N} \models \varphi(\bar{a}, b)$, then there is $b' \in \mathcal{M}$ with $\mathcal{M} \models \varphi(\bar{a}, b')$, and*
- (2) *For all formulas $\varphi(x, \bar{y})$ and all $\bar{a} \in \mathcal{M}$, measure is preserved in the sense that $\mu(\varphi(N, \bar{a}) \cap M) = \mu(\varphi(N, \bar{a}))$ and $v(\varphi(N, \bar{a}) \cap M) = v(\varphi(N, \bar{a}))$*

PROOF. If \mathcal{M} is an elementary substructure of \mathcal{N} . Then both conditions follow. The opposite implication follows by the usual induction. \square

THEOREM 2.5.5 ([223]). *Let \mathcal{M} be an L structure, with $X \subseteq \mathcal{N}$. Then there is an elementary substructure of \mathcal{M} of cardinality at most $\max(|X|, |L|, \aleph_0)$.*

PROOF. We start with X and iteratively close under the functions of L , as usual. The set X_ω obtained in this way respects the cardinality bound. We define \mathcal{M}_0 on X_ω by defining the measurable sets according to intersection with X_ω . By the Tarski-Vaught test (Theorem 2.5.4), this is an elementary substructure. \square

Goldbring and Towsner develop a significant theory of types in approximate measure logic. One significant additional observation is the notion of *approximation by definable stable sets*. A set S is said to be approximated by definable stable sets if for every $\epsilon > 0$ there is a definable stable set whose symmetric difference from S is less than ϵ . Here, as in the previous section on adapted spaces, approximation of formulas has its place.

2.6. Continuous First-Order Logic and Metric Structures

2.6.1. Continuous First-Order Logic. Perhaps the great counterexample in applications of stability theory is the Hilbert space $L^2(\mathbb{C})$. This structure has a well-behaved (and important) independence theory. On the other hand, the norm guarantees the interpretability of an infinite linear ordering, guaranteeing instability. While we could, of course, pursue the matter from the perspective of thorn forking, another approach has proved productive.

The idea of this approach is that for many structures, collectively called *metric structures*, there is a metric (expressed or implied) that could import instability, even when the combinatorics of the structure otherwise act like a stable structure.

A seminal 2008 paper of Ben Yaacov, Berenstein, Henson, and Usvyatsov merged several approaches current at the time, including compact abstract theories. The approach here is to absorb the metric into the logic, so that the definable sets of stable theories work like those in first-order stable theories, without interference from the metric.

A *metric structure* is a many-sorted structure in which each sort is a complete metric space of finite diameter. Key examples include Hilbert spaces, Banach spaces, probability spaces, and probability spaces with a distinguished automorphism. Those which do not natively have finite diameter may be treated as increasing sequences of balls. Probability spaces lend themselves naturally to this treatment, as we will see shortly, as do several more involved random structures, as we will see in Chapter 6.

The logic for metric structures, called *continuous first-order logic*, an extension of Lukasiewicz propositional logic, takes truth values in $[0, 1]$. Indeed, it is sometimes more conventional to shy away from calling these “truth” values at all; many authors would interpret the values in a geometrical sense, and not a credal one — an interesting contrast with the humble Jansenist origins of probability.

To the extent that the values do represent truth, they follow the slightly unusual convention of using 0 as a numerical value for True (or acceptance) and 1 as a numerical value for False (or rejection). The authors of [64] chose this convention to emphasize the metric nature of their logic.

Continuous first-order logic builds on work of Keisler and Henson (see [61] for a more detailed history). The following definitions are from [63].

DEFINITION 2.6.1. A *continuous signature* is an object of the form $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{G}, n, d)$ where

- (1) \mathcal{R} (which we will call the set of *predicate symbols*) and \mathcal{F} (which we will call the *function symbols*) are disjoint and \mathcal{R} is nonempty, and
- (2) n is a function associating to each member of $\mathcal{R} \cup \mathcal{F}$ its arity
- (3) \mathcal{G} has the form $\{\delta_{s,i} : (0, 1] \rightarrow (0, 1] : s \in \mathcal{R} \cup \mathcal{F} \text{ and } i < n_s\}$, and
- (4) d is a special symbol.

Because of the context, we will use the notation $n_S = n(S)$ for the arity function. We now define the class of structures. The similarity to the AML structures of the previous section will be evident.

DEFINITION 2.6.2. Let $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{G}, n, d)$ be a continuous signature. A *continuous \mathcal{L} -pre-structure* is an ordered pair $\mathfrak{M} = (M, \rho)$, where M is a non-empty set, and ρ is a function on $\mathcal{R} \cup \mathcal{F} \cup \{d\}$ taking values in sets of maps defined on Cartesian powers of M , such that

- (1) To each function symbol f , the function ρ assigns a mapping $f^{\mathfrak{M}} : M^{n_f} \rightarrow M$
- (2) To each predicate symbol P , the function ρ assigns a mapping $P^{\mathfrak{M}} : M^{n_P} \rightarrow [0, 1]$.
- (3) The function ρ assigns d to a pseudo-metric $d^{\mathfrak{M}} : M \times M \rightarrow [0, 1]$. (Note: a pseudo-metric is a symmetric nonnegative function into the reals satisfying the triangle inequality.)
- (4) For each $f \in \mathcal{F}$ for each $i < n_f$, the element $\delta_{f,i} \in \mathcal{G}$ is a modulus of continuity for f in the i th argument; that is, for each $\epsilon \in (0, 1]$, we have

$$\forall \bar{a}, \bar{b}, c, e \left[d^{\mathfrak{M}}(c, e) < \delta_{f,i}(\epsilon) \Rightarrow d^{\mathfrak{M}}(f^{\mathfrak{M}}(\bar{a}, c, \bar{b}), f^{\mathfrak{M}}(\bar{a}, e, \bar{b})) < \epsilon \right]$$

where $lh(\bar{a}) = i$ and $lh(\bar{a}) + lh(\bar{b}) = n_f - 1$.

- (5) For each $P \in \mathcal{R}$ for each $i < n_P$, the element $\delta_{P,i} \in \mathcal{G}$ is a modulus of continuity for P in the i th argument; that is, for each $\epsilon \in (0, 1]$, we have

$$\forall \bar{a}, \bar{b}, c, e \left[d^{\mathfrak{M}}(c, e) < \delta_{P,i}(\epsilon) \Rightarrow |P^{\mathfrak{M}}(\bar{a}, c, \bar{b}) - P^{\mathfrak{M}}(\bar{a}, e, \bar{b})| < \epsilon \right]$$

where $lh(\bar{a}) = i$ and $lh(\bar{a}) + lh(\bar{b}) = n_P - 1$.

DEFINITION 2.6.3. A *continuous weak \mathcal{L} -structure* is a continuous \mathcal{L} -pre-structure such that ρ assigns to d a metric.

Since we are concerned here with countable structures, we will not use the stronger notion of a *continuous \mathcal{L} -structure* common in the literature, which requires that d be assigned to a *complete* metric. However, it is possible, given a continuous weak structure (even a pre-structure), to pass to a completion [63].

We now proceed to define formulas and their evaluation. The following definition may be found in [61], and is, in some ways similar to the adapted functions of Section 2.4.

DEFINITION 2.6.4. Let \mathcal{L} be a continuous signature.

- (1) The \mathcal{L} -terms are defined inductively as follows:
 - (a) Every variable is a term.
 - (b) If t_1, \dots, t_n are terms and f is an n -ary function symbol, then $f(t_1, \dots, t_n)$ is a term.
- (2) The atomic \mathcal{L} -formulas are the expressions $P(t_1, \dots, t_n)$, where P is an n -ary predicate symbol, and the expressions $d(t_1, t_2)$, where t_i are terms.
- (3) The quantifier-free \mathcal{L} -formulas are the smallest class containing all atomic \mathcal{L} -formulas and satisfying the following closure condition:
 - If $u : [0, 1]^n \rightarrow [0, 1]$ is continuous and φ_i are formulas, then $u(\varphi_1, \dots, \varphi_n)$ is a formula.
- (4) The \mathcal{L} -formulas are the smallest class containing all the atomic \mathcal{L} -formulas and closed under both the previous condition and the following one:

- If φ is an \mathcal{L} -formula and x is a variable, then $\sup_x \varphi$ and $\inf_x \varphi$ are both \mathcal{L} -formulas.

In practice, a much smaller set of formulas is entirely adequate — exactly as we saw for adapted spaces and approximate measure logic. Any \mathcal{L} -formula may be approximated to arbitrary accuracy, for instance, by a formula where u is taken from \div , $x \mapsto 1 - x$, and $x \mapsto \frac{1}{2}x$. This result, and many others like it, are documented in [61]. Throughout the rest of the present paper, then, we will work only with this reduced set of formulas.

DEFINITION 2.6.5. Let V denote the set of variables, and let $\sigma : V \rightarrow M$. Let φ be a formula.

- (1) The *interpretation under σ* of a term t (written $t^{\mathfrak{M}, \sigma}$) is defined by replacing each variable x in t by $\sigma(x)$.
- (2) Let φ be a formula. We then define the *value of φ in \mathfrak{M} under σ* (written $\mathfrak{M}(\varphi, \sigma)$) as follows:
 - (a) $\mathfrak{M}(P(\bar{t}), \sigma) := P^{\mathfrak{M}}(\overline{t^{\mathfrak{M}, \sigma}})$
 - (b) $\mathfrak{M}(\alpha \div \beta, \sigma) := \max(\mathfrak{M}(\alpha, \sigma) - \mathfrak{M}(\beta, \sigma), 0)$
 - (c) $\mathfrak{M}(\neg \alpha, \sigma) := 1 - \mathfrak{M}(\alpha, \sigma)$
 - (d) $\mathfrak{M}(\frac{1}{2}\alpha, \sigma) := \frac{1}{2}\mathfrak{M}(\alpha, \sigma)$
 - (e) $\mathfrak{M}(\sup_x \alpha, \sigma) := \sup_{a \in M} \mathfrak{M}(\alpha, \sigma_x^a)$, where σ_x^a is equal to σ except that $\sigma_x^a(x) = a$.
- (3) We write $(\mathfrak{M}, \sigma) \models \varphi$ exactly when $\mathfrak{M}(\varphi, \sigma) = 0$.

Of course, if φ has no free variables, then the value of $\mathfrak{M}(\varphi, \sigma)$ is independent of σ .

A sound and complete proof system has been described for continuous first-order logic. Of course, it is only complete up to approximation. The following axiom system was given in [63], and soundness and completeness results were proved there. The axioms are stated here for completeness, since they are not well-known.

- (A1) $(\varphi \div \psi) \div \varphi$
- (A2) $((\chi \div \varphi) \div (\chi \div \psi)) \div (\psi \div \varphi)$
- (A3) $(\varphi \div (\varphi \div \psi)) \div (\psi \div (\psi \div \varphi))$
- (A4) $(\varphi \div \psi) \div (\neg \psi \div \neg \varphi)$
- (A5) $\frac{1}{2}\varphi \div (\varphi \div \frac{1}{2}\varphi)$
- (A6) $(\varphi \div \frac{1}{2}\varphi) \div \frac{1}{2}\varphi$
- (A7) $(\sup_x \psi \div \sup_x \varphi) \div \sup_x (\psi \div \varphi)$
- (A8) $\varphi[t/x] \div \sup_x \varphi$ where no variable in t is bound by a quantifier in φ .
- (A9) $\sup_x \varphi \div \varphi$, wherever x is not free in φ .
- (A10) $d(x, x)$
- (A11) $d(x, y) \div d(y, x)$
- (A12) $(d(x, z) \div d(x, y)) \div d(y, z)$
- (A13) For each $f \in \mathcal{F}$, each $\epsilon \in (0, 1]$, and each $r, q \in \mathcal{D}$ with $r > \epsilon$ and $q < \delta_{f,i}(\epsilon)$, the axiom $(q \div d(z, w)) \wedge (d(f(\bar{x}, z, \bar{y}), f(\bar{x}, w, \bar{y})) \div r)$, where $lh(\bar{x}) + lh(\bar{y}) = n_f - 1$.
- (A14) For each $P \in \mathcal{R}$, each $\epsilon \in (0, 1]$, and each $r, q \in \mathcal{D}$ with $r > \epsilon$ and $q < \delta_{P,i}(\epsilon)$, the axiom $(q \div d(z, w)) \wedge ((P(\bar{x}, z, \bar{y}) \div P(\bar{x}, w, \bar{y})) \div r)$, where $lh(\bar{x}) + lh(\bar{y}) = n_P - 1$.

Axioms A1–A4 are those of Łukasiewicz propositional logic, and axioms A5–A6 complete the propositional part of continuous logic (primarily prescribing the behavior of $\frac{1}{2}$). Axioms A7–A9 describe the role of the quantifiers. Axioms A10–A12 guarantee that d is a pseudo-metric, and axioms A13–A14 guarantee uniform continuity of functions and predicates. We write $\Gamma \vdash_Q \varphi$ whenever φ is provable from Γ in continuous first-order logic, as axiomatized above. Where no confusion is likely, we will write $\Gamma \vdash \varphi$.

THEOREM 2.6.6 ([63]). *Let L be a continuous signature, and Γ a set of L -formulas.*

- (1) *For every formula φ , if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.*
- (2) *If Γ is satisfiable, then Γ is consistent.*
- (3) *If Γ is consistent, then Γ is satisfiable.*

Every classical first-order structure can be considered as a metric structure. We take the discrete metric on the elements of the structure, and modulus of continuity $D(\epsilon) = \epsilon$. As usual, it is very easy to be continuous in a discrete topology. As a technicality, we consider the natural continuous versions of first-order formulas.

DEFINITION 2.6.7. Let L be a classical first-order signature, and $\varphi(\bar{x})$ an L -formula. We define a continuous first-order formula $\tilde{\varphi}$ by induction on the form of φ .

- (1) If P is a relation symbol of L and $\varphi(\bar{x})$ is of the form $P(t_1(\bar{x}), t_2(\bar{x}), \dots, t_n(\bar{x}))$, for L -terms t_i , then $\tilde{\varphi}(\bar{x})$ is $\varphi(\bar{x})$.
- (2) If $\varphi(\bar{x})$ is of the form $t_1(\bar{x}) = t_2(\bar{x})$, then $\tilde{\varphi}(\bar{x})$ is $d(t_1(\bar{x}), t_2(\bar{x}))$.
- (3) If $\varphi(\bar{x})$ is of the form $\neg\psi(\bar{x})$, then $\tilde{\varphi}(\bar{x})$ is $1 \div \tilde{\psi}(\bar{x})$.
- (4) If $\varphi(\bar{x})$ is of the form $(\psi_1 \wedge \psi_2)(\bar{x})$, then $\tilde{\varphi}(\bar{x})$ is $\max\{\tilde{\psi}_1(\bar{x}), \tilde{\psi}_2(\bar{x})\}$.
- (5) If $\varphi(\bar{x})$ is of the form $\exists y\psi(\bar{x}, y)$, then $\tilde{\varphi}(\bar{x})$ is $\inf_y \tilde{\psi}(\bar{x}, y)$.

This transformation preserves truth values in the following sense.

LEMMA 2.6.8. *Let \mathcal{M} be a classical first-order L -structure, with $\bar{a} \in \mathcal{M}$, and $\varphi(\bar{x})$ an L -formula. Then if $\mathcal{M} \models \varphi(\bar{a})$, then $\tilde{\varphi}(\bar{a})^{\mathcal{M}} = 0$.*

The truth preservation also works at the scale of structures.

PROPOSITION 2.6.9 ([222]). *Let \mathcal{M}, \mathcal{N} be a classical first-order structure. Then if $\mathcal{M} \equiv \mathcal{N}$ as classical first-order structures, then $\mathcal{M} \equiv \mathcal{N}$ as discrete metric structures.*

PROOF. We may assume without loss of generality that $\mathcal{N} \preceq \mathcal{M}$ as classical structures. Now we may show by induction on a CFO formula φ that there is only a finite set $R \subseteq [0, 1]$ of values it may have on the classical structure \mathcal{M} . Moreover, for any $r \in R$, there is a classical formula ψ_r such that $\varphi(\bar{a})$ has value r on a classical structure \mathcal{A} if and only if $\mathcal{A} \models \psi_r(\bar{a})$. From this, the result follows. \square

Definability of elements is understood in metric structures in the sense of approximation.

DEFINITION 2.6.10. Let \mathcal{M} be a metric structure, $A \subseteq \mathcal{M}$, and $b \in \mathcal{M}$. We say that b is *definable* over A if and only if there is a sequence $(\bar{a}_n : n \in \mathbb{N})$ and a sequence of CFO formulas φ_n such that the sequence $(\mathcal{M}(\varphi_n(x, \bar{a}_n)) : n \in \mathbb{N})$ converges to $d(x, b)$.

2.6.2. Some Metric Structures. Now let \mathcal{H} be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. We treat \mathcal{H} as a many-sorted structure where the sorts are balls of increasing radius, and where the signature includes function symbols for the inclusions among these balls, scaling, addition, subtraction, and the inner product.

In this signature, we can formulate continuous first-order formulas which have value zero if and only if the structure satisfies the axioms (other than completeness) of a Hilbert space. For instance, the axiom that

$$\forall x \forall y \langle x, y \rangle = \langle y, x \rangle$$

can be written as

$$\begin{aligned} & \sup_x \sup_y \langle x, y \rangle \div \langle y, x \rangle \\ & \sup_x \sup_y \langle y, x \rangle \div \langle x, y \rangle. \end{aligned}$$

We interpret distance on this structure in the usual way, by $d(x, y) = \sqrt{\langle x - y, x - y \rangle}$. At this point, a CFO structure in which all of the relevant axioms evaluate to zero will also be a complete metric space, and so will be a Hilbert space. We could also axiomatize infinite dimension by requiring, for each n , the condition

$$\inf_{x_1, \dots, x_n} \max_{1 \leq i, j \leq n} |\langle x_i, x_j \rangle - \delta_{ij}|.$$

The CFO theory of infinite-dimensional Hilbert spaces is κ -categorical for every infinite cardinal κ .

Returning to the original question, we sketch the stability theory on Hilbert spaces. A (complete) n -type p over a set A is a set of CFO formulas in free variables x_1, \dots, x_n , with parameters from A , such that there is some structure \mathcal{M} and some n -tuple \bar{b} such that p is the set of all formulas whose value in \mathcal{M} on \bar{b} is zero. The set of all such types we denote by $S_n(A)$.

For a theory T and set A , let \mathcal{M}_S be a model of T realizing every type in $S_n(A)$ for every n . Now the metric d on \mathcal{M}_S induces a natural metric on $S_n(A)$ in the following way. For any $p, q \in S_n(A)$, and for tuples \bar{b}, \bar{c} realizing p and q respectively, we set $d(p, q, \bar{b}, \bar{c})$ to be the maximum of $d(b_i, c_i)$ as i ranges from 1 to n . We define $d(p, q) = \inf_{\bar{b}, \bar{c}} d(p, q, \bar{b}, \bar{c})$.

DEFINITION 2.6.11. Let T be a CFO theory. We say that T is λ -stable if for any $\mathcal{M} \models T$ and any set A with cardinality at most λ , there is a d -dense set $D \subseteq S_1(A)$ with cardinality at most λ .

From this perspective, the following result shows that the transition to continuous logic has resolved the difficulty with which the present section started.

PROPOSITION 2.6.12 ([61]). *The CFO theory of infinite-dimensional Hilbert spaces is ω -stable.*

One can also observe that the notion of stability here does generalize stability for classical first-order structures. As we have seen, any first-order theory can be viewed as a continuous first-order theory, and its models viewed as metric structures, by giving the structures the discrete metric.

THEOREM 2.6.13. *If T is a classical theory, then T is stable iff T is stable as a discrete metric theory.*

Up to this point in the exposition, the connection of continuous first-order logic and metric structures to probability may seem distant, or, at most, contrived. The next example, however, will begin to show the applicability. Let (Ω, \mathcal{B}, P) be a probability space. We say that $S \in \mathcal{B}$ is an *atom* if it has positive measure and no subset has properly smaller nonzero measure. Further assume that (Ω, \mathcal{B}, P) has no atoms (that is, it is atomless).

Let $\hat{\mathcal{B}}$ be the quotient of \mathcal{B} by the equivalence relation that makes A_1 and A_2 equivalent if and only if they have symmetric difference of measure zero. We can then consider a metric structure with universe $\hat{\mathcal{B}}$, with metric given by $d(A_1, A_2) = P(A_1 \triangle A_2)$, and with functions \cap, \cup , complement, and a predicate for P . This structure satisfies the axioms of Boolean algebras, the Kolmogorov axioms for P , and the additional axiom

$$\sup_x \sup_y |d(x, y) - P(x \triangle y)|.$$

Moreover, because Ω is atomless, the structure also satisfies the axiom

$$\sup_x \inf_y |P(x \cap y) - P(x \cap y^c)|.$$

Conversely, any metric structure satisfying all of these axioms arises from an atomless probability space in this way.

PROPOSITION 2.6.14 ([61]). *Let $\mathcal{M}_1, \mathcal{M}_2$ be separable models of the axioms of atomless probability algebras as above. Then $\mathcal{M}_1 \cong \mathcal{M}_2$.*

This result is proved by a routine back-and-forth argument. Ben Yaacov [58] proved that in a sufficiently saturated, sufficiently homogeneous atomless probability algebra, $tp(\bar{a}/C) = tp(\bar{b}/C)$ if and only if for every function $u : \hat{\mathcal{B}}^n \rightarrow \hat{\mathcal{B}}$ mapping \bar{x} to an intersection of literals of \bar{x} , we have $P(u(\bar{a})|C) = P(u(\bar{b})|C)$.

With this in mind, we can show the following.

THEOREM 2.6.15 ([61]). *The CFO theory of atomless probability algebras is ω -stable.*

PROOF. Let \mathcal{U} be a large saturated model, and let $(X^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}}, P^{\mathcal{M}})$ be the corresponding probability space. Let $C \subseteq \mathcal{M}$ be a countable set, closed under Boolean combinations, and let \mathcal{C} be a countable Boolean algebra of subsets of $X^{\mathcal{M}}$ corresponding to C .

We describe a dense set of 1-types over C . Let \mathcal{F} be the set of types of form $tp(a/C)$ where $P(a|C)$ is an integrable step function on (X, \mathcal{C}, P) . \square

An important element in modern probability is the theory of measure-preserving functions on probability spaces. We formalize this in the following way. An automorphism of the probability space (X, \mathcal{B}, P) consists of an invertible measure-preserving transformation $\Phi : C_1 \rightarrow C_2$, where $C_i \in \mathcal{B}$ with $P(C_i) = 1$. Such a map induces a map $\hat{\Phi} : \hat{\mathcal{B}} \rightarrow \hat{\mathcal{B}}$ preserving both the measure and the algebra structure.

For any atomless probability algebra \mathcal{A} and any automorphism τ induced by an automorphism of the probability space, the following formulas will have value 0.

- (1) $\sup_x |P(x) - P(\tau(x))|$
- (2) $\sup_x \inf_y |P(x \triangle y)|$
- (3) $\sup_x \sup_y |P(\tau(x \cup y) \triangle (\tau(x) \cup \tau(y)))|$

$$(4) \sup_x \sup_y |P(\tau(x \cap y) \triangle (\tau(x) \cap \tau(y)))|$$

We add the following condition: an automorphism τ of probability spaces is said to be *aperiodic* if and only if for every $n \in \mathbb{N}$ and every $y \in \mathcal{B}$, we have $P(y \cap \tau^n(y)) = 0$. Rokhlin's lemma allows this to be translated to a condition on automorphisms of a probability algebra. This condition, as a CFO formula, is written

$$\inf_y \max \left\{ \left| \frac{1}{n} - P(y) \right|, P(y \cap \tau^i(y)) : i \leq n \right\}$$

for each $n \geq 1$. The theory of atomless probability algebras with an aperiodic automorphism is complete.

THEOREM 2.6.16 ([61]). *The CFO theory of atomless probability algebras with a generic automorphism is stable, but not ω -stable.*

2.7. Continuous First Order Logic as a Generalization

To a great extent, many of the systems described here can be expressed in continuous first-order logic. We have already seen that the ω -stable theory of atomless probability algebras includes the Kolmogorov axioms as CFO formulas.

Toward formulating Gaifman's measure models, let (U, μ) be a measure model in signature L . We will construct a metric structure in a metric signature L' , using a technique that will be developed further in Section 6.2. The structure will have one sort, and B . For each first-order L -sentence φ , we will have a constant symbol $\llbracket \varphi \rrbracket \in B$. We will also have a unary predicate P , and function symbols for the classical Boolean connectives on B , and Boolean operations on B , and a metric on B .

We arrange that for all quantifier-free φ , we have $P(\llbracket \varphi \rrbracket) = \mu(\varphi)$. We will also axiomatize (hopefully the formalism is familiar by now) that every element $x \in B$ has some φ_x such that $P(x \triangle \llbracket \varphi_x \rrbracket) = 0$ and that the Boolean connectives in B correspond to Boolean connectives in L . Further, we require that

$$P(\exists x \varphi(x)) = \sup_{\bar{a} \subseteq U} \left\{ P \left(\bigwedge_{i=1}^n \varphi(a_i) \right) \right\}.$$

We define a metric on B , as usual, by the P -measure of the symmetric difference.

Gaifman does not seem to have defined a notion of morphism of measure models, but it seems reasonable that a homomorphism $f : (U_1, \mu_1) \rightarrow (U_2, \mu_2)$ would be a function $f : U_1 \rightarrow U_2$ that respects the measure of quantifier-free formulas. Under this definition, the morphisms of measure models would exactly correspond to the morphisms of the metric structures representing them.

It seems difficult to believe that the reverse interpretability could be true: that there is a uniform transformation from metric structures to measure models such that the morphisms of the metric structures correspond exactly to the morphisms of the measure models. Probabilistic argumentation systems could be encoded as metric structures in much the same way, with the prospects of a reverse encoding even more daunting.

The same technique used to create metric structures to encode measure models also give metric structures for structures in approximate measure logic. This pair is interesting, because a significant fragment of type theory and stability has been worked out in both approximate measure logic and continuous first order logic. It

seems unknown at present to what extent these agree: Do the types of the CFO theory correspond to those of the AML theory? Are forking, dividing, and stability conserved?

The challenge to incorporate Bayesian networks into metric structures arises from their hybrid nature — the underlying probability space and the events on it are continuous, while the graph structure is discrete. To address this, let (Ω, \mathcal{B}, P) be a probability space, and let G be a Bayesian network of P . Let \mathcal{A} be the probability algebra (atomless or not) corresponding to (Ω, \mathcal{B}, P) . We add another sort V of vertices with a discrete metric, along with a function $\nu : \hat{\mathcal{B}} \rightarrow V$, which we will prescribe to be a bijection, and a symmetric irreflexive binary relation on V . In this signature, we can, excepting probability zero events, write CFO specifications for $P(\nu^{-1}(x) | \nu^{-1}(y_1, \dots, y_n))$. In this way, all necessary data about a Bayesian network is encoded in a metric structure.

In the other direction, we recall the use of Bayesian networks to compute inference in probabilistic argumentation systems, so that Bayesian networks can encode at least any metric structure arising as a representation of a probabilistic argumentation system.

Despite the obvious similarities, continuous first-order logic does not appear to generalize the logic of adapted spaces in an immediately meaningful way. Certainly there are bounded continuous functions which are not uniformly continuous. Likely some interesting cases on bounded adapted spaces can be formalized in continuous first-order logic, but this does not seem to have been explored in the literature.

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