Math 305  

First Order Linear Equations

A first order differential equation can be linear in either \( y \) or \( t \). An ODE is linear in \( y \) if it can ultimately be written in the form

\[
y' + p(t)y = f(t)
\]

Steps:

1. Identify the coefficient of \( y' \).

2. Is the derivative of this function equal to the coefficient of \( y \)? If yes, go to 7. If no, go to 3.

3. Divide by coefficient of \( y' \).

4. Now identify \( p(t) \) (the coefficient of \( y \)).

5. Evaluate \( \int_t p(s)ds \).

6. Multiply the ODE through by \( e^{\int_t p(s)ds} \).

7. The left-hand side can be written as the product of the coefficient of \( y' \) times \( y \). Write it like this.

8. Integrate both sides.

9. Plug in IC.

Example A. \( t^2y' + 2ty = \sin t \). \( y\left(\frac{\pi}{2}\right) = 1 \).

1. The coefficient of \( y' \) is \( t^2 \).

2. \( \frac{d}{dt} t^2 = 2t = \text{coefficient of } y \). Go to 7.

7. \( t^2y' + 2ty = \frac{d}{dt}(t^2y) = \sin t \)

8. \( \int d(t^2y) = \int \sin tdt \) so that \( t^2y = -\cos t + c \).

9. Plug in IC: \( \frac{\pi^2}{4} \cdot 1 = 0 + c \). Thus \( t^2y = -\cos t + \frac{\pi^2}{4} \).
Example B. \((\sin t)y' + (\cos t)y = e^t\) \(y\left(\frac{\pi}{4}\right) = 0\)

1. The coefficient of \(y'\) is \(\sin t\).

2. \(\frac{d}{dt}\sin t = \cos t = \text{coefficient of } y. \) Go to 7.

7. \((\sin t)y' + (\cos t)y = \frac{d}{dt}(y \sin t) = e^t.\)

8. \(\int d(y \sin t) = \int e^t \, dt\) so that \(y \sin t = e^t + c.\)

9. Plug in IC: \(0 \sin \frac{\pi}{4} = e^\pi + c: \) Thus \(y \sin t = e^t - e^{\pi/4}.\)

Example C. \(2y' + y = t\) \(\quad y(0) = 1.\)

1. The coefficient of \(y'\) is 2.

2. \(\frac{d}{dt}2 = 0 \neq \text{coefficient of } y.\)

3. Divide ODE by 2.

4. \(p(t) = \frac{1}{2}.\)

5. \(\int \frac{1}{2}ds = \frac{1}{2}t\)

6. \(e^{t/2}y' + \frac{1}{2}e^{t/2}y = (e^{t/2})\frac{t}{2}\)

7. \(e^{t/2}y' + \frac{1}{2}e^{t/2}y = \frac{d}{dt}(e^{t/2}y) = (e^{t/2})\frac{t}{2}\)

8. \(\int d(e^{t/2}y) = \int \frac{t}{2}e^{t/2}dt\) so that

\[ ye^{1/2t} = te^{1/2t} - 2e^{1/2t} + c \]

9. Plugging in IC \(1 = 0 - 2 + c.\) Thus \(ye^{t/2} = te^{t/2} - 2e^{t/2} + 3.\)

An equation is linear in \(t\) if it can be written as

\[ t' + p(y)t = f(y). \]

The procedure is the same as the above replacing \(t\) by \(y\) and \(y\) by \(t\).
Example D. \( t' + (\tan y)t = \sin y. \quad y(0) = 0 \)

1. The coefficient of \( t' \) is 1.
2. \( \frac{d}{dy}(1) = 0 \neq \tan y = \text{coefficient of } t. \)
3. Divide ODE by 1.
4. \( p(y) = \tan y \)

5. \( \int p(s)ds = -\ln \cos y = \ln \sec y \)

6. Note \( e^{\ln \sec y} = \sec y. \) This is what we will multiply the ODE through by \( (\sec y)t' + (\tan y \sec y)t = \sin y \sec y. \)

7. \( (\sec y)t' + (\tan y \sec y)t = \frac{d}{dy}(\sec y)t = \sin y \sec y \)

8. \( \int d(t \sec y) = \int \frac{\sin y}{\cos y}dy \) so that \( t \sec y = -\ln \cos y + c. \)

9. Plugging in IC: \( 0 \sec 0 = -\ln \cos 0 + c. \) Thus \( t \sec y = -\ln \cos y. \)

Problems:

1. \( e^{t^2} y' + 2te^{t^2} y = \tan t \quad y(0) = 1 \)

2. \( t^{-3} y' - \frac{3}{t^4} y = t \quad y(1) = 3 \)

3. \( (\sin y)t' + (\cos y)t = y^2 + 1 \quad y(0) = \frac{\pi}{2} \)

4. \( y^2 t' + 2yt = \sin y \quad y(0) = \frac{\pi}{4} \)

5. \( (\cos t)y' + (\sin t)y = 1 \quad y\left(\frac{\pi}{4}\right) = 2 \)

6. \( t' + \frac{1}{y} t = \cos y \quad y(2) = 1 \)

7. \( \frac{1}{2} y' - ty = e^{t^2} \sin^2 t \quad y(0) = 1 \)

8. \( (\sin y)t' - (\cos y)t = \sin y \quad y(1) = \frac{\pi}{4} \)
**Hint I:** Note that when an equation is linear in $y$, and you have written the left-hand side as the derivative of a product, the right-hand side should only involve $t$! If not, something is wrong.

**Hint II:** By now you should realize that $t$, $y$, $x$, $z$, etc. are what we call “dummy” variables. You can have equations linear in any of these variables with one of the other variables the independent variable. A first order ODE is said to be linear if the ODE is linear in the dependent variable.

Examples:

9. $y^3y' = \frac{y^4}{3t + y^5}$
10. $z' + (\cot s)z = 1$
11. $y^2x' + 2yx = e^y$
12. $t' = \frac{t}{t^3 - w}$
13. $t' - \frac{1}{x}t = x^3 \sin x$

Solutions:

1. $y = e^{-t^2} \ln |\sec t| + e^{-t^2}$
2. $y = \frac{1}{2}t^5 + \frac{5}{2}y^3$
3. $t = \frac{1}{3}y^3 \csc y + y \csc y - \left(\frac{\pi}{2} + \frac{\pi}{3} \left(\frac{\pi}{2}\right)^3\right) \csc y$
4. $t = -\frac{1}{y^2} \cos y + \frac{\sqrt{2}}{2y^2}$
5. $y = \sin t + (2\sqrt{2} - 1) \cos t$
6. $t = \sin y + \frac{1}{y} \cos y + (2 - \sin 1 - \cos 1) \frac{1}{y}$
7. $y = te^{t^2} - \frac{1}{2}e^{t^2} \sin 2t + e^{t^2}$
8. $t = -\sin y \ln |\csc y + \cot y| + (\sqrt{2} + \ln(\sqrt{2} + 1)) \sin y$
9. $t = \frac{1}{2}y^5 + cy^3$
10. $z = -\cos s + c \csc s$
11. $x = y^{-2}e^y + cy^{-2}$
12. $w = \frac{t^3}{4} + \frac{c}{t}$
13. $t = -x^3 \cos x + 2x^2 \sin x + 2x \cos x + cx$