Some Math 580 Statistical Inference qualifying exam problems, often with solutions. Most solutions are from David Olive, but a few solutions were contributed by Bhaskar Bhattacharya, Abdel Mugdadi, and Yaser Samadi.

2.68∗. (Aug. 2000 Qual): The number of defects per yard, Y of a certain fabric is known to have a Poisson distribution with parameter λ. However, λ is a random variable with pdf

$$f(\lambda) = e^{-\lambda} I(\lambda > 0).$$

a) Find E(Y).
b) Find Var(Y).

Solution. Note that the pdf for λ is the EXP(1) pdf, so $\lambda \sim$ EXP(1).
a) $E(Y) = E[E(Y|\lambda)] = E(\lambda) = 1$.
b) $V(Y) = E[V(Y|\lambda)] + V[E(Y|\lambda)] = E(\lambda) + V(\lambda) = 1 + 1^2 = 2$.

4.27. (Jan. 2003 Qual) Let X_1 and X_2 be iid Poisson (λ) random variables. Show that $T = X_1 + 2X_2$ is not a sufficient statistic for λ. (Hint: the Factorization Theorem uses the word *iff*. Alternatively, find a minimal sufficient statistic S and show that S is not a function of T.)

See 4.38 solution.

4.28. (Aug. 2002 Qual): Suppose that $X_1, ..., X_n$ are iid $N(\sigma, \sigma)$ where $\sigma > 0$.
a) Find a minimal sufficient statistic for σ.
b) Show that $(\overline{X}, S^2)$ is a sufficient statistic but is not a complete sufficient statistic for σ.

4.31. (Aug. 2004 Qual): Let $X_1, ..., X_n$ be iid beta(θ, θ). (Hence $\delta = \nu = \theta$.)
a) Find a minimal sufficient statistic for θ.
b) Is the statistic found in a) complete? (prove or disprove)

Solution.

$$f(x) = \frac{\Gamma(2\theta)}{\Gamma(\theta)\Gamma(\theta)} x^{\theta-1}(1-x)^{\theta-1} = \frac{\Gamma(2\theta)}{\Gamma(\theta)\Gamma(\theta)} \exp[(\theta-1)(\log(x) + \log(1-x))],$$

for $0 < x < 1$, a 1 parameter exponential family. Hence $\sum_{i=1}^n (\log(X_i) + \log(1 - X_i))$ is a complete minimal sufficient statistic.

4.32. (Sept. 2005 Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables with probability mass function

$$f(x) = P(X = x) = \frac{1}{x^\nu \zeta(\nu)}$$

where $\nu > 1$ and $x = 1, 2, 3, ...$ Here the zeta function

$$\zeta(\nu) = \sum_{x=1}^{\infty} \frac{1}{x^\nu}$$

for $\nu > 1$.

a) Find a minimal sufficient statistic for ν.

b) Is the statistic found in a) complete? (prove or disprove)

c) Give an example of a sufficient statistic that is strictly not minimal.

Solution. a) and b)

$$f(x) = I_{\{1,2,...\}}(x)\frac{1}{\zeta(\nu)}\exp[-\nu\log(x)]$$

is a 1 parameter regular exponential family with $\Omega = (-\infty, -1)$. Hence $\sum_{i=1}^n \log(X_i)$ is a complete minimal sufficient statistic.

c) By the Factorization Theorem, $\boldsymbol{W} = (X_1, ..., X_n)$ is sufficient, but $\boldsymbol{W}$ is not minimal since $\boldsymbol{W}$ is not a function of $\sum_{i=1}^n \log(X_i)$.

4.36. (Aug. 2009 Qual): Let $X_1, ..., X_n$ be iid uniform$(\theta, \theta+1)$ random variables where θ is real.

a) Find a minimal sufficient statistic for θ.

b) Show whether the minimal sufficient statistic is complete or not.

Solution. Now

$$f_X(x) = I(\theta < x < \theta + 1)$$

and

$$\frac{f(\boldsymbol{x})}{f(\boldsymbol{y})} = \frac{I(\theta < x_{(1)} \le x_{(n)} < \theta + 1)}{I(\theta < y_{(1)} \le y_{(n)} < \theta + 1)}$$

which is constant for all real θ iff $(x_{(1)}, x_{(n)}) = (y_{(1)}, y_{(n)})$. Hence $\boldsymbol{T} = (X_{(1)}, X_{(n)})$ is a minimal sufficient statistic by the LSM theorem. To show that $\boldsymbol{T}$ is not complete, first find $E(\boldsymbol{T})$. Now

$$F_X(t) = \int_\theta^t dx = t - \theta$$

for $\theta < t < \theta + 1$. Hence

$$f_{X_{(n)}}(t) = n[F_X(t)]^{n-1} f_x(t) = n(t-\theta)^{n-1}$$

for $\theta < t < \theta + 1$ and

$$E_\theta(X_{(n)}) = \int t f_{X_{(n)}}(t) dt = \int_\theta^{\theta+1} tn(t-\theta)^{n-1} dt.$$

Use u–substitution with $u = t - \theta$, $t = u + \theta$ and $dt = du$. Hence $t = \theta$ implies $u = 0$, and $t = \theta + 1$ implies $u = 1$. Thus

$$E_\theta(X_{(n)}) = \int_0^1 n(u+\theta)u^{n-1} du = \int_0^1 nu^n du + \int_0^1 n\theta u^{n-1} du =$$

$$n\frac{u^{n+1}}{n+1}\Big|_0^1 + \theta\, n\, \frac{u^n}{n}\Big|_0^1 = \frac{n}{n+1} + \frac{n\theta}{n} = \theta + \frac{n}{n+1}.$$

Now

$$f_{X_{(1)}}(t) = n[1 - F_X(t)]^{n-1} f_x(t) = n(1 - t + \theta)^{n-1}$$

for $\theta < t < \theta + 1$ and thus

$$E_\theta(X_{(1)}) = \int_\theta^{\theta+1} tn(1 - t + \theta)^{n-1} dt.$$

Use u–substitution with $u = (1 - t + \theta)$ and $t = 1 - u + \theta$ and $du = -dt$. Hence $t = \theta$ implies $u = 1$, and $t = \theta + 1$ implies $u = 0$. Thus

$$E_\theta(X_{(1)}) = -\int_1^0 n(1 - u + \theta)u^{n-1} du = n(1 + \theta) \int_0^1 u^{n-1} du - n \int_0^1 u^n du =$$

$$n(1 + \theta) \frac{u^n}{n}\Big|_0^1 - n \frac{u^{n+1}}{n+1}\Big|_0^1 = (\theta + 1)\frac{n}{n} - \frac{n}{n+1} = \theta + \frac{1}{n+1}.$$

To show that $\boldsymbol{T}$ is not complete try showing $E_\theta(aX_{(1)} + bX_{(n)} + c) = 0$ for some constants a, b and c. Note that $a = -1$, $b = 1$ and $c = -\frac{n-1}{n+1}$ works. Hence

$$E_\theta(-X_{(1)} + X_{(n)} - \frac{n-1}{n+1}) = 0$$

for all real θ but

$$P_\theta(g(\boldsymbol{T}) = 0) = P_\theta(-X_{(1)} + X_{(n)} - \frac{n-1}{n+1} = 0) = 0 < 1$$

for all real θ. Hence $\boldsymbol{T}$ is not complete.

4.37. (Sept. 2010 Qual): Let $Y_1, ..., Y_n$ be iid from a distribution with pdf

$$f(y) = 2\,\tau\,y\,e^{-y^2}\,(1 - e^{-y^2})^{\tau-1}$$

for $y > 0$ and $f(y) = 0$ for $y \le 0$ where $\tau > 0$.

a) Find a minimal sufficient statistic for τ.

b) Is the statistic found in a) complete? Prove or disprove.

Solution. Note that

$$f(y) = I(y > 0)\ 2y\ e^{-y^2}\ \tau\ \exp[(1 - \tau)(-\log(1 - e^{-y^2}))]$$

is a 1 parameter exponential family with minimal and complete sufficient statistic $-\sum_{i=1}^n \log(1 - e^{-Y_i^2})$.

4.38. (Aug. 2016 qual) a) Let $X_1, ..., X_n$ be independent identically distributed gamma(α, β), and, independently, $Y_1, ..., Y_m$ independent identically distributed gamma$(\alpha, k\beta)$ where k is known, and $\alpha, \beta > 0$ are parameters. Find a two dimensional sufficient statistic for (α, β).

b) Let X_1, X_2 be independent identically distributed Poisson(θ). Show $T = X_1 + 2X_2$ is not sufficient for θ.

Solution: a) $f(\boldsymbol{x}, \boldsymbol{y}) =$

$$\left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n (\prod_{i=1}^n x_i)^{\alpha-1} \exp[-\sum_{i=1}^n x_i/\beta] \left(\frac{1}{\Gamma(\alpha)(k\beta)^\alpha}\right)^m (\prod_{j=1}^m y_j)^{\alpha-1} \exp[-\sum_{j=1}^m y_j/(k\beta)]$$

$$= \left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n \left(\frac{1}{\Gamma(\alpha)(k\beta)^\alpha}\right)^m \left[(\prod_{i=1}^n x_i)(\prod_{j=1}^m y_j)\right]^{\alpha-1} \exp\left[-\left(\frac{\sum_{i=1}^n x_i + \sum_{j=1}^m y_j/k}{\beta}\right)\right].$$

By Factorization,

$$\left((\prod_{i=1}^n X_i)(\prod_{j=1}^m Y_j), \sum_{i=1}^n X_i + \sum_{j=1}^m Y_j/k\right) \text{ or}$$

$$\left(\sum_{i=1}^n \log(X_i) + \sum_{j=1}^m \log(Y_j), \sum_{i=1}^n X_i + \sum_{j=1}^m Y_j/k\right)$$

is sufficient.

b) The minimal sufficient statistic $X_1 + X_2$ is not a function of T, thus T is not sufficient. Alternatively, the Factorization Theorem says T is sufficient iff $f(\boldsymbol{x}|\theta) = g(T(\boldsymbol{x})|\theta)h(\boldsymbol{x})$ where $h(\boldsymbol{x})$ does not depend on θ and g depends on $\boldsymbol{x}$ only through $T(\boldsymbol{x})$. No such factorization exists.

4.39. (Feb. 2023 qual): Let $W_1, ..., W_n$ be iid from a Weibull(ϕ, λ) distribution where if $W \sim$ Weibull(ϕ, λ), then the pdf of W is

$$f(w) = \frac{\phi}{\lambda} w^{\phi-1} e^{-\frac{w^\phi}{\lambda}}$$

where λ, w, and ϕ are all positive. So $f(w) = 0$ for $w \leq 0$.

a) If ϕ is known, find a complete sufficient statistic for λ.

b) If both ϕ and λ are unknown, find a minimal sufficient statistic.

Solution. a) If ϕ is known, then

$$f(w) = w^{\phi-1} I(w \geq 0) \ \frac{\phi}{\lambda} \ \exp\left[\frac{-1}{\lambda} w^\phi\right]$$

is a one parameter regular exponential family in λ. Hence $T(\boldsymbol{W}) = \sum_{i=1}^n W_i^\phi$ is complete.

b) $\prod_{i=1}^n f(y_i) = \prod_{i=1}^n \frac{\phi}{\lambda} y_i^{\phi-1} e^{-y_i^\phi/\lambda} I(y_i > 0) =$

$$\left(\frac{\phi}{\lambda}\right)^n \prod_{i=1}^n y_i^{\phi-1} \exp\left(\frac{-1}{\lambda}\sum_{i=1}^n y_i^\phi\right) \prod_{i=1}^n I(y_i > 0).$$

Since ϕ is unknown, by the Factorization theorem, a permutation of the data is the lowest dimensional sufficient statistic, including the order statistics. Let $\boldsymbol{Z}_n = \boldsymbol{z}_n$ be an arbitrary permuatation of the data $\boldsymbol{Y}_n = \boldsymbol{y}_n$. Let $t_1(\boldsymbol{z}_n)$ be a one to one and onto function so that $t_1^{-1}(t_1(\boldsymbol{z}_n)) = \boldsymbol{z}_n$. Let $t_o(\boldsymbol{z}_n) = (y_{(1)}, ..., y_{(n)})$ be the observed order statistics and $t_o(\boldsymbol{Z}_n) = (Y_{(1)}, ..., Y_{(n)})$ be the order statistics. Then any observed sufficient statistic has the form $(R_n(\boldsymbol{y}_n), t_1(\boldsymbol{z}_n))$ where the possible vector valued statistic $R_n(\boldsymbol{y}_n)$ is redundant and so not needed. Here $R_n(\boldsymbol{y}_n)$ comes from $\prod_{i=1}^n y_i$. Then the order statistics $t_o(\boldsymbol{z}_n) = t(R_n(\boldsymbol{y}_n), t_1(\boldsymbol{z}_n)) = t_o(t_1^{-1}(t_1(\boldsymbol{z}_n))$ are a function of any of the sufficient statistics. Hence the order statistics are minimal sufficient.

(Using LSM is much harder for this distribution.

$$\text{So } \mathrm{R}_{\boldsymbol{x},\boldsymbol{y}}(\phi,\lambda) = \frac{\mathrm{f}(\boldsymbol{x}|\phi,\lambda)}{\mathrm{f}(\boldsymbol{y}|\phi,\lambda)} =$$

$$\frac{\prod_{i=1}^n I(x_i > 0) \prod_{i=1}^n x_i^{\phi-1}}{\prod_{i=1}^n I(y_i > 0) \prod_{i=1}^n y_i^{\phi-1}} \exp\left(\frac{1}{\lambda}(\sum_{i=1}^n x_i^\phi - \sum_{i=1}^n y_i^\phi)\right) =$$

$$\frac{\prod_{i=1}^n I(x_i > 0)}{\prod_{i=1}^n I(y_i > 0)} \prod_{i=1}^n \left(\frac{x_{(i)}}{y_{(i)}}\right)^{\phi-1} \exp\left(\frac{1}{\lambda}(\sum_{i=1}^n x_{(i)}^\phi - \sum_{i=1}^n y_{(i)}^\phi)\right) = c_{\boldsymbol{x},\boldsymbol{y}}$$

$\forall \phi > 0, \lambda > 0$ iff $\sum_{i=1}^n x_{(i)}^\phi - \sum_{i=1}^n y_{(i)}^\phi = 0$ as λ varies and

$$\prod_{i=1}^n \left(\frac{x_{(i)}}{y_{(i)}}\right)^{\phi-1}$$

is a constant as ϕ varies. When ϕ is unknown, $\sum_{i=1}^n x_{(i)}^\phi$ is not a statistic, but if the order statistics are equal, then $R_{\boldsymbol{x},\boldsymbol{y}}(\phi,\lambda)$ is constant. Now suppose the order statistics are not equal. If $n = 1$, then $R_{\boldsymbol{x},\boldsymbol{y}}(\phi,\lambda)$ will not be constant. Hence assume that for $n = k$, $R_{\boldsymbol{x},\boldsymbol{y}}(\phi,\lambda)$ is constant iff the order statistics are equal. Now let $n = k+1$. Let $\boldsymbol{x}'_k$ and $\boldsymbol{y}'_k$ correspond to the first k order statistics. Then $x'_{k+1} = x_{(n)}$ and $y'_{k+1} = y_{(n)}$. Since $\boldsymbol{x}'_k$ and $\boldsymbol{y}'_k$ are in the support (each entry is positive), by the induction hypothesis, $x_{(1)} = y_{(1)}, ..., x_{(k)} = y_{(k)}$ or $R_{\boldsymbol{x}'_k,\boldsymbol{y}'_k}$ is not constant. For $R_{\boldsymbol{x},\boldsymbol{y}}$ to be constant, need

$$a = \prod_{i=1}^k \left(\frac{x_{(i)}}{y_{(i)}}\right)^{\phi-1} \left(\frac{x_{(n)}}{y_{(n)}}\right)^{\phi-1} = 1$$

as ϕ varies and $b = \sum_{i=1}^k x_{(i)}^\phi - \sum_{i=1}^k y_{(i)}^\phi + x_{(n)}^\phi - y_{(n)}^\phi = 0$. Thus

$$\frac{x_{(n)}}{y_{(n)}} = \prod_{i=1}^k \left(\frac{y_{(i)}}{x_{(i)}}\right)$$

for $a = 1$. For $b = 0$, need

$$\sum_{i=1}^k x_{(i)}^\phi - \sum_{i=1}^k y_{(i)}^\phi + \left[y_{(n)} \prod_{i=1}^k \left(\frac{y_{(i)}}{x_{(i)}}\right) - y_{(n)}\right]^\phi = \sum_{i=1}^k x_{(i)}^\phi - \sum_{i=1}^k y_{(i)}^\phi + d^\phi = 0$$

as ϕ varies, which is impossible unless $d = 0$ or $d = 1$. The case $d = 1$ requires $\sum_{i=1}^k x_{(i)}^\phi - \sum_{i=1}^k y_{(i)}^\phi = -1$ for all ϕ which is impossible. Then by the induction hypothesis,

$$\sum_{i=1}^k x_{(i)}^\phi - \sum_{i=1}^k y_{(i)}^\phi = 0$$

for $b = 0$. Thus $x_{(i)} = y_{(i)}$ for $i = 1, ..., n = k+1$. Hence by induction the order statistics need to be equal for $R_{\boldsymbol{x},\boldsymbol{y}}$ to be constant. Thus the order statistics are minimal sufficient by LSM.)

4.40. (Aug. 2024 Qual): Suppose $X_1, \ldots, X_n$ is a random sample from a population with pdf

$$f(x|\theta) = \frac{e^{-(x-\theta)}}{(1+e^{-(x-\theta)})^2},$$

where $x, \theta \in \mathbb{R}$. Find a minimal sufficient statistic for θ.

Solution:

$$\frac{f(\boldsymbol{x})}{f(\boldsymbol{y})} = \frac{e^{-\sum(x_i-\theta)}}{\prod[1+e^{-(x_i-\theta)}]^2} \frac{\prod[1+e^{-(y_i-\theta)}]^2}{e^{-\sum(y_i-\theta)}} = \frac{\prod[1+e^{-(y_{(i)}-\theta)}]^2}{\prod[1+e^{-(x_{(i)}-\theta)}]^2} \frac{e^{-\sum x_i} e^{n\theta}}{e^{-\sum y_i} e^{n\theta}}$$

$= ad \equiv c$ for all θ iff $a =$

$$\prod_{i=1}^n \left[\frac{1+e^{-(y_{(i)}-\theta)}}{1+e^{-(x_{(i)}-\theta)}}\right]^2 \equiv c'$$

for all θ iff the order statistics are equal. So the order statistics $X_{(1)}, ..., X_{(n)}$ are minimal sufficient by LSM.

5.2. (1989 Univ. of Minn. and Aug. 2000 SIU Qual): Let (X, Y) have the bivariate density

$$f(x,y) = \frac{1}{2\pi} \exp(\frac{-1}{2}[(x-\rho\cos\theta)^2 + (y-\rho\sin\theta)^2]).$$

Suppose that there are n independent pairs of observations (X_i, Y_i) from the above density and that ρ is known. Assume that $0 \le \theta \le 2\pi$. Find a candidate for the maximum likelihood estimator $\hat{\theta}$ by differentiating the log likelihood $\log(L(\theta))$. (Do not show that the candidate is the MLE, it is difficult to tell whether the candidate, 0 or 2π is the MLE without the actual data.)

Solution. The likelihood function $L(\theta) =$

$$\frac{1}{(2\pi)^n} \exp(\frac{-1}{2}[\sum(x_i-\rho\cos\theta)^2 + \sum(y_i-\rho\sin\theta)^2]) =$$

$$\frac{1}{(2\pi)^n} \exp(\frac{-1}{2}[\sum x_i^2 - 2\rho\cos\theta\sum x_i + \rho^2\cos^2\theta + \sum y_i^2 - 2\rho\sin\theta\sum y_i + \rho^2\sin^2\theta])$$

$$= \frac{1}{(2\pi)^n} \exp(\frac{-1}{2}[\sum x_i^2 + \sum y_i^2 + \rho^2]) \exp(\rho\cos\theta\sum x_i + \rho\sin\theta\sum y_i).$$

Hence the log likelihood log $L(\theta)$

$$= c + \rho\cos\theta\sum x_i + \rho\sin\theta\sum y_i.$$

The derivative with respect to θ is

$$-\rho\sin\theta\sum x_i + \rho\cos\theta\sum y_i.$$

Setting this derivative to zero gives

$$\rho \sum y_i \cos\theta = \rho \sum x_i \sin\theta$$

or

$$\frac{\sum y_i}{\sum x_i} = \tan\theta.$$

Thus

$$\hat{\theta} = \tan^{-1}\left(\frac{\sum y_i}{\sum x_i}\right).$$

Now the boundary points are $\theta = 0$ and $\theta = 2\pi$. Hence $\hat{\theta}_{MLE}$ equals 0, 2π, or $\hat{\theta}$ depending on which value maximizes the likelihood.

5.23. (Jan. 2001 Qual): Let $X_1, ..., X_n$ be a random sample from a normal distribution with **known** mean μ and unknown variance τ.

a) Find the maximum likelihood estimator of the variance τ.

b) Find the maximum likelihood estimator of the standard deviation $\sqrt{\tau}$. Explain how the MLE was obtained.

Solution. a) The log likelihood is $\log L(\tau) = -\frac{n}{2}\log(2\pi\tau) - \frac{1}{2\tau}\sum_{i=1}^n (X_i - \mu)^2$. The derivative of the log likelihood is equal to $-\frac{n}{2\tau} + \frac{1}{2\tau^2}\sum_{i=1}^n (X_i - \mu)^2$. Setting the derivative equal to 0 and solving for τ gives the MLE $\hat{\tau} = \frac{\sum_{i=1}^n (X_i - \mu)^2}{n}$. Now the likelihood is only defined for $\tau > 0$. As τ goes to 0 or ∞, $\log L(\tau)$ tends to $-\infty$. Since there is only one critical point, $\hat{\tau}$ is the MLE.

b) By the invariance principle, the MLE is $\sqrt{\frac{\sum_{i=1}^n (X_i - \mu)^2}{n}}$.

5.28. (Aug. 2002 Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables from a half normal HN(μ, σ^2) distribution with pdf

$$f(x) = \frac{2}{\sqrt{2\pi}\ \sigma} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

where $\sigma > 0$ and $x > \mu$ and μ is real. **Assume that μ is known.**

a) Find the maximum likelihood estimator of σ^2.

b) What is the maximum likelihood estimator of σ? Explain.

Solution. This problem is nearly the same as finding the MLE of σ^2 when the data are iid $N(\mu, \sigma^2)$ when μ is known. See Problem 5.23 and Section 10.23. The MLE in a) is $\sum_{i=1}^n (X_i - \mu)^2/n$. For b) use the invariance principle and take the square root of the answer in a).

5.29. (Jan. 2003 Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables from a lognormal (μ, σ^2) distribution with pdf

$$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(\log(x) - \mu)^2}{2\sigma^2}\right)$$

where $\sigma > 0$ and $x > 0$ and μ is real. **Assume that σ is known.**

a) Find the maximum likelihood estimator of μ.

b) What is the maximum likelihood estimator of μ^3? Explain.

Solution. a)

$$\hat{\mu} = \frac{\sum \log(X_i)}{n}$$

To see this note that

$$L(\mu) = (\prod \frac{1}{x_i\sqrt{2\pi\sigma^2}}) \exp(\frac{-\sum(\log(x_i) - \mu)^2}{2\sigma^2}.$$

So

$$\log(L(\mu)) = \log(c) \quad - \quad \frac{\sum(\log(x_i) - \mu)^2}{2\sigma^2}$$

and the derivative of the log likelihood wrt μ is

$$\frac{\sum 2(\log(x_i) - \mu)}{2\sigma^2}.$$

Setting this quantity equal to 0 gives $n\mu = \sum \log(x_i)$ and the solution $\hat{\mu}$ is unique. The second derivative is $-n/\sigma^2 < 0$, so $\hat{\mu}$ is indeed the global maximum.

b)

$$\left(\frac{\sum \log(X_i)}{n}\right)^3$$

by invariance.

5.30. (Aug. 2004 Qual): Let X be a single observation from a normal distribution with mean θ and with variance θ^2, where $\theta > 0$. Find the maximum likelihood estimator of θ^2.

Solution.

$$L(\theta) = \frac{1}{\theta\sqrt{2\pi}} e^{-(x-\theta)^2/2\theta^2}$$

$$ln(L(\theta)) = -ln(\theta) - ln(\sqrt{2\pi}) - (x-\theta)^2/2\theta^2$$

$$\begin{aligned} \frac{dln(L(\theta))}{d\theta} &= \frac{-1}{\theta} + \frac{x-\theta}{\theta^2} + \frac{(x-\theta)^2}{\theta^3} \\ &= \frac{x^2}{\theta^3} - \frac{x}{\theta^2} - \frac{1}{\theta} \stackrel{set}{=} 0 \end{aligned}$$

by solving for θ,

$$\theta = \frac{x}{2} * (-1 + \sqrt{5}),$$

and

$$\theta = \frac{x}{2} * (-1 - \sqrt{5}).$$

But, $\theta > 0$. Thus, $\hat{\theta} = \frac{x}{2} * (-1 + \sqrt{5})$, when $x > 0$, and $\hat{\theta} = \frac{x}{2} * (-1 - \sqrt{5})$, when $x < 0$.

To check with the second derivative

$$\begin{aligned} \frac{d^2 ln(L(\theta))}{d\theta^2} &= -\frac{2\theta + x}{\theta^3} + \frac{3(\theta^2 + \theta x - x^2)}{\theta^4} \\ &= \frac{\theta^2 + 2\theta x - 3x^2}{\theta^4} \end{aligned}$$

but the sign of the θ^4 is always positive, thus the sign of the second derivative depends on the sign of the numerator. Substitute $\hat{\theta}$ in the numerator and simplify, you get $\frac{x^2}{2}(-5 \pm \sqrt{5})$, which is always negative. Hence by the invariance principle, the MLE of θ^2 is $\hat{\theta}^2$.

5.31. (Sept. 2005 Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables with probability density function

$$f(x) = \frac{\sigma^{1/\lambda}}{\lambda} \exp\left[-(1 + \frac{1}{\lambda}) \log(x)\right] I[x \geq \sigma]$$

where $x \geq \sigma$, $\sigma > 0$, and $\lambda > 0$. The indicator function $I[x \geq \sigma] = 1$ if $x \geq \sigma$ and 0, otherwise. Find the maximum likelihood estimator (MLE) $(\hat{\sigma}, \hat{\lambda})$ of (σ, λ) with the following steps.

a) Explain why $\hat{\sigma} = X_{(1)} = \min(X_1, ..., X_n)$ is the MLE of σ regardless of the value of $\lambda > 0$.

b) Find the MLE $\hat{\lambda}$ of λ if $\sigma = \hat{\sigma}$ (that is, act as if $\sigma = \hat{\sigma}$ is known).

Solution. a) For any $\lambda > 0$, the likelihood function

$$L(\sigma, \lambda) = \sigma^{n/\lambda}\ I[x_{(1)} \geq \sigma]\ \frac{1}{\lambda^n}\ \exp\left[-(1 + \frac{1}{\lambda}) \sum_{i=1}^n \log(x_i)\right]$$

is maximized by making σ as large as possible. Hence $\hat{\sigma} = X_{(1)}$.

b)

$$L(\hat{\sigma}, \lambda) = \hat{\sigma}^{n/\lambda}\ I[x_{(1)} \geq \hat{\sigma}]\ \frac{1}{\lambda^n}\ \exp\left[-(1 + \frac{1}{\lambda}) \sum_{i=1}^n \log(x_i)\right].$$

Hence $\log L(\hat{\sigma}, \lambda) =$

$$\frac{n}{\lambda} \log(\hat{\sigma}) - n \log(\lambda) - (1 + \frac{1}{\lambda}) \sum_{i=1}^n \log(x_i).$$

Thus

$$\frac{d}{d\lambda} \log L(\hat{\sigma}, \lambda) = \frac{-n}{\lambda^2} \log(\hat{\sigma}) - \frac{n}{\lambda} + \frac{1}{\lambda^2} \sum_{i=1}^n \log(x_i) \stackrel{set}{=} 0,$$

or $-n \log(\hat{\sigma}) + \sum_{i=1}^n \log(x_i) = n\lambda$. So

$$\hat{\lambda} = -\log(\hat{\sigma}) + \frac{\sum_{i=1}^n \log(x_i)}{n} = \frac{\sum_{i=1}^n \log(x_i/\hat{\sigma})}{n}.$$

Now

$$\frac{d^2}{d\lambda^2} \log L(\hat\sigma, \lambda) = \frac{2n}{\lambda^3} \log(\hat\sigma) + \frac{n}{\lambda^2} - \frac{2}{\lambda^3} \sum_{i=1}^n \log(x_i) \Bigg|_{\lambda=\hat\lambda}$$

$$= \frac{n}{\hat\lambda^2} - \frac{2}{\hat\lambda^3} \sum_{i=1}^n \log(x_i/\hat\sigma) = \frac{-n}{\hat\lambda^2} < 0.$$

Hence $(\hat\sigma, \hat\lambda)$ is the MLE of (σ, λ).

5.32. (Aug. 2003 Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables with pdf

$$f(x) = \frac{1}{\lambda} \exp\left[-(1 + \frac{1}{\lambda}) \log(x)\right]$$

where $\lambda > 0$ and $x \geq 1$.

a) Find the maximum likelihood estimator of λ.

b) What is the maximum likelihood estimator of λ^8 ? Explain.

Solution. a) the likelihood

$$L(\lambda) = \frac{1}{\lambda^n} \exp\left[-(1 + \frac{1}{\lambda}) \sum \log(x_i)\right],$$

and the log likelihood

$$\log(L(\lambda)) = -n \log(\lambda) - (1 + \frac{1}{\lambda}) \sum \log(x_i).$$

Hence

$$\frac{d}{d\lambda} \log(L(\lambda)) = \frac{-n}{\lambda} + \frac{1}{\lambda^2} \sum \log(x_i) \stackrel{set}{=} 0,$$

or $\sum \log(x_i) = n\lambda$ or

$$\hat\lambda = \frac{\sum \log(X_i)}{n}.$$

Notice that

$$\frac{d^2}{d\lambda^2} \log(L(\lambda)) = \frac{n}{\lambda^2} - \frac{2 \sum \log(x_i)}{\lambda^3} \Bigg|_{\lambda=\hat\lambda} =$$

$$\frac{n}{\hat\lambda^2} - \frac{2n\hat\lambda}{\hat\lambda^3} = \frac{-n}{\hat\lambda^2} < 0.$$

Hence $\hat\lambda$ is the MLE of λ.

b) By invariance, $\hat\lambda^8$ is the MLE of λ^8.

5.33. (Jan. 2004 Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables with probability mass function

$$f(x) = e^{-2\theta} \frac{1}{x!} \exp[\log(2\theta) x],$$

for $x = 0, 1, \ldots,$ where $\theta > 0$. Assume that at least one $X_i > 0$.

a) Find the maximum likelihood estimator of θ.

b) What is the maximum likelihood estimator of $(\theta)^4$? Explain.

Solution. a) The likelihood

$$L(\theta) = c \; e^{-n2\theta} \exp[\log(2\theta) \sum x_i],$$

and the log likelihood

$$\log(L(\theta)) = d - n2\theta + \log(2\theta) \sum x_i.$$

Hence

$$\frac{d}{d\theta} \log(L(\theta)) = -2n + \frac{2}{2\theta} \sum x_i \overset{set}{=} 0,$$

or $\sum x_i = 2n\theta$, or

$$\hat{\theta} = \overline{X}/2.$$

Notice that

$$\frac{d^2}{d\theta^2} \log(L(\theta)) = \frac{-\sum x_i}{\theta^2} < 0$$

unless $\sum x_i = 0$.

b) $(\hat{\theta})^4 = (\overline{X}/2)^4$ by invariance.

5.34. (Jan. 2006 Qual): Let $X_1, ..., X_n$ be iid with one of two probability density functions. If $\theta = 0$, then

$$f(x|\theta) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

If $\theta = 1$, then

$$f(x|\theta) = \begin{cases} \frac{1}{2\sqrt{x}}, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find the maximum likelihood estimator of θ.

Solution. $L(0|\boldsymbol{x}) = 1$ for $0 < x_i < 1$, and $L(1|\boldsymbol{x}) = \prod_{i=1}^n \frac{1}{2\sqrt{x_i}}$ for $0 < x_i < 1$. Thus the MLE is 0 if $1 \ge \prod_{i=1}^n \frac{1}{2\sqrt{x_i}}$ and the MLE is 1 if $1 < \prod_{i=1}^n \frac{1}{2\sqrt{x_i}}$.

Warning: Variants of the following question often appear on qualifying exams.

5.35. (Aug. 2006 Qual): Let $Y_1, ..., Y_n$ denote a random sample from a $N(a\theta, \theta)$ population.

a) Find the MLE of θ when $a = 1$.

b) Find the MLE of θ when a is known but arbitrary.

Solution. a) Notice that $\theta > 0$ and

$$f(y) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\theta}} \exp\left(\frac{-(y-\theta)^2}{2\theta}\right).$$

Hence the likelihood

$$L(\theta) = c \, \frac{1}{\theta^{n/2}} \exp\left[\frac{-1}{2\theta} \sum (y_i - \theta)^2\right]$$

and the log likelihood

$$\log(L(\theta)) = d \; - \; \frac{n}{2}\log(\theta) - \frac{1}{2\theta}\sum (y_i - \theta)^2 =$$

$$d \; - \; \frac{n}{2}\log(\theta) \; - \; \frac{1}{2}\sum_{i=1}^{n}\left(\frac{y_i^2}{\theta} \; - \; \frac{2y_i\theta}{\theta} \; + \; \frac{\theta^2}{\theta}\right)$$

$$= d \; - \; \frac{n}{2}\log(\theta) \; - \; \frac{1}{2}\frac{\sum_{i=1}^n y_i^2}{\theta} \; + \; \sum_{i=1}^n y_i \; - \; \frac{1}{2}n\theta.$$

Thus

$$\frac{d}{d\theta}\log(L(\theta)) = \frac{-n}{2}\frac{1}{\theta} + \frac{1}{2}\sum_{i=1}^n y_i^2 \frac{1}{\theta^2} - \frac{n}{2} \overset{set}{=} 0,$$

or

$$\frac{-n}{2}\theta^2 - \frac{n}{2}\theta + \frac{1}{2}\sum_{i=1}^n y_i^2 = 0,$$

or

$$n\theta^2 \; + \; n\theta \; - \; \sum_{i=1}^n y_i^2 = 0. \tag{1}$$

Now the quadratic formula states that for $a \neq 0$, the quadratic equation $ay^2 + by + c = 0$ has roots

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Applying the quadratic formula to (1) gives

$$\theta = \frac{-n \pm \sqrt{n^2 + 4n\sum_{i=1}^n y_i^2}}{2n}.$$

Since $\theta > 0$, a candidate for the MLE is

$$\hat{\theta} = \frac{-n + \sqrt{n^2 + 4n\sum_{i=1}^n Y_i^2}}{2n} = \frac{-1 + \sqrt{1 + 4\frac{1}{n}\sum_{i=1}^n Y_i^2}}{2}.$$

Since $\hat{\theta}$ satisfies (1),

$$n\hat{\theta} - \sum_{i=1}^n y_i^2 = -n\hat{\theta}^2. \tag{2}$$

Note that

$$\frac{d^2}{d\theta^2}\log(L(\theta)) = \frac{n}{2\theta^2} - \frac{\sum_{i=1}^n y_i^2}{\theta^3} = \frac{1}{2\theta^3}[n\theta - 2\sum_{i=1}^n y_i^2]\Big|_{\theta=\hat\theta} =$$

$$\frac{1}{2\hat\theta^3}[n\hat\theta - \sum_{i=1}^n y_i^2 - \sum_{i=1}^n y_i^2] = \frac{1}{2\hat\theta^3}[-n\hat\theta^2 - \sum_{i=1}^n y_i^2] < 0$$

by (2). Since $L(\theta)$ is continuous with a unique root on $\theta > 0$, $\hat\theta$ is the MLE.

5.37. (Aug. 2006 Qual): Let $X_1, ..., X_n$ be independent identically distributed (iid) random variables with probability density function

$$f(x) = \frac{2}{\lambda\sqrt{2\pi}} e^x \exp\left(\frac{-(e^x-1)^2}{2\lambda^2}\right)$$

where $x > 0$ and $\lambda > 0$.

a) Find the maximum likelihood estimator (MLE) $\hat\lambda$ of λ.

b) What is the MLE of λ^2? Explain.

Solution. a) $L(\lambda) = c\frac{1}{\lambda^n}\exp\left(\frac{-1}{2\lambda^2}\sum_{i=1}^n (e^{x_i}-1)^2\right)$.
Thus

$$\log(L(\lambda)) = d - n\log(\lambda) - \frac{1}{2\lambda^2}\sum_{i=1}^n (e^{x_i}-1)^2.$$

Hence

$$\frac{d\log(L(\lambda))}{d\lambda} = \frac{-n}{\lambda} + \frac{1}{\lambda^3}\sum(e^{x_i}-1)^2 \stackrel{set}{=} 0,$$

or $n\lambda^2 = \sum(e^{x_i}-1)^2$, or

$$\hat\lambda = \sqrt{\frac{\sum(e^{X_i}-1)^2}{n}}.$$

Now

$$\frac{d^2\log(L(\lambda))}{d\lambda^2} = \frac{n}{\lambda^2} - \frac{3}{\lambda^4}\sum(e^{x_i}-1)^2\Big|_{\lambda=\hat\lambda}$$

$$= \frac{n}{\hat\lambda^2} - \frac{3n}{\hat\lambda^4}\hat\lambda^2 = \frac{n}{\hat\lambda^2}[1-3] < 0.$$

So $\hat\lambda$ is the MLE.

5.38. (Jan. 2007 Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables from a distribution with pdf

$$f(x) = \frac{2}{\lambda\sqrt{2\pi}}\ \frac{1}{x}\ \exp\left[\frac{-(\log(x))^2}{2\lambda^2}\right]$$

where $\lambda > 0$ where and $0 \le x \le 1$.

a) Find the maximum likelihood estimator (MLE) of λ.

b) Find the MLE of λ^2.

Solution. a) The likelihood

$$L(\lambda) = \prod f(x_i) = c \left(\prod \frac{1}{x_i} \right) \frac{1}{\lambda^n} \exp\left[\frac{\sum -(\log x_i)^2}{2\lambda^2} \right],$$

and the log likelihood

$$\log(L(\lambda)) = d - \sum \log(x_i) - n\log(\lambda) - \frac{\sum (\log x_i)^2}{2\lambda^2}.$$

Hence

$$\frac{d}{d\lambda} \log(L(\lambda)) = \frac{-n}{\lambda} + \frac{\sum (\log x_i)^2}{\lambda^3} \stackrel{set}{=} 0,$$

or $\sum (\log x_i)^2 = n\lambda^2$, or

$$\hat{\lambda} = \sqrt{\frac{\sum (\log x_i)^2}{n}}.$$

This solution is unique.

Notice that

$$\frac{d^2}{d\lambda^2} \log(L(\lambda)) = \frac{n}{\lambda^2} - \left. \frac{3\sum (\log x_i)^2}{\lambda^4} \right|_{\lambda=\hat{\lambda}}$$

$$= \frac{n}{\hat{\lambda}^2} - \frac{3n\hat{\lambda}^2}{\hat{\lambda}^4} = \frac{-2n}{\hat{\lambda}^2} < 0.$$

Hence

$$\hat{\lambda} = \sqrt{\frac{\sum (\log X_i)^2}{n}}$$

is the MLE of λ.

b)

$$\hat{\lambda}^2 = \frac{\sum (\log X_i)^2}{n}$$

is the MLE of λ^2 by invariance.

5.41. (Jan. 2009 Qual): Suppose that X has probability density function

$$f_X(x) = \frac{\theta}{x^{1+\theta}}, \ \ x \ge 1$$

where $\theta > 0$.

a) If $U = X^2$, derive the probability density function $f_U(u)$ of U.

b) Find the method of moments estimator of θ.

c) Find the method of moments estimator of θ^2.

5.42. (Jan. 2009 Qual): Suppose that the joint probability distribution function of $X_1, ..., X_k$ is

$$f(x_1, x_2, ..., x_k|\theta) = \frac{n!}{(n-k)!\theta^k} \exp\left(\frac{-[(\sum_{i=1}^k x_i) + (n-k)x_k]}{\theta} \right)$$

where $0 \le x_1 \le x_2 \le \cdots \le x_k$ and $\theta > 0$.

a) Find the maximum likelihood estimator (MLE) for θ.

b) What is the MLE for θ^2? Explain briefly.

Solution. a) Let $t = [(\sum_{i=1}^k x_i) + (n-k)x_k]$. $L(\theta) = f(\boldsymbol{x}|\theta)$ and $\log(L(\theta)) = \log(f(\boldsymbol{x}|\theta)) =$

$$d - k\log(\theta) - \frac{t}{\theta}.$$

Hence

$$\frac{d}{d\theta}\log(L(\theta)) = \frac{-k}{\theta} + \frac{t}{\theta^2} \stackrel{set}{=} 0.$$

Hence

$$k\theta = t$$

or

$$\hat{\theta} = \frac{t}{k}.$$

This is a unique solution and

$$\frac{d^2}{d\theta^2}\log(L(\theta)) = \frac{k}{\theta^2} - \frac{2t}{\theta^3}\bigg|_{\theta=\hat{\theta}} = \frac{k}{\hat{\theta}^2} - \frac{2k\hat{\theta}}{\hat{\theta}^3} = -\frac{k}{\hat{\theta}^2} < 0.$$

Hence $\hat{\theta} = T/k$ is the MLE where $T = [(\sum_{i=1}^k X_i) + (n-k)X_k]$.

b) $\hat{\theta}^2$ by the invariance principle.

5.43. (Jan. 2010 Qual): Let $X_1, ..., X_n$ be iid with pdf

$$f(x) = \frac{\cos(\theta)}{2\cosh(\pi x/2)}\exp(\theta x)$$

where x is real and $|\theta| < \pi/2$.

a) Find the maximum likelihood estimator (MLE) for θ.

b) What is the MLE for $\tan(\theta)$? Explain briefly.

Solution. a) $L(\theta) = \frac{[\cos(\theta)]^n \exp(\theta \sum x_i)}{\prod 2\cosh(\pi x_i/2)}$. So $\log(L(\theta)) = c + n\log(\cos(\theta)) + \theta \sum x_i$, and

$$\frac{d\log(L(\theta))}{d\theta} = n\frac{1}{\cos(\theta)}[-\sin(\theta)] + \sum x_i \stackrel{set}{=} 0,$$

or $\tan(\theta) = \overline{x}$, or $\hat{\theta} = \tan^{-1}(\overline{X})$.

Since

$$\frac{d^2\log(L(\theta))}{d\theta^2} = -n\sec^2(\theta) < 0$$

for $|\theta| < 1/2$, $\hat{\theta}$ is the MLE.

b) The MLE is $\tan(\hat{\theta}) = \tan(\tan^{-1}(\overline{X})) = \overline{X}$ by the invariance principle.

(By properties of the arctan function, $\hat{\theta} = \tan^{-1}(\overline{X})$ iff $\tan(\hat{\theta}) = \overline{X}$ and $-\pi/2 < \hat{\theta} < \pi/2$.)

5.44. (Aug. 2009 Qual): Let $X_1, ..., X_n$ be a random sample from a population with pdf

$$f(x) = \frac{1}{\sigma} \exp\left(- \quad \frac{x-\mu}{\sigma} \right), \ x \geq \mu,$$

where $-\infty < \mu < \infty$, $\sigma > 0$.

a) Find the maximum likelihood estimator of μ and σ.

b) Evaluate $\tau(\mu, \sigma) = P_{\mu,\sigma}[X_1 \geq t]$ where $t > \mu$. Find the maximum likelihood estimator of $\tau(\mu, \sigma)$.

Solution. a) This is a two parameter exponential distribution. So see Section 10.14 where $\sigma = \lambda$ and $\mu = \theta$.

b)

$$1 - F(x) = \tau(\mu, \sigma) = \exp\left[- \left(\frac{x-\mu}{\sigma} \right) \right].$$

By the invariance principle, the MLE of $\tau(\mu, \sigma) = \tau(\hat{\mu}, \hat{\sigma})$

$$= \exp\left[- \left(\frac{x - X_{(1)}}{\overline{X} - X_{(1)}} \right) \right].$$

5.45. (Sept. 2010 Qual): Let $Y_1, ..., Y_n$ be independent identically distributed (iid) random variables from a distribution with probability density function (pdf)

$$f(y) = \frac{1}{2\sqrt{2\pi}} \ \left(\frac{1}{\theta}\sqrt{\frac{\theta}{y}} + \frac{\theta}{y^2}\sqrt{\frac{y}{\theta}} \right) \ \frac{1}{\nu} \ \exp\left[\frac{-1}{2\nu^2} \left(\frac{y}{\theta} \ + \ \frac{\theta}{y} \ - 2 \right) \right]$$

where $y > 0, \theta > 0$ is **known** and $\nu > 0$.

a) Find the maximum likelihood estimator (MLE) of ν.

b) Find the MLE of ν^2.

Solution. a) Let

$$w = t(y) = \frac{y}{\theta} + \frac{\theta}{y} - 2.$$

Then the likelihood

$$L(\nu) = d \frac{1}{\nu^n} \exp\left(\frac{-1}{2\nu^2} \sum_{i=1}^n w_i\right),$$

and the log likelihood

$$\log(L(\nu)) = c - n \log(\nu) - \frac{1}{2\nu^2} \sum_{i=1}^n w_i.$$

Hence

$$\frac{d}{d\nu} \log(L(\nu)) = \frac{-n}{\nu} + \frac{1}{\nu^3} \sum_{i=1}^n w_i \stackrel{set}{=} 0,$$

or

$$\hat{\nu} = \sqrt{\frac{\sum_{i=1}^n w_i}{n}}.$$

This solution is unique and

$$\frac{d^2}{d\nu^2}\log(L(\nu)) = \frac{n}{\nu^2} - \left.\frac{3\sum_{i=1}^n w_i}{\nu^4}\right|_{\nu=\hat{\nu}} = \frac{n}{\hat{\nu}^2} - \frac{3n\hat{\nu}^2}{\hat{\nu}^4} = \frac{-2n}{\hat{\nu}^2} < 0.$$

Thus

$$\hat{\nu} = \sqrt{\frac{\sum_{i=1}^n W_i}{n}}$$

is the MLE of ν if $\hat{\nu} > 0$.

b) $\hat{\nu}^2 = \dfrac{\sum_{i=1}^n W_i}{n}$ by invariance.

5.46. (Sept. 2011 Qual): Let $Y_1, ..., Y_n$ be independent identically distributed (iid) random variables from a distribution with probability density function (pdf)

$$f(y) = \phi\ y^{-(\phi+1)} \frac{1}{1+y^{-\phi}}\ \frac{1}{\lambda}\ \exp[\frac{-1}{\lambda}\log(1+y^{-\phi})]$$

where $y > 0, \phi > 0$ is **known** and $\lambda > 0$.

a) Find the maximum likelihood estimator (MLE) of λ.

b) Find the MLE of λ^2.

Solution. a) The likelihood

$$L(\lambda) = c\frac{1}{\lambda^n}\exp\left[-\frac{1}{\lambda}\sum_{i=1}^n \log(1+y_i^{-\phi})\right],$$

and the log likelihood $\log(L(\lambda)) = d - n\log(\lambda) - \frac{1}{\lambda}\sum_{i=1}^n \log(1+y_i^{-\phi})$. Hence

$$\frac{d}{d\lambda}\log(L(\lambda)) = \frac{-n}{\lambda} + \frac{\sum_{i=1}^n \log(1+y_i^{-\phi})}{\lambda^2} \stackrel{set}{=} 0,$$

or $\sum_{i=1}^n \log(1+y_i^{-\phi}) = n\lambda$ or

$$\hat{\lambda} = \frac{\sum_{i=1}^n \log(1+y_i^{-\phi})}{n}.$$

This solution is unique and

$$\frac{d^2}{d\lambda^2}\log(L(\lambda)) = \frac{n}{\lambda^2} - \left.\frac{2\sum_{i=1}^n \log(1+y_i^{-\phi})}{\lambda^3}\right|_{\lambda=\hat{\lambda}} = \frac{n}{\hat{\lambda}^2} - \frac{2n\hat{\lambda}}{\hat{\lambda}^3} = \frac{-n}{\hat{\lambda}^2} < 0.$$

Thus

$$\hat{\lambda} = \frac{\sum_{i=1}^n \log(1+Y_i^{-\phi})}{n}$$

is the MLE of λ if ϕ is known.

b) The MLE is $\hat{\lambda}^2$ by invariance.

5.47. (Aug. 2012 Qual): Let $Y_1, ..., Y_n$ be independent identically distributed (iid) random variables from an inverse half normal distribution with probability density function (pdf)

$$f(y) = \frac{2}{\sigma\ \sqrt{2\pi}}\ \frac{1}{y^2}\ \exp\left(\frac{-1}{2\sigma^2 y^2}\right)$$

where $y > 0$ and $\sigma > 0$.

a) Find the maximum likelihood estimator (MLE) of σ^2.

b) Find the MLE of σ.

Solution. a) The likelihood

$$L(\sigma^2) = c\ \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \exp\left[\frac{-1}{2\sigma^2}\sum_{i=1}^n \frac{1}{y_i^2}\right],$$

and the log likelihood

$$\log(L(\sigma^2)) = d - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^n \frac{1}{y_i^2}.$$

Hence

$$\frac{d}{d(\sigma^2)}\log(L(\sigma^2)) = \frac{-n}{2(\sigma^2)} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^n \frac{1}{y_i^2} \stackrel{set}{=} 0,$$

or $\sum_{i=1}^n \frac{1}{y_i^2} = n\sigma^2$ or

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n \frac{1}{y_i^2}.$$

This solution is unique and

$$\frac{d^2}{d(\sigma^2)^2}\log(L(\sigma^2)) =$$

$$\frac{n}{2(\sigma^2)^2} - \left.\frac{\sum_{i=1}^n \frac{1}{y_i^2}}{(\sigma^2)^3}\right|_{\sigma^2=\hat{\sigma}^2} = \frac{n}{2(\hat{\sigma}^2)^2} - \frac{n\hat{\sigma}^2}{(\hat{\sigma}^2)^3}\frac{2}{2} = \frac{-n}{2\hat{\sigma}^4} < 0.$$

Thus

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n \frac{1}{Y_i^2}$$

is the MLE of σ^2.

b) By invariance, $\hat{\sigma} = \sqrt{\hat{\sigma}^2}$.

5.48. (Jan. 2013 Qual): Let $Y_1, ..., Y_n$ be independent identically distributed (iid) random variables from a distribution with probability density function (pdf)

$$f(y) = \frac{\theta}{y^2}\exp\left(\frac{-\theta}{y}\right)$$

where $y > 0$ and $\theta > 0$.

a) Find the maximum likelihood estimator (MLE) of θ.

b) Find the MLE of $1/\theta$.

Solution. a) The likelihood $L(\theta) = c\ \theta^n \exp\left[-\theta \sum_{i=1}^n \frac{1}{y_i}\right]$, and the log likelihood $\log(L(\theta)) = d + n\log(\theta) - \theta \sum_{i=1}^n \frac{1}{y_i}$. Hence

$$\frac{d}{d\theta}\log(L(\theta)) = \frac{n}{\theta} - \sum_{i=1}^n \frac{1}{y_i} \stackrel{set}{=} 0, \quad \text{or} \quad \hat{\theta} = \frac{\mathrm{n}}{\sum_{\mathrm{i=1}}^{\mathrm{n}} \frac{1}{\mathrm{y_i}}}.$$

Since this solution is unique and $\frac{d^2}{d\theta^2}\log(L(\theta)) = \frac{-n}{\theta^2} < 0$, $\hat{\theta} = \frac{n}{\sum_{i=1}^n \frac{1}{Y_i}}$ is the MLE of θ.

b) By invariance, the MLE is $1/\hat{\theta} = \frac{\sum_{i=1}^n \frac{1}{Y_i}}{n}$.

5.49. (Aug. 2013 Qual): Let $Y_1, ..., Y_n$ be independent identically distributed (iid) random variables from a Lindley distribution with probability density function (pdf)

$$f(y) = \frac{\theta^2}{1+\theta}(1+y)e^{-\theta y}$$

where $y > 0$ and $\theta > 0$.

a) Find the maximum likelihood estimator (MLE) of θ. You may assume that

$$\left.\frac{d^2}{d\theta^2}\log(L(\theta))\right|_{\theta=\hat{\theta}} < 0.$$

b) Find the MLE of $1/\theta$.

Solution: a) The likelihood

$$L(\theta) = c\ \left(\frac{\theta^2}{1+\theta}\right)^n \exp(-\theta \sum_{i=1}^n y_i),$$

and the log likelihood

$$\log(L(\theta)) = d + n\log\left(\frac{\theta^2}{1+\theta}\right) - \theta \sum_{i=1}^n y_i.$$

Always use properties of logarithms to simplify the log likelihood before taking derivatives. Note that

$$\log(L(\theta)) = d + 2n\log(\theta) - n\log(1+\theta) - \theta \sum_{i=1}^n y_i.$$

Hence
$$\frac{d}{d\theta}\log(L(\theta)) = \frac{2n}{\theta} \ - \ \frac{n}{(1+\theta)} \ - \ \sum_{i=1}^n y_i \stackrel{set}{=} 0,$$
or
$$\frac{2(1+\theta)-\theta}{\theta(1+\theta)} - \overline{y} = 0 \ \text{ or } \ \frac{2+\theta}{\theta(1+\theta)} - \overline{y} = 0$$
or $2+\theta = \overline{y}(\theta+\theta^2)$ or $\overline{y}\theta^2 + \theta(\overline{y}-1) - 2 = 0$. So
$$\hat{\theta} = \frac{-(\overline{Y}-1) + \sqrt{(\overline{Y}-1)^2 + 8\overline{Y}}}{2\overline{Y}}.$$

b) By invariance, the MLE is $1/\hat{\theta}$.

5.53. (Jan. 2015 QUAL): b) $\hat{\sigma} = \sqrt{\hat{\sigma}^2} = \sqrt{Q/n}$ by invariance.

5.54. (Aug. 2016 QUAL): Suppose $X_1, ..., X_n$ are random variables with likelihood function
$$L(\theta) = \frac{\left[\prod_{i=1}^{n-k} \frac{1}{\theta} e^{-x_i/\theta}\right]\left[\prod_{i=n-k+1}^{n} e^{-x_i/\theta}\right]}{\prod_{i=1}^{n-k} e^{-d_i/\theta}}$$
where $\theta > 0$, $x_i > 0$, and $x_i > d_i > 0$ for $i = 1, ..., n-k$. The d_i and k are known constants. Find the maximum likelihood estimator (MLE) of θ.

Solution:
$$L(\theta) = \frac{1}{\theta^{n-k}} \prod_{i=1}^{n-k} e^{-(x_i-d_i)/\theta} \prod_{i=n-k+1}^{n} e^{-x_i/\theta}.$$
Hence
$$\log(L(\theta)) = -(n-k)\log(\theta) - \frac{1}{\theta}\sum_{i=1}^{n-k}(x_i - d_i) - \frac{1}{\theta}\sum_{i=n-k+1}^{n} x_i =$$
$$-(n-k)\log(\theta) - \frac{1}{\theta}\sum_{i=1}^{n-k} x_i + \frac{1}{\theta}\sum_{i=1}^{n-k} d_i - \frac{1}{\theta}\sum_{i=n-k+1}^{n} x_i =$$
$$-(n-k)\log(\theta) - \frac{f}{\theta}$$
where $f = \sum_{i=1}^{n-k} x_i - \sum_{i=1}^{n-k} d_i + \sum_{i=n-k+1}^{n} x_i = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n-k} d_i = \sum_{i=1}^{n-k}(x_i - d_i) + \sum_{i=n-k+1}^{n} x_i > 0$. So
$$\frac{d\log(L(\theta)}{d\theta} = \frac{-(n-k)}{\theta} + \frac{f}{\theta^2} \stackrel{set}{=} 0$$
or $(n-k)\theta = f$ or
$$\hat{\theta} = \frac{f}{n-k}.$$

This solution was unique and

$$\frac{d^2 \log(L(\theta)}{d\theta^2} = \frac{n-k}{\theta^2} - \frac{2f}{\theta^3}\bigg|_{\hat{\theta}} = \frac{n-k}{\hat{\theta}^2} - \frac{2(n-k)\hat{\theta}}{\hat{\theta}^3} = \frac{-(n-k)}{\hat{\theta}^2} < 0.$$

Hence $\hat{\theta}$ is the MLE.

5.55. (Jan. 2018 QUAL): Suppose $X_1, ..., X_n$ are iid from a Kumaraswamy distribution with probability density function (pdf)

$$f(x) = \theta x^{\theta-1}\beta(1-x^\theta)^{\beta-1}$$

where $\theta > 0$ is **known**, $\beta > 0$, and $0 < x < 1$.

a) Find a complete sufficient statistic for β.

b) Find the maximum likelihood estimator of β.

Solution: a) Note that $f(x) = \theta x^{\theta-1} I(0 < x < 1)\beta \exp[(\beta-1)\log(1-x^\theta)]$ is the pdf of a 1PREF with $h(x) = \theta x^{\theta-1} I(0 < x < 1)$, $c(\beta) = \beta$, $\eta = w(\beta) = \beta - 1$, $t(x) = \log(1-x^\theta)$ and $\Omega = (-1, \infty)$. Hence the complete sufficient statistic is $\sum_{i=1}^n \log(1 - X_i^\theta)$.

b) $L(\beta) = d\beta^n \exp[(\beta-1)\sum_{i=1}^n \log(1-x_i^\theta)]$. Hence $\log(L(\beta)) = c + n\log(\beta) + (\beta - 1)\sum_{i=1}^n \log(1-x_i^\theta)$. So

$$\frac{d\log(L(\beta))}{d\beta} = \frac{n}{\beta} + \sum_{i=1}^n \log(1-x_i^\theta) \stackrel{set}{=} 0,$$

or

$$\hat{\beta} = \frac{-n}{\sum_{i=1}^n \log(1-X_i^\theta)}$$

which is unique. Then $\hat{\beta}$ is the MLE since

$$\frac{d^2\log(L(\beta))}{d\beta^2} = \frac{-n}{\beta^2} < 0.$$

5.56. (Jan. 2018 QUAL): Assume $X_1, ..., X_n$ are i.i.d from Gamma distribution with parameters α and β ($Gamma(\alpha, \beta)$) where both α and β are unknown.

a) Find the method of moments estimators for α and β.

b) Show that the estimators obtained in part (a) for α and β are always non-negative.

Solution: a) We have

$$\mu_1 = E[X] = \alpha\beta = m_1, \quad \text{where} \quad m_1 = \frac{1}{n}\sum_{i=1}^n X_i$$

$$\mu_2 = E[X^2] = \alpha\beta^2 + \alpha^2\beta^2 = \alpha(\alpha+1)\beta^2 = m_2, \quad \text{where} \quad m_2 = \frac{1}{n}\sum_{i=1}^n X_i^2.$$

Then

$$\frac{m_2}{m_1^2} = \frac{\alpha(\alpha+1)\beta^2}{\alpha^2\beta^2} = 1 + \frac{1}{\alpha} \Longrightarrow \quad \frac{1}{\alpha} = \frac{m_2 - m_1^2}{m_1^2}$$

or

$$\widehat{\alpha} = \frac{m_1^2}{m_2 - m_1^2} \quad \& \quad \widehat{\beta} = \frac{m_1}{\widehat{\alpha}} = \frac{m_2 - m_1^2}{m_1}$$

b) Since $X_i > 0$, therefore $m_1 > 0$. Also we have

$$\begin{aligned} m_2 - m_1^2 &= \frac{1}{n}\sum_{i=1}^n X_i^2 - \bar{X}^2 \\ &= \frac{1}{n}\Big[\sum_{i=1}^n X_i^2 - n\bar{X}^2\Big] \\ &= \frac{1}{n}\Big[\sum_{i=1}^n (X_i - \bar{X})^2\Big] \geq 0 \end{aligned}$$

Therefore $\widehat{\alpha} \geq 0$ and $\widehat{\beta} \geq 0$.

5.57. (Jan. 2020 QUAL): Let $X_1, ..., X_n$ be independent and identically distributed with probability density function (pdf)

$$f(x|\theta) = A(\theta)B(x)$$

for $0 < x \leq \theta$, and $f(x|\theta) = 0$, otherwise. Here the parameter $\theta > 0$, the function $A(\theta) > 0$ for $\theta > 0$ and the function $B(x) > 0$ for $0 < x \leq \theta$.

a) Is the family of distributions (with pdf $f(x|\theta)$ for $\theta > 0$) an exponential family? Explain.

b) Find a minimal sufficient statistic for θ, and show that it is so.

c) If $A(\theta) = 1/\theta$ and $B(x) \equiv 1$ for $0 < x \leq \theta$, find the maximum likelihood estimator of θ.

Solution: a) No, the support depends on θ.

b)

$$\frac{f(\boldsymbol{x})}{f(\boldsymbol{y})} = \frac{[A(\theta)]^n[\prod_{i=1}^n B(x_i)]I[\max(x_i) \leq \theta]}{[A(\theta)]^n[\prod_{i=1}^n B(y_i)]I[\max(y_i) \leq \theta]} \equiv c \ \ \forall\theta \ \text{ iff}$$

$$\frac{I[\max(x_i) \leq \theta]}{I[\max(y_i) \leq \theta]} \equiv d \ \ \forall\theta$$

Hence $\max(X_i)$ is the minimal sufficient statistic by LSM.

c) $X \sim U(0, \theta)$ with $L(\theta) = \theta^{-n} I[\max(x_i) \geq \theta]$ which is maximized at $\theta = \max(x_i)$. Hence $\max(X_i)$ is the maximum likelihood estimator.

6.2. (Aug. 2002 QUAL): Let $X_1, ..., X_n$ be independent identically distributed random variable from a $N(\mu, \sigma^2)$ distribution. Hence $E(X_1) = \mu$ and $VAR(X_1) = \sigma^2$. Consider estimators of σ^2 of the form

$$S^2(k) = \frac{1}{k}\sum_{i=1}^n (X_i - \overline{X})^2$$

where $k > 0$ is a constant to be chosen. Determine the value of k which gives the smallest mean square error. (Hint: Find the MSE as a function of k, then take derivatives with respect to k. Also, use Theorem 4.1c and Remark 5.1 VII.)

6.7. (Jan. 2001 Qual): Let $X_1, ..., X_n$ be independent, identically distributed $N(\mu, 1)$ random variables where μ is unknown and $n \geq 2$. Let t be a fixed real number. Then the expectation

$$E_\mu(I_{(-\infty,t]}(X_1)) = P_\mu(X_1 \leq t) = \Phi(t - \mu)$$

for all μ where $\Phi(x)$ is the cumulative distribution function of a $N(0, 1)$ random variable.

a) Show that the sample mean $\overline{X}$ is a sufficient statistic for μ.

b) Explain why (or show that) $\overline{X}$ is a complete sufficient statistic for μ.

c) Using the fact that the conditional distribution of X_1 given $\overline{X} = \overline{x}$ is the $N(\overline{x}, 1 - 1/n)$ distribution where the second parameter $1 - 1/n$ is the variance of conditional distribution, find

$$E_\mu(I_{(-\infty,t]}(X_1)|\overline{X} = \overline{x}) = E_\mu[I_{(-\infty,t]}(W)]$$

where $W \sim N(\overline{x}, 1 - 1/n)$. (Hint: your answer should be $\Phi(g(\overline{x}))$ for some function g.)

d) What is the uniformly minimum variance unbiased estimator for $\Phi(t - \mu)$?

Solution. a) The joint density

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}} \exp[-\frac{1}{2}\sum(x_i - \mu)^2]$$

$$= \frac{1}{(2\pi)^{n/2}} \exp[-\frac{1}{2}(\sum x_i^2 - 2\mu\sum x_i + n\mu^2)]$$

$$= \frac{1}{(2\pi)^{n/2}} \exp[-\frac{1}{2}\sum x_i^2] \exp[n\mu\overline{x} - \frac{n\mu^2}{2}].$$

Hence by the factorization theorem $\overline{X}$ is a sufficient statistic for μ.

b) $\overline{X}$ is sufficient by a) and complete since the $N(\mu, 1)$ family is a regular one parameter exponential family.

c) $E(I_{-(\infty,t]}(X_1)|\overline{X} = \overline{x}) = P(X_1 \leq t|\overline{X} = \overline{x}) = \Phi(\frac{t-\overline{x}}{\sqrt{1-1/n}})$.

d) By the LSU theorem,

$$\Phi(\frac{t - \overline{X}}{\sqrt{1 - 1/n}})$$

is the UMVUE.

6.14. (Jan. 2003 Qual): Let $X_1, ..., X_n$ be independent, identically distributed exponential(θ) random variables where $\theta > 0$ is unknown. Consider the class of estimators of θ

$$\{T_n(c) = c\sum_{i=1}^{n} X_i \ \mid \ c > 0\}.$$

Determine the value of c that minimizes the mean square error MSE. Show work and prove that your value of c is indeed the global minimizer.

Solution. Note that $\sum X_i \sim G(n, \theta)$. Hence MSE(c) $= Var_\theta(T_n(c)) + [E_\theta T_n(c) - \theta]^2$ $= c^2 Var_\theta(\sum X_i) + [ncE_\theta X - \theta]^2 = c^2 n\theta^2 + [nc\theta - \theta]^2$.

So
$$\frac{d}{dc} MSE(c) = 2cn\theta^2 + 2[nc\theta - \theta]n\theta.$$
Set this equation to 0 to get $2n\theta^2[c + nc - 1] = 0$ or $c(n+1) = 1$. So $c = 1/(n+1)$.

The second derivative is $2n\theta^2 + 2n^2\theta^2 > 0$ so the function is convex and the local min is in fact global.

6.19. (Aug. 2000 SIU, 1995 Univ. Minn. Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables from a $N(\mu, \sigma^2)$ distribution. Hence $E(X_1) = \mu$ and $VAR(X_1) = \sigma^2$. Suppose that μ is known and consider estimates of σ^2 of the form
$$S^2(k) = \frac{1}{k}\sum_{i=1}^{n}(X_i - \mu)^2$$
where k is a constant to be chosen. Note: $E(\chi^2_m) = m$ and $VAR(\chi^2_m) = 2m$. Determine the value of k which gives the smallest mean square error. (Hint: Find the MSE as a function of k, then take derivatives with respect to k.)

Solution.
$$W \equiv S^2(k)/\sigma^2 \sim \chi^2_n/k$$
and
$$MSE(S^2(k)) = MSE(W) = VAR(W) + (E(W) - \sigma^2)^2$$
$$= \frac{\sigma^4}{k^2}2n + (\frac{\sigma^2 n}{k} - \sigma^2)^2$$
$$= \sigma^4[\frac{2n}{k^2} + (\frac{n}{k} - 1)^2] = \sigma^4\frac{2n + (n-k)^2}{k^2}.$$
Now the derivative $\frac{d}{dk}MSE(S^2(k))/\sigma^4 =$
$$\frac{-2}{k^3}[2n + (n-k)^2] + \frac{-2(n-k)}{k^2}.$$
Set this derivative equal to zero. Then
$$2k^2 - 2nk = 4n + 2(n-k)^2 = 4n + 2n^2 - 4nk + 2k^2.$$
Hence
$$2nk = 4n + 2n^2$$
or $k = n + 2$.

Should also argue that $k = n + 2$ is the global minimizer. Certainly need $k > 0$ and the absolute bias will tend to ∞ as $k \to 0$ and the bias tends to σ^2 as $k \to \infty$, so $k = n + 2$ is the unique critical point and is the global minimizer.

6.20. (Aug. 2001 Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables with pdf

$$f(x|\theta) = \frac{2x}{\theta} e^{-x^2/\theta}, \quad x > 0$$

and $f(x|\theta) = 0$ for $x \leq 0$.

a) Show that X_1^2 is an unbiased estimator of θ. (Hint: use the substitution $W = X^2$ and find the pdf of W or use u-substitution with $u = x^2/\theta$.)

b) Find the Cramer-Rao lower bound for the variance of an unbiased estimator of θ.

c) Find the uniformly minimum variance unbiased estimator (UMVUE) of θ.

Solution. a) Let $W = X^2$. Then $f(w) = f_X(\sqrt{w}) \;\; 1/(2\sqrt{w}) = (1/\theta) \exp(-w/\theta)$ and $W \sim EXP(\theta)$. Hence $E_\theta(X^2) = E_\theta(W) = \theta$.

b) This is an exponential family and

$$\log(f(x|\theta)) = \log(2x) - \log(\theta) - \frac{1}{\theta} x^2$$

for $x > 0$. Hence

$$\frac{\partial}{\partial \theta} f(x|\theta) = \frac{-1}{\theta} + \frac{1}{\theta^2} x^2$$

and

$$\frac{\partial^2}{\partial \theta^2} f(x|\theta) = \frac{1}{\theta^2} + \frac{-2}{\theta^3} x^2.$$

Hence

$$I_1(\theta) = -E_\theta[\frac{1}{\theta^2} + \frac{-2}{\theta^3} x^2] = \frac{1}{\theta^2}$$

by a). Now

$$CRLB = \frac{[\tau'(\theta)]^2}{n I_1(\theta)} = \frac{\theta^2}{n}$$

where $\tau(\theta) = \theta$.

c) This is a regular exponential family so $\sum_{i=1}^n X_i^2$ is a complete sufficient statistic. Since

$$E_\theta[\frac{\sum_{i=1}^n X_i^2}{n}] = \theta,$$

the UMVUE is $\frac{\sum_{i=1}^n X_i^2}{n}$.

6.21. (Aug. 2001 Qual): See Mukhopadhyay (2000, p. 377). Let $X_1, ..., X_n$ be iid $N(\theta, \theta^2)$ normal random variables with mean θ and variance θ^2. Let

$$T_1 = \overline{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

and let

$$T_2 = c_n S = c_n \sqrt{\frac{\sum_{i=1}^n (X_i - \overline{X})^2}{n-1}}$$

where the constant c_n is such that $E_\theta[c_n S] = \theta$. You do not need to find the constant c_n. Consider estimators $W(\alpha)$ of θ of the form.

$$W(\alpha) = \alpha T_1 + (1-\alpha) T_2$$

where $0 \le \alpha \le 1$.

a) Find the variance

$$Var_\theta[W(\alpha)] = Var_\theta(\alpha T_1 + (1-\alpha) T_2).$$

b) Find the mean square error of $W(\alpha)$ in terms of $Var_\theta(T_1), Var_\theta(T_2)$ and α.

c) Assume that

$$Var_\theta(T_2) \approx \frac{\theta^2}{2n}.$$

Determine the value of α that gives the smallest mean square error. (Hint: Find the MSE as a function of α, then take the derivative with respect to α. Set the derivative equal to zero and use the above approximation for $Var_\theta(T_2)$. Show that your value of α is indeed the global minimizer.)

Solution. a) In normal samples, $\overline{X}$ and S are independent, hence

$$Var_\theta[W(\alpha)] = \alpha^2 Var_\theta(T_1) + (1-\alpha)^2 Var_\theta(T_2).$$

b) $W(\alpha)$ is an unbiased estimator of θ. Hence MSE$[W(\alpha)] \equiv MSE(\alpha) = Var_\theta[W(\alpha)]$ which is found in part a).

c) Now

$$\frac{d}{d\alpha} MSE(\alpha) = 2\alpha Var_\theta(T_1) - 2(1-\alpha) Var_\theta(T_2) \stackrel{set}{=} 0.$$

Hence

$$\hat{\alpha} = \frac{Var_\theta(T_2)}{Var_\theta(T_1) + Var_\theta(T_2)} \approx \frac{\frac{\theta^2}{2n}}{\frac{2\theta^2}{2n} + \frac{\theta^2}{2n}} = 1/3$$

using the approximation and the fact that $\text{Var}(\bar{X}) = \theta^2/n$. Note that the second derivative

$$\frac{d^2}{d\alpha^2} MSE(\alpha) = 2[Var_\theta(T_1) + Var_\theta(T_2)] > 0,$$

so $\alpha = 1/3$ is a local min. The critical value was unique, hence $1/3$ is the global min.

6.22. (Aug. 2003 Qual): Suppose that $X_1, ..., X_n$ are iid normal distribution with mean 0 and variance σ^2. Consider the following estimators: $T_1 = \frac{1}{2}|X_1 - X_2|$ and $T_2 = \sqrt{\frac{1}{n}\sum_{i=1}^n X_i^2}$.

a) Is T_1 unbiased for σ? Evaluate the mean square error (MSE) of T_1.

b) Is T_2 unbiased for σ? If not, find a suitable multiple of T_2 which is unbiased for σ.

Solution. a) $X_1 - X_2 \sim N(0, 2\sigma^2)$. Thus,

$$\begin{aligned} E(T_1) &= \int_0^\infty u \frac{1}{\sqrt{4\pi\sigma^2}} e^{\frac{-u^2}{4\sigma^2}} du \\ &= \frac{\sigma}{\sqrt{\pi}}. \end{aligned}$$

$$\begin{aligned} E(T_1^2) &= \frac{1}{2}\int_0^\infty u^2 \frac{1}{\sqrt{4\pi\sigma^2}} e^{\frac{-u^2}{4\sigma^2}} du \\ &= \frac{\sigma^2}{2}. \end{aligned}$$

$V(T_1) = \sigma^2(\frac{1}{2} - \frac{1}{\pi})$ and

$$MSE(T_1) = \sigma^2[(\frac{1}{\sqrt{\pi}}) - 1)^2 + \frac{1}{2} - \frac{1}{\pi}] = \sigma^2[\frac{3}{2} - \frac{2}{\sqrt{\pi}}].$$

b) $\frac{X_i}{\sigma}$ has a N(0,1) and $\frac{\sum_{i=1}^n X_i^2}{\sigma^2}$ has a chi square distribution with n degrees of freedom. Thus

$$E(\sqrt{\frac{\sum_{i=1}^n X_i^2}{\sigma^2}}) = \frac{\sqrt{2}\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})},$$

and

$$E(T_2) = \frac{\sigma}{\sqrt{n}} \frac{\sqrt{2}\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}.$$

Therefore,

$$E(\frac{\sqrt{n}}{\sqrt{2}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} T_2) = \sigma.$$

6.23. (Aug. 2003 Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables with pdf (probability density function)

$$f(x) = \frac{1}{\lambda} \exp\left(-\frac{x}{\lambda}\right)$$

where x and λ are both positive. Find the uniformly minimum variance unbiased estimator (UMVUE) of λ^2.

Solution. This is a regular one parameter exponential family with complete sufficient statistic $T_n = \sum_{i=1}^n X_i \sim G(n, \lambda)$. Hence $E(T_n) = n\lambda$, $E(T_n^2) = V(T_n) + (E(T_n))^2 = n\lambda^2 + n^2\lambda^2$, and $T_n^2/(n+n^2)$ is the UMVUE of λ^2.

6.24. (Jan. 2004 Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables with pdf (probability density function)

$$f(x) = \sqrt{\frac{\sigma}{2\pi x^3}} \exp\left(-\frac{\sigma}{2x}\right)$$

where x and σ are both positive. Then $X_i = \dfrac{\sigma}{W_i}$ where $W_i \sim \chi_1^2$. Find the uniformly minimum variance unbiased estimator (UMVUE) of $\dfrac{1}{\sigma}$.

Solution.

$$\frac{1}{X_i} = \frac{W_i}{\sigma} \sim \frac{\chi_1^2}{\sigma}.$$

Hence if

$$T = \sum_{i=1}^n \frac{1}{X_i}, \text{ then } \mathrm{E}(\frac{\mathrm{T}}{\mathrm{n}}) = \frac{\mathrm{n}}{\mathrm{n}\sigma},$$

and T/n is the UMVUE since $f(x)$ is an exponential family with complete sufficient statistic $1/X$.

6.25. (Jan. 2004 Qual): Let $X_1, ..., X_n$ be a random sample from the distribution with density

$$f(x) = \begin{cases} \frac{2x}{\theta^2}, & 0 < x < \theta \\ 0 & \text{elsewhere} \end{cases}$$

Let $T = max(X_1, ..., X_n)$. To estimate θ consider estimators of the form CT. Determine the value of C which gives the smallest mean square error.

Solution. The pdf of T is

$$g(t) = \frac{2nt^{2n-1}}{\theta^{2n}}$$

for $0 < t < \theta$.
$E(T) = \frac{2n}{2n+1}\theta$ and $E(T^2) = \frac{2n}{2n+2}\theta^2$.

$$MSE(CT) = (C\frac{2n}{2n+1}\theta - \theta)^2 + C^2[\frac{2n}{2n+2}\theta^2 - (\frac{2n}{2n+1}\theta)^2]$$

$$\frac{dMSE(CT)}{dC} = 2[\frac{2Cn\theta}{2n+1} - \theta][\frac{2n\theta}{2n+1}] + 2C[\frac{2n\theta^2}{2n+2} - \frac{4n^2\theta^2}{(2n+1)^2}].$$

Solve $\frac{dMSE(CT)}{dC} \overset{set}{=} 0$ to get

$$C = 2\frac{n+1}{2n+1}.$$

The MSE is a quadratic in C and the coefficient on C^2 is positive, hence the local min is a global min.

6.26. (Aug. 2004 Qual): Let $X_1, ..., X_n$ be a random sample from a distribution with pdf

$$f(x) = \frac{2x}{\theta^2}, \ 0 < x < \theta.$$

Let $T = c\overline{X}$ be an estimator of θ where c is a constant.

a) Find the mean square error (MSE) of T as a function of c (and of θ and n).

b) Find the value c that minimizes the MSE. Prove that your value is the minimizer.

Solution. a) $E(X_i) = 2\theta/3$ and $V(X_i) = \theta^2/18$. So bias of $T = B(T) = Ec\overline{X} - \theta = c\frac{2}{3}\theta - \theta$ and Var$(T) =$

$$Var(\frac{c\sum X_i}{n}) = \frac{c^2}{n^2}\sum Var(X_i) = \frac{c^2}{n^2}\frac{n\theta^2}{18}.$$

So MSE = Var(T) $+[B(T)]^2 =$

$$\frac{c^2\theta^2}{18n} + (\frac{2\theta}{3}c - \theta)^2.$$

b)

$$\frac{dMSE(c)}{dc} = \frac{2c\theta^2}{18n} + 2(\frac{2\theta}{3}c - \theta)\frac{2\theta}{3}.$$

Set this equation equal to 0 and solve, so

$$\frac{\theta^2 2c}{18n} + \frac{4}{3}\theta(\frac{2}{3}\theta c - \theta) = 0$$

or

$$c[\frac{2\theta^2}{18n} + \frac{8}{9}\theta^2] = \frac{4}{3}\theta^2$$

or

$$c(\frac{1}{9n} + \frac{8}{9})\theta^2 = \frac{4}{3}\theta^2$$

or

$$c(\frac{1}{9n} + \frac{8n}{9n}) = \frac{4}{3}$$

or

$$c = \frac{9n}{1+8n}\frac{4}{3} = \frac{12n}{1+8n}.$$

This is a global min since the MSE is a quadratic in c^2 with a positive coefficient, or because

$$\frac{d^2MSE(c)}{dc^2} = \frac{2\theta^2}{18n} + \frac{8\theta^2}{9} > 0.$$

6.27. (Aug. 2004 Qual): Suppose that $X_1, ..., X_n$ are iid Bernoulli(p) where $n \geq 2$ and $0 < p < 1$ is the unknown parameter.

a) Derive the UMVUE of $\nu(p)$, where $\nu(p) = e^2(p(1-p))$.

b) Find the Cramér Rao lower bound for estimating $\nu(p) = e^2(p(1-p))$.

Solution. a) Consider the statistic $W = X_1(1-X_2)$ which is an unbiased estimator of $\psi(p) = p(1-p)$. The statistic $T = \sum_{i=1}^n X_i$ is both complete and sufficient. The possible values of W are 0 or 1. Let $U = \phi(T)$ where

$$\begin{aligned}
\phi(t) &= E[X_1(1-X_2)|T=t] \\
&= 0P[X_1(1-X_2)=0|T=t] + 1P[X_1(1-X_2)=1|T=t] \\
&= P[X_1(1-X_2)=1|T=t] \\
&= \frac{P[X_1=1, X_2=0 \text{ and } \sum_{\rm i=1}^{\rm n} \rm X_i = t]}{P[\sum_{i=1}^n X_i = t]} \\
&= \frac{P[X_1=1]P[X_2=0]P[\sum_{i=3}^n X_i = t-1]}{P[\sum_{i=1}^n X_i = t]}.
\end{aligned}$$

Now $\sum_{i=3}^n X_i$ is $Bin(n-2,p)$ and $\sum_{i=1}^n X_i$ is $Bin(n,p)$. Thus

$$\phi(t) = \frac{p(1-p)[\binom{n-2}{t-1}p^{t-1}(1-p)^{n-t-1}]}{\binom{n}{t}p^t(1-p)^{n-t}}$$

$$= \frac{\binom{n-2}{t-1}}{\binom{n}{t}} = \frac{(n-2)!}{(t-1)!(n-2-t+1)!} \; \frac{t(t-1)!(n-t)(n-t-1)!}{n(n-1)(n-2)!} = \frac{t(n-t)}{n(n-1)}$$

$$= \frac{\frac{t}{n}(n - n\frac{t}{n})}{n-1} = \frac{\frac{t}{n}n(1-\frac{t}{n})}{n-1} = \frac{n}{n-1}\overline{x}(1-\overline{x}).$$

Thus $\frac{n}{n-1}\overline{X}(1-\overline{X})$ is the UMVUE of $p(1-p)$ and $e^2U = e^2\frac{n}{n-1}\overline{X}(1-\overline{X})$ is the UMVUE of $\tau(p) = e^2p(1-p)$.

Alternatively, $\overline{X}$ is a complete sufficient statistic, so try an estimator of the form $U = a(\overline{X})^2 + b\overline{X} + c$. Then U is the UMVUE if $E_p(U) = e^2p(1-p) = e^2(p-p^2)$. Now $E(\overline{X}) = E(X_1) = p$ and $V(\overline{X}) = V(X_1)/n = p(1-p)/n$ since $\sum X_i \sim Bin(n,p)$. So $E[(\overline{X})^2] = V(\overline{X}) + [E(\overline{X})]^2 = p(1-p)/n + p^2$. So $E_p(U) = a[p(1-p)/n] + ap^2 + bp + c$

$$= \frac{ap}{n} - \frac{ap^2}{n} + ap^2 + bp + c = (\frac{a}{n} + b)p + (a - \frac{a}{n})p^2 + c.$$

So $c = 0$ and $a - \frac{a}{n} = a\frac{n-1}{n} = -e^2$ or

$$a = \frac{-n}{n-1}e^2.$$

Hence $\frac{a}{n} + b = e^2$ or

$$b = e^2 - \frac{a}{n} = e^2 + \frac{n}{n(n-1)}e^2 = \frac{n}{n-1}e^2.$$

So

$$U = \frac{-n}{n-1} e^2 (\overline{X})^2 + \frac{n}{n-1} e^2 \overline{X} = \frac{n}{n-1} e^2 \overline{X}(1 - \overline{X}).$$

b) The FCRLB for $\tau(p)$ is $[\tau'(p)]^2/nI_1(p)$. Now $f(x) = p^x(1-p)^{1-x}$, so $\log f(x) = x\log(p) + (1-x)\log(1-p)$. Hence

$$\frac{\partial \log f}{\partial p} = \frac{x}{p} - \frac{1-x}{1-p}$$

and

$$\frac{\partial^2 \log f}{\partial p^2} = \frac{-x}{p^2} - \frac{1-x}{(1-p)^2}.$$

So

$$I_1(p) = -E(\frac{\partial^2 \log f}{\partial p^2}) = -(\frac{-p}{p^2} - \frac{1-p}{(1-p)^2}) = \frac{1}{p(1-p)}.$$

So

$$FCRLB_n = \frac{[e^2(1-2p)]^2}{\frac{n}{p(1-p)}} = \frac{e^4(1-2p)^2 p(1-p)}{n}.$$

6.30. (Jan. 2009 Qual): Suppose that $Y_1, ..., Y_n$ are independent binomial(m_i, ρ) where the $m_i \geq 1$ are known constants. Let

$$T_1 = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n m_i} \quad \text{and} \quad \mathrm{T}_2 = \frac{1}{\mathrm{n}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{\mathrm{Y}_\mathrm{i}}{\mathrm{m}_\mathrm{i}}$$

be estimators of ρ.

a) Find MSE(T_1).

b) Find MSE(T_2).

c) Which estimator is better?

Hint: by the arithmetic–geometric–harmonic mean inequality,

$$\frac{1}{n} \sum_{i=1}^n m_i \geq \frac{n}{\sum_{i=1}^n \frac{1}{m_i}}.$$

Solution. a)

$$E(T_1) = \frac{\sum_{i=1}^n E(Y_i)}{\sum_{i=1}^n m_i} = \frac{\sum_{i=1}^n m_i \rho}{\sum_{i=1}^n m_i} = \rho,$$

so MSE(T_1) $= V(T_1) =$

$$\frac{1}{(\sum_{i=1}^n m_i)^2} V(\sum_{i=1}^n Y_i) = \frac{1}{(\sum_{i=1}^n m_i)^2} \sum_{i=1}^n V(Y_i) = \frac{1}{(\sum_{i=1}^n m_i)^2} \sum_{i=1}^n m_i \rho(1-\rho)$$

$$= \frac{\rho(1-\rho)}{\sum_{i=1}^n m_i}.$$

b)

$$E(T_2) = \frac{1}{n}\sum_{i=1}^{n} \frac{E(Y_i)}{m_i} = \frac{1}{n}\sum_{i=1}^{n} \frac{m_i \rho}{m_i} = \frac{1}{n}\sum_{i=1}^{n} \rho = \rho,$$

so $\text{MSE}(T_2) = V(T_2) =$

$$\frac{1}{n^2} V(\sum_{i=1}^{n} \frac{Y_i}{m_i}) = \frac{1}{n^2}\sum_{i=1}^{n} V(\frac{Y_i}{m_i}) = \frac{1}{n^2}\sum_{i=1}^{n} \frac{V(Y_i)}{(m_i)^2} = \frac{1}{n^2}\sum_{i=1}^{n} \frac{m_i\rho(1-\rho)}{(m_i)^2}$$

$$= \frac{\rho(1-\rho)}{n^2}\sum_{i=1}^{n} \frac{1}{m_i}.$$

c) The hint

$$\frac{1}{n}\sum_{i=1}^{n} m_i \geq \frac{n}{\sum_{i=1}^{n} \frac{1}{m_i}}$$

implies that

$$\frac{n}{\sum_{i=1}^{n} m_i} \leq \frac{\sum_{i=1}^{n} \frac{1}{m_i}}{n} \quad \text{and} \quad \frac{1}{\sum_{\rm i=1}^{\rm n} \rm m_i} \leq \frac{\sum_{\rm i=1}^{\rm n} \frac{1}{\rm m_i}}{\rm n^2}.$$

Hence $\text{MSE}(T_1) \leq \text{MSE}(T_2)$, and T_1 is better.

6.31. (Sept. 2010 Qual): Let $Y_1, ..., Y_n$ be iid gamma($\alpha = 10, \beta$) random variables. Let $T = c\overline{Y}$ be an estimator of β where c is a constant.

a) Find the mean square error (MSE) of T as a function of c (and of β and n).

b) Find the value c that minimizes the MSE. Prove that your value is the minimizer.

Solution. a) $E(T) = cE(Y) = c\alpha\beta = 10c\beta$.
$V(T) = c^2 V(\overline{Y}) = c^2\alpha\beta^2/n = 10c^2\beta^2/n$.
$MSE(T) = V(T) + [B(T)]^2 = 10c^2\beta^2/n + (10c\beta - \beta)^2$.

$$b) \quad \frac{d \;\; MSE(c)}{dc} = \frac{2c10\beta^2}{n} + 2(10c\beta - \beta)10\beta \overset{\text{set}}{=} 0$$

or $[20\beta^2/n] \;\; c + 200\beta^2 \;\; c - 20\beta^2 = 0$
or $c/n + 10c - 1 = 0$ or $c(1/n \; + \; 10) = 1$
or

$$c = \frac{1}{\frac{1}{n} + 10} = \frac{n}{10n + 1}.$$

This value of c is unique, and

$$\frac{d^2 \;\; MSE(c)}{dc^2} = \frac{20\beta^2}{n} + 200\beta^2 > 0,$$

so c is the minimizer.

6.32. (Jan. 2011 Qual): Let $Y_1, ..., Y_n$ be independent identically distributed random variables with pdf (probability density function)

$$f(y) = (2-2y)I_{(0,1)}(y) \;\nu\; \exp[(1-\nu)(-\log(2y - y^2))]$$

where $\nu > 0$ and $n > 1$. The indicator $I_{(0,1)}(y) = 1$ if $0 < y < 1$ and $I_{(0,1)}(y) = 0$, otherwise.

a) Find a complete sufficient statistic.

b) Find the Fisher information $I_1(\nu)$ if $n = 1$.

c) Find the Cramer Rao lower bound (CRLB) for estimating $1/\nu$.

d) Find the uniformly minimum unbiased estimator (UMVUE) of ν.

Hint: You may use the fact that $T_n = -\sum_{i=1}^n \log(2Y_i - Y_i^2) \sim G(n, 1/\nu)$, and

$$E(T_n^r) = \frac{1}{\nu^r} \frac{\Gamma(r+n)}{\Gamma(n)}$$

for $r > -n$. Also $\Gamma(1+x) = x\Gamma(x)$ for $x > 0$.

Solution. a) Since this distribution is a one parameter regular exponential family, $T_n = -\sum_{i=1}^n \log(2Y_i - Y_i^2)$ is complete.

b) Note that $\log(f(y|\nu)) = \log(\nu) + \log(2-2y) + (1-\nu)[-\log(2y-y^2)]$. Hence

$$\frac{d \ \log(f(y|\nu))}{d\nu} = \frac{1}{\nu} + \log(2y - y^2)$$

and

$$\frac{d^2 \ \log(f(y|\nu))}{d\nu^2} = \frac{-1}{\nu^2}.$$

Since this family is a 1P-REF, $I_1(\nu) = -E\left(\frac{-1}{\nu^2}\right) = \frac{1}{\nu^2}$.

c) $\dfrac{[\tau'(\nu)]^2}{nI_1(\nu)} = \dfrac{\nu^2}{\nu^4 \ n} = \dfrac{1}{n\nu^2}$.

d) $E[T_n^{-1}] = \dfrac{1}{\nu^{-1}} \dfrac{\Gamma(-1+n)}{\Gamma(n)} = \dfrac{\nu}{n-1}$. So $(n-1)/T_n$ is the UMVUE of ν by LSU.

6.33. (Sept. 2011 Qual): Let $Y_1, ..., Y_n$ be iid random variables from a distribution with pdf

$$f(y) = \frac{\theta}{2(1+|y|)^{\theta+1}}$$

where $\theta > 0$ and y is real. Then $W = \log(1+|Y|)$ has pdf $f(w) = \theta e^{-w\theta}$ for $w > 0$.

a) Find a complete sufficient statistic.

b) Find the (Fisher) information number $I_1(\theta)$.

c) Find the uniformly minimum variance unbiased estimator (UMVUE) for θ.

Solution. a) Since $f(y) = \dfrac{\theta}{2}[\exp[-(\theta+1)\log(1+|y|)]$ is a 1P-REF, $T = \sum_{i=1}^n \log(1+|Y_i|)$ is a complete sufficient statistic.

b) Since this is an exponential family, $\log(f(y|\theta)) = \log(\theta/2) - (\theta+1)\log(1+|y|)$ and

$$\frac{\partial}{\partial\theta} \log(f(y|\theta)) = \frac{1}{\theta} - \log(1+|y|).$$

Hence

$$\frac{\partial^2}{\partial\theta^2} \log(f(y|\theta)) = \frac{-1}{\theta^2}$$

and
$$I_1(\theta) = -E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log(f(Y|\theta))\right] = \frac{1}{\theta^2}.$$

c) The complete sufficient statistic $T \sim G(n, 1/\theta)$. Hence the UMVUE of θ is $(n-1)/T$ since for $r > -n$,
$$E(T^r) = E(T^r) = \left(\frac{1}{\theta}\right)^r \frac{\Gamma(r+n)}{\Gamma(n)}.$$
So
$$E(T^{-1}) = \theta\frac{\Gamma(n-1)}{\Gamma(n)} = \theta/(n-1).$$

6.34. (Similar to Sept. 2010 Qual): Suppose that $X_1, X_2, ..., X_n$ are independent identically distributed random variables from normal distribution with unknown mean μ and known variance σ^2. Consider the parametric function $g(\mu) = e^{2\mu}$.

a) Derive the uniformly minimum variance unbiased estimator (UMVUE) of $g(\mu)$.

b) Find the Cramer-Rao lower bound (CRLB) for the variance of an unbiased estimator of $g(\mu)$.

c) Is the CRLB attained by the variance of the UMVUE of $g(\mu)$?

Solution. a) Note that $\overline{X}$ is a complete and sufficient statistic for μ and $\overline{X} \sim N(\mu, n^{-1}\sigma^2)$. We know that $E(e^{2\overline{X}})$, the mgf of $\overline{X}$ when $t = 2$, is given by $e^{2\mu+2n^{-1}\sigma^2}$. Thus the UMVUE of $e^{2\mu}$ is $e^{-2n^{-1}\sigma^2}e^{2\overline{X}}$.

b) The CRLB for the variance of unbiased estimator of $g(\mu)$ is given by $4n^{-1}\sigma^2 e^{4\mu}$ whereas

$$\begin{aligned} V(e^{-2n^{-1}\sigma^2}e^{2\bar{X}}) &= e^{-4n^{-1}\sigma^2}E(e^{4\bar{X}}) - e^{4\mu} \\ &= e^{-4n^{-1}\sigma^2}e^{4\mu+\frac{1}{2}16n^{-1}\sigma^2} - e^{4\mu} \\ &= e^{4\mu}[e^{4n^{-1}\sigma^2} - 1] \\ &> 4n^{-1}\sigma^2 e^{4\mu} \end{aligned} \quad (3)$$

since $e^x > 1 + x$ for all $x > 0$. Hence the CRLB is not attained.

6.36. (Aug. 2012 Qual): Let $Y_1, ..., Y_n$ be iid from a one parameter exponential family with pdf or pmf $f(y|\theta)$ with complete sufficient statistic $T(\boldsymbol{Y}) = \sum_{i=1}^n t(Y_i)$ where $t(Y_i) \sim \theta X$ and X has a known distribution with known mean $E(X)$ and known variance $V(X)$. Let $W_n = cT(\boldsymbol{Y})$ be an estimator of θ where c is a constant.

a) Find the mean square error (MSE) of W_n as a function of c (and of n, $E(X)$ and $V(X)$).

b) Find the value of c that minimizes the MSE. Prove that your value is the minimizer.

c) Find the uniformly minimum variance unbiased estimator (UMVUE) of θ.

Solution. See Theorem 6.5.

a) $E(W_n) = c\sum_{i=1}^n E(t(Y_i)) = cn\theta E(X)$, and
$V(W_n) = c^2\sum_{i=1}^n V(t(Y_i)) = c^2 n\theta^2 V(X)$. Hence $MSE(c) \equiv MSE(W_n) =$
$V(W_n) + [E(W_n) - \theta]^2 = c^2 n\theta^2 V(X) + (cn\theta E(X) - \theta)^2$.

b) Thus

$$\frac{d\ MSE(c)}{dc} = 2cn\theta^2 V(X) + 2(cn\theta E(X) - \theta)n\theta E(X) \stackrel{set}{=} 0,$$

or

$$c(n\theta^2 V(X) + n^2\theta^2 [E(X)]^2) = n\theta^2 E(X),$$

or

$$c_M = \frac{E(X)}{V(X) + n[E(X)]^2},$$

which is unique. Now

$$\frac{d^2\ MSE(c)}{dc^2} = 2[n\theta^2 V(X) + n^2\theta^2 [E(X)]^2] > 0.$$

So $MSE(c)$ is convex and $c = c_M$ is the minimizer.

c) Let $c_U = \frac{1}{nE(X)}$. Then $E[c_U T(\boldsymbol{Y})] = \theta$, hence $c_U T(\boldsymbol{Y})$ is the UMVUE of θ by the Lehmann Scheffe theorem.

6.37. (Jan. 2013 qual):Let $X_1, ..., X_n$ be a random sample from a Poisson (λ) distribution. Let $\overline{X}$ and S^2 denote the sample mean and the sample variance, respectively.

a) Show that $\overline{X}$ is uniformly minimum variance unbiased (UMVU) estimator of λ.

b) Show that $E(S^2|\overline{X}) = \overline{X}$.

c) Show that $\text{Var}(S^2) > \text{Var}(\overline{X})$.

Solution: a) Since $f(x) = \frac{1}{x!}\exp[\log(\lambda)x] I(x \in \{0, 1, ...\})$ is a 1P-REF, $\sum_{i=1}^n X_i$ is a complete sufficient statistic and $E(\overline{X}) = \lambda$. Hence $\overline{X} = (\sum_{i=1}^n X_i)/n$ is the UMVUE of λ by the LSU theorem.

b) $E(S^2) = \lambda$ is an unbiased estimator of λ. Hence $E(S^2|\overline{X})$ is the unique UMVUE of λ by the LSU theorem. Thus $E(S^2|\overline{X}) = \overline{X}$ by part a).

c) By Steiner's formula, $V(S^2) = V(E(S^2|\overline{X})) + E(V(S^2|\overline{X})) = V(\overline{X}) + E(V(S^2|\overline{X})) > V(\overline{X})$. (To show $V(S^2) \geq V(\overline{X})$, note that $\overline{X}$ is the UMVUE and S^2 is an unbiased estimator of λ. Hence $V(\overline{X}) \leq V(S^2)$ by the definition of a UMVUE, and the inequality is strict for at least one value of λ since the UMVUE is unique.)

6.38. (Aug. 2012 Qual): Let $X_1, ..., X_n$ be a random sample from a Poisson distribution with mean θ.

a) Show that $T = \sum_{i=1}^n X_i$ is complete sufficient statistic for θ.

b) For $a > 0$, find the uniformly minimum variance unbiased estimator (UMVUE) of $g(\theta) = e^{a\theta}$.

c) Prove the identity:

$$E\left[2^{X_1}|T\right] = \left(1 + \frac{1}{n}\right)^T.$$

Solution: a) See solution to Problem 6.37 a).

b) The complete sufficient statistic $T = \sum_{i=1}^n X_i \sim$ Poisson $(n\theta)$. Hence the mgf of T is

$$E(e^{tT}) = m_T(t) = \exp[n\theta(e^t - 1)].$$

Thus $n(e^t - 1) = a$, or $e^t = a/n + 1$, or $e^t = (a+n)/n$, or $t = \log[(a+n)/n]$. Thus

$$e^{tT} = (e^t)^T = \left(\frac{a+n}{n}\right)^T = \exp[(T \log(\frac{a+n}{n})]$$

is the UMVUE of $e^{a\theta}$ by the LSU theorem.

c) Let $X = X_1$, and note that 2^X is an unbiased estimator of e^θ since

$$2^X = e^{\log(2^X)} = e^{(\log 2)X},$$

$$\text{and} \quad \mathrm{E}(2^{\mathrm{X}}) = \mathrm{m_X}(\log 2) = \exp[\theta(\mathrm{e}^{\log 2} - 1)] = \mathrm{e}^\theta.$$

Thus $E[2^X|T]$ is the UMVUE of $E(2^X) = e^\theta$ by the LSU theorem. By part b) with $a = 1$,

$$E[2^X|T] = \left(\frac{1+n}{n}\right)^T.$$

6.39. (Aug. 2013 Qual): Let $X_1, ..., X_n$ be independent identically distributed from a $\mathrm{N}(\mu, \sigma^2)$ population, where σ^2 is known. Let $\overline{X}$ be the sample mean.

a) Find $E(\overline{X} - \mu)^2$.

b) Using a), find the UMVUE of μ^2.

c) Find $E(\overline{X} - \mu)^3$. [Hint: Show that if Y is a $N(0, \sigma^2)$ random variable, then $E(Y^3) = 0$].

d) Using c), find the UMVUE of μ^3.

Solution. a) $E(\overline{X} - \mu)^2 = \mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n}$.

b) From a), $E(\overline{X}^2 - 2\mu\overline{X} + \mu^2) = E(\overline{X}^2) - \mu^2 = \frac{\sigma^2}{n}$, or $E(\overline{X}^2) - \frac{\sigma^2}{n} = \mu^2$, or $E(\overline{X}^2 - \frac{\sigma^2}{n}) = \mu^2$.

Since $\overline{X}$ is a complete and sufficient statistic, and $\overline{X}^2 - \frac{\sigma^2}{n}$ is an unbiased estimator of μ^2 and is a function of $\overline{X}$, the UMVUE of μ^2 is $\overline{X}^2 - \frac{\sigma^2}{n}$ by the Lehmann-Scheffé Theorem.

c) Let $Y = \overline{X} - \mu \sim \mathrm{N}(0, \tau^2 = \sigma^2/n)$. Then $E(Y^3) = \int_{-\infty}^{\infty} h(y)dy = 0$, because $h(y)$ is an odd function.

d) $E(\overline{X} - \mu)^3 = E(\overline{X}^3 - 3\mu\overline{X}^2 + 3\mu^2\overline{X} - \mu^3) = E(\overline{X}^3) - 3\mu E(\overline{X}^2) + 3\mu^2 E(\overline{X}) - \mu^3$ $= E(\overline{X}^3) - 3\mu\left(\frac{\sigma^2}{n} + \mu^2\right) + 3\mu^3 - \mu^3 = E(\overline{X}^3) - 3\mu\frac{\sigma^2}{n} - \mu^3$.

Thus $E(\overline{X}^3) - 3\mu\frac{\sigma^2}{n} - \mu^3 = 0$, so replacing μ with its unbiased estimator $\overline{X}$ in the middle term, we get

$$E\left[\overline{X}^3 - 3\overline{X}\frac{\sigma^2}{n}\right] = \mu^3.$$

Since $\overline{X}$ is a complete and sufficient statistic, and $\overline{X}^3 - 3\overline{X}\frac{\sigma^2}{n}$ is an unbiased estimator of μ^3 and is a function of $\bar{X}$, the UMVUE of μ^3 is $\overline{X}^3 - 3\overline{X}\frac{\sigma^2}{n}$ by the Lehmann-Scheffé Theorem.

6.40. (Jan. 2014 Qual): Let $Y_1, ..., Y_n$ be iid from a uniform $U(0, \theta)$ distribution where $\theta > 0$. Then $T = \max(Y_1, ..., Y_n)$ is a complete sufficient statistic.

a) Find $E(T^k)$ for $k > 0$.

b) Find the UMVUE of θ^k for $k > 0$.

Solution: a) The pdf of T is $f(t) = \frac{nt^{n-1}}{\theta^n} I(0 < t < \theta)$. Hence $E(T^k) = \displaystyle\int_0^\theta t^k \frac{nt^{n-1}}{\theta^n} dt = \int_0^\theta \frac{nt^{k+n-1}}{\theta^n} dt = \frac{n\theta^{k+n}}{(k+n)\theta^n} = \frac{n}{k+n}\theta^k$.

b) Thus the UMVUE of θ^k is $\dfrac{k+n}{n} T^k$.

6.41. (Jan. 2014 Qual): Let $Y_1, ..., Y_n$ be iid from a distribution with probability distribution function (pdf)

$$f(y) = \frac{\theta}{(1+y)^{\theta+1}}$$

where $y > 0$ and $\theta > 0$.

a) Find a minimal sufficient statistic for θ.

b) Is the statistic found in a) complete? (prove or disprove)

c) Find the Fisher information $I_1(\theta)$ if $n = 1$.

d) Find the Cramer Rao lower bound (CRLB) for estimating θ^2.

6.42. (Aug. 2014 and Jan. 2024 Quals): Let $X_1, ..., X_n$ be iid from a distribution with pdf

$$f(x|\theta) = \theta x^{\theta-1} I(0 < x < 1), \quad \theta > 0.$$

a) Show $W = -\log(X) \sim$ exponential$(1/\theta)$.

b) Find the method of moments estimator of θ.

c) Find the UMVUE of $1/\theta^2$.

d) Find the Fisher information $I_1(\theta)$.

e) Find the Cramér Rao lower bound for unbiased estimators of $\tau(\theta) = 1/\theta^2$.

Solution. a) Show $f(w) = \theta e^{-w\theta}$ for $w > 0$.

b) $E(X) = \int_0^1 \theta x^\theta dx = \theta/(\theta+1) \stackrel{set}{=} \overline{X}$. So $\theta = \theta\overline{X} + \overline{X}$, or $\theta(1 - \overline{X}) = \overline{X}$. So $\hat{\theta} = \dfrac{\overline{X}}{1 - \overline{X}}$.

c) $T = -\sum_{i=1}^n \log(X_i) \sim G(n, 1/\theta)$ is complete and

$$E(T^2) = \frac{\frac{1}{\theta^2}\Gamma(2+n)}{\Gamma(n)} = \frac{n(n+1)}{\theta^2}.$$

Or use

$$E(T^2) = V(T) + [E(T)]^2 = \frac{n}{\theta^2} + \left(\frac{n}{\theta}\right)^2 = \frac{n^2+n}{\theta^2}.$$

Hence

$$\frac{\Gamma(n)}{\Gamma(2+n)} T^2 = \frac{T^2}{n(n+1)}$$

is the UMVUE of θ^2 by LSU.

d) Now $\log(f(x|\theta) = \log(\theta) + (\theta - 1)\log(x)$. So $\frac{d}{d\theta}\log(f(x|\theta) = \frac{1}{\theta} + \log(x)$, and $\frac{d^2}{d\theta^2}\log(f(x|\theta) = \frac{-1}{\theta^2}$. This familay is a 1P-REF, so

$$I_1(\theta) = -E_\theta\left[\frac{d^2}{d\theta^2}\log(f(x|\theta)\right] = 1/\theta^2.$$

e) Now $\tau'(\theta) = -2\theta^{-3}$, and CRLB $= \frac{[\tau'(\theta)]^2}{nI_1(\theta)} = \frac{4}{n\theta^4}$.

6.43. (Jan. 2015 QUAL): $(\overline{Y}, S^2)$ is complete sufficient.

a) $\overline{Y} + S^2$ by LSU.

b) Using $\overline{Y} \perp\!\!\!\perp S^2$, get $E\left(\frac{c_n\overline{Y}}{S^2}\right) = \frac{\mu}{\sigma^2} = \frac{c_n\mu}{\sigma^2} = c_n\mu E\left(\frac{1}{S^2}\right)$.

Using $\frac{n-1}{\sigma^2}S^2 \sim \chi^2_{n-1}$, show $E\left(\frac{1}{S^2}\right) = \frac{n-1}{n-3}\,\frac{1}{\sigma^2}$. So $c_n = \frac{n-3}{n-1}$.

6.44. (Aug. 2016 Qual): Assume the service time of a customer at a store follows a Pareto distribution with minimum waiting time equal to θ minutes. The maximum length of the service is dependent on the type of the service. Suppose $X_1, ..., X_n$ is a random sample of service times of n customers, where each X_i has a Pareto density given by

$$f(x|\theta) = \begin{cases} 4\theta^4 x^{-5} & x \geq \theta, \\ 0 & x < \theta \end{cases}$$

for an unknown $\theta > 0$. WE are interested in estimating the parameter θ.

a) Write the likelihood function for observed values $x_1, ..., x_n$ of $X_1, ..., X_n$.

b) Find a sufficient statistics for θ. Call it $\widehat{\theta}$.

c) Derive the distribution function and probability density function of $\widehat{\theta}$.

d) Determine the bias and mean squared error of $\widehat{\theta}$.

e) Derive the value of a_n that makes $a_n\widehat{\theta}$ an unbiased estimator of θ.

f) Is the unbiased estimator $a_n\widehat{\theta}$ the uniformly minimum-variance unbiased estimator (UMVUE) of θ ? explain.

Solution:

a)

$$L(\theta|x_1, \ldots, x_n) = \prod_{i=1}^n f_\theta(x_i) = \prod_{i=1}^n 4\theta^4 x_i^{-5} I_{[\theta,\infty)}(x_i) = 4^n\theta^{4n} I_{[\theta,\infty)}(x_{(1)}) \prod_{i=1}^n x_i^{-5}$$

where $x_{(1)}$ is the first order statistics.

b) By factorization theorem, we have

$$f(\mathbf{x}|\theta) = 4^n\theta^{4n} I_{[\theta,\infty)}(x_{(1)}) \prod_{i=1}^n x_i^{-5} = g(T(\mathbf{x})|\theta)h(\mathbf{x})$$

where $T(\mathbf{x}) = X_{(1)}$. Therefore, $\widehat{\theta} = X_{(1)}$ is a sufficient statistics.

c) The distribution function:

$$\begin{aligned}
F_{X_{(1)}}(t) &= P(X_{(1)} \le t) = 1 - P(X_{(1)} > t) = 1 - P(X_1 > t, \dots, X_n > t) \\
&= 1 - P(X_1 > t)\dots P(X_n > t) = 1 - \prod_{i=1}^{n} P(X_i > t) \\
&= 1 - (1 - F_X(t))^n = 1 - (\frac{\theta}{t})^{4n}
\end{aligned}$$

The density function:

$$f_{X_{(1)}}(t) = \frac{d}{dt}F_{X_{(1)}}(t) = \begin{cases} 4n\theta^{4n}t^{-4n-1} & t \ge \theta, \\ 0 & t < \theta \end{cases}$$

d)

$$\text{Bias}(\widehat{\theta}) = E[\widehat{\theta}] - \theta = E[T(\mathbf{x})] - \theta = \frac{4n\theta}{4n-1} - \theta = \frac{1}{4n-1}\theta$$

$$\begin{aligned}
MSE(\widehat{\theta}) = MSE(T(\mathbf{x})) &= Var(T(\mathbf{x})) + \text{Bias}(T(\mathbf{x}))^2 \\
&= \frac{4n\theta^2}{(4n-1)^2(4n-2)} + \frac{\theta^2}{(4n-1)^2}
\end{aligned}$$

e) From part d) we have

$$E[\widehat{\theta}] = E[T(\mathbf{x})] = \frac{4n\theta}{4n-1},$$

therefore,

$$E[\frac{4n-1}{4n}\widehat{\theta}] = E[\frac{4n-1}{4n}T(\mathbf{x})] = \theta,$$

that is, $a_n = \frac{4n-1}{4n}$.

f) Yes, $a_n\widehat{\theta}$ is the UMVUE of θ. Because it is an unbiased estimator, and also it is a function of the complete sufficient statistics $X_{(1)}$. Then, based on the Lehmann-Scheffe Theorem, it is the UMVUE.

To show that $T(\mathbf{x}) = X_{(1)}$ is complete, we need to show if $E[g(T)] = 0$ for all θ, implies $P(g(T) = 0) = 1$ for all θ. The following "method" is not quite right because there may be functions g such that $E(g(T))$ is not differentiable.

Suppose for all θ, $g(t)$ is a function that satisfying

$$E[g(T)] = \int_{\theta}^{\infty} g(t)4n\theta^{4n}\frac{1}{t^{4n+1}}dt = 0$$

Then, by taking derivative on both sides of this, we have

$$
\begin{aligned}
0 = \frac{d}{d\theta}E[g(T)] &= \frac{d}{d\theta}\int_{\theta}^{\infty} g(t)4n\theta^{4n}\frac{1}{t^{4n+1}}dt \\
&= \theta^{4n}\frac{d}{d\theta}\int_{\theta}^{\infty} g(t)4n\frac{1}{t^{4n+1}}dt + \big(\frac{d}{d\theta}\theta^{4n}\big)\int_{\theta}^{\infty} g(t)4n\frac{1}{t^{4n+1}}dt \\
&= -\theta^{4n}g(\theta)4n\frac{1}{\theta^{4n+1}} + 0 \\
&= -4ng(\theta)\frac{1}{\theta}
\end{aligned}
$$

Since $4ng(\theta)\frac{1}{\theta} = 0$, and $4n\frac{1}{\theta} \neq 0$, it must be that $g(\theta) = 0$, and this is true for every $\theta > 0$, therefore $T(\mathbf{x}) = X_{(1)}$ is a complete statistics.

6.45. (Jan. 2018 Qual): Let $Y_1, ..., Y_n$ be independent identically distributed random variables with pdf

$$f(y|\theta) = \theta y^{\theta-1} I(0 < y < 1), \quad \theta > 0.$$

Then $-\log(Y) \sim$ exponential$(1/\theta)$.

a) Find $I_1(\theta)$.

b) Find the Cramer Rao lower bound (CRLB) for unbiased estimators of θ^2.

c) Find the uniformly minimum unbiased estimator (UMVUE) of $1/\theta^2$.

Solution: a) Note that $f(y|\theta) = \theta I(0 < y < 1)\exp[(\theta - 1)\log(y)]$ is the pdf of a 1PREF. Now $\log(f(y|\theta)) = \log(\theta) + (\theta - 1)\log(y)$, and

$$\frac{d}{d\theta}log(f(y|\theta)) = \frac{1}{\theta} + \log(y).$$

So

$$\frac{d^2}{d\theta^2}log(f(y|\theta)) = \frac{-1}{\theta^2}.$$

For a 1PREF, $I_1(\theta) = -E(1/\theta^2) = 1/\theta^2$.

b) If $\tau(\theta) = \theta^2$, then $\tau'(\theta) = 2\theta$, and

$$CRLB = \frac{[\tau'(\theta)]^2}{nI_1(\theta)]} = \frac{4\theta^2}{n\frac{1}{\theta^2}} = \frac{4\theta^4}{n}.$$

c) Let $W_i = -\log(Y_i)$ and $T_n = \overline{W} = \frac{-1}{n}\sum \log(Y_i) \sim \frac{1}{n}G(n, 1/\theta)$. Then

$$E(T_n^2) = W[(\overline{W})^2] = V(\overline{W}) + [E(\overline{W})]^2 = \frac{V(W)}{n} + [E(W)]^2 = \frac{1}{n\theta^2} + \frac{1}{\theta^2}\frac{n}{n} = \frac{n+1}{n\theta^2}.$$

Hence by LSU, the UMVUE is

$$\frac{n}{n+1}T_n^2 = \frac{n}{n+1}\left(\frac{-\sum \log(Y_i)}{n}\right)^2.$$

6.46. (Jan 2019 Qual): Let $X_1, ..., X_n$ be independent and identically distributed (iid) from a distribution with probability density function (pdf)

$$f(x) = e^{-(x-\theta)}$$

where $\theta \leq x < \infty$. Hence $f(x) = 0$ for $x < \theta$.

a) Find the method of moments estimator of θ.

b) Let $T = \min(X_1, ..., X_n) = X_{(1)}$. To estimate θ, consider estimators of the form $T + c$. Determine the value of c which gives the smallest mean square error.

c) T is the MLE of θ and T is a complete sufficient statistic for θ. Find the UMVUE of θ.

Solution: $Y = X - \theta \sim EXP(1)$

a) $E(Y) = 1$ so $E(X) = \theta + 1 \stackrel{set}{=} \overline{X}$. Thus $\hat{\theta}_{MM} = \overline{X} - 1$.

b) Let $T = X_{(1)}$. Then show $f_T(t) = ne^{-(t-\theta)n}$ for $t > 0$. Hence $T \sim EXP(\theta, 1/n)$ and $E(T) = \theta + 1/n$. Then $MSE(T + c) = V_\theta(T + c) = [bias_\theta(T + c)]^2 =$
$V_\theta(T) + [E_\theta(T + c) - \theta]^2 = a + (\theta + 1/n + c - \theta)^2$ which is minimized by $c = -1/n$.

c) $E(T - 1/n) = \theta$ so $T - 1/n = X_{(1)} - 1/n$ is the UMVUE by LSU.

6.47. (Sept. 2022 Qual): Let $Y_1, ..., Y_n$ be iid with probability density function (pdf)

$$f(y) = \frac{2}{\sigma\sqrt{2\pi}} \frac{\theta y^{\theta-1}}{1 - y^\theta} \exp\left(\frac{-1}{2\sigma^2}[\log(1 - y^\theta)]^2\right)$$

where $0 < y < 1$, $\theta > 0$ **is known**, and $\sigma^2 > 0$. Then $t(Y) = [\log(1 - Y^\theta)]^2 \sim G\left(\frac{1}{2}, 2\sigma^2\right)$.

a) Find a complete sufficient statistic for σ.

b) Find $I_1(\sigma)$.

c) Find the UMVUE of σ^2.

d) Find the Cramér Rao lower bound (CRLB) for unbiased estimators of σ^2.

Solution: a) This family is a 1PREF. Hence $T(\boldsymbol{Y}) = \sum_{i=1}^n [\log(1 - Y_i^\theta)]^2$ is a complete sufficient statistic.

b)

$$\log(f(y)) = d - \log(\sigma) - \frac{1}{2\sigma^2} t(y)$$

$$\frac{d}{d\sigma} \log(f(y)) = \frac{-1}{\sigma} + \frac{t(y)}{\sigma^3}$$

$$\frac{d^2}{d\sigma^2} \log(f(y)) = \frac{1}{\sigma^2} - \frac{3t(y)}{\sigma^4}$$

$$I_1(\sigma) = \frac{-1}{\sigma^2} + \frac{3}{\sigma^4} E(t(Y)) = \frac{-1}{\sigma^2} + \frac{3}{\sigma^4}\sigma^2 = 2/\sigma^2.$$

c) $T(\boldsymbol{Y}) = \sum_{i=1}^n t(Y_i) \sim G(n/2, 2\sigma^2)$. Thus $E(T(\boldsymbol{Y})/n) = \sigma^2$ and $T(\boldsymbol{Y})/n$ is the UMVUE of σ^2.

d) Let $\tau(\sigma) = \sigma^2$. then $\tau'(\sigma) = 2\sigma$ and

$$CRLB = \frac{[\tau'(\sigma)]^2}{nI_1(\sigma)} = \frac{4\sigma^2\sigma^2}{2n} = \frac{2\sigma^4}{n}.$$

6.48. (Feb. 2023 Qual): Suppose $Y_1, ..., Y_n$ are iid with pdf

$$\sqrt{\frac{2}{\pi}} \; \frac{1}{\sigma^3} \; y^2 I(y > 0) \; \exp\left(\frac{-1}{2\sigma^2} \; y^2 \right)$$

where $\sigma^2 > 0$ and $W = Y^2 \sim G(3/2, 2\sigma^2)$.

a) Let $T_n = c\overline{W}_n$ be an estimator of σ^2. Find the mean square error (MSE) of T_n as a function of c (and of σ and n).

b) Find the value c that minimizes the MSE. Prove that your value is the minimizer.

Solution: a) $E(T_n) = cE(\overline{W}_n) = cE(W) = c(3/2)2\sigma^2 = 3c\sigma^2$.
$V(T_n) = c^2 V(\overline{W}_n) = c^2 V(W)/n = c^2(3/2)(4\sigma^4/n) = 6c^2\sigma^4/n$.

$$MSE(T_n) = V(T_n) + (E(T_n) - \sigma^2)^2 = 6c^2\sigma^4/n + (3c\sigma^2 - \sigma^2)^2 = \frac{6c^2\sigma^4}{n} + \sigma^4(3c-1)^2 = MSE(c).$$

$$b) \; \frac{dMSE(c)}{dc} = \frac{12c\sigma^4}{n} + 2\sigma^4(3c-1)3 \overset{set}{=} 0$$

$$\text{or } \mathrm{c}\sigma^4 \frac{12}{\mathrm{n}} + \mathrm{c}\sigma^4 18 - 6\sigma^4 = 0$$

$$\text{or c} \left(\frac{12}{\mathrm{n}} + 18 \right) = 6$$

$$\text{or c} = \frac{6}{(12/\mathrm{n}) + 18} = \frac{6\mathrm{n}}{12 + 18\mathrm{n}} = \frac{\mathrm{n}}{2 + 3\mathrm{n}}.$$

$$\frac{d^2 MSE(c)}{dc^2} = \frac{12\sigma^4}{n} + 18\sigma^4 > 0,$$

so c is the minimizer.

6.49. (Jan. 2025 Qual): Suppose Y is a random variable with probability density function (pdf)

$$f(y) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \; \frac{(y-c)^{\alpha-1}(d-y)^{\beta-1}}{(d-c)^{\alpha+\beta-1}}$$

where $\alpha > 0$, $\beta > 0$ and $c \le y \le d$ where $c < d$ are known real numbers. Also, $f(y) = 0$ for $y < c$ and $y > d$.

a) Show that the family of distributions $f(y) = f(y|\alpha, \beta)$ is a two parameter regular exponential family.

b) Find a complete sufficient statistic for (α, β) if the sample size is n.

c) Now suppose $\beta = 1$ and $d = c + 1$ so the pdf of Y is

$$f(y) = \alpha(y - c)^{\alpha-1}$$

for $c \le y \le c + 1$. Let $Y_1, ..., Y_n$ be independent and identically distributed (iid) random variables from this distribution. Find the uniformly minimum variance unbiased estimator (UMVUE) of $1/\alpha$. You may use the fact that

$$T_n = -\sum_{i=1}^{n} \log(Y_i - c) \sim G(n, 1/\alpha).$$

Solution. a)

$$f(y) = I_{[c,d]}(y) \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \; \frac{1}{(d-c)^{\alpha+\beta-1}} \exp[(\alpha-1)\log(y-c) + (\beta-1)\log(d-y)]$$

is a 2-parameter regular exponential family with $\eta_1 = \alpha - 1$, $\eta_2 = \beta - 1$ and $\Omega = (-1, \infty) \times (-1, \infty)$.

b) $(\sum_{i=1}^n \log(Y_i - c), \sum_{j=1}^n \log(d - Y_j))$ by exponential family theory.

c) Then we have a 1 parameter regular exponential family with T_n a complete sufficient statistic. Hence T_n/n is the UMVUE of $1/\alpha$ since $E(T_n/n) = 1/\alpha$.

7.6. (Aug. 2002 Qual): Let $X_1, ..., X_n$ be independent, identically distributed random variables from a distribution with a beta(θ, θ) pdf

$$f(x|\theta) = \frac{\Gamma(2\theta)}{\Gamma(\theta)\Gamma(\theta)} [x(1-x)]^{\theta-1}$$

where $0 < x < 1$ and $\theta > 0$.

a) Find the UMP (uniformly most powerful) level α test for $H_o : \theta = 1$ vs. $H_1 : \theta = 2$.

b) If possible, find the UMP level α test for $H_o : \theta = 1$ vs. $H_1 : \theta > 1$.

Solution. For both a) and b), the test is reject Ho iff $\prod_{i=1}^n x_i(1-x_i) > c$ where $P_{\theta=1}[\prod_{i=1}^n x_i(1-x_i) > c] = \alpha$.

7.10. (Jan. 2001 SIU and 1990 Univ. MN Qual): Let $X_1, ..., X_n$ be a random sample from the distribution with pdf

$$f(x, \theta) = \frac{x^{\theta-1} e^{-x}}{\Gamma(\theta)}, \;\; x > 0, \;\; \theta > 0.$$

Find the uniformly most powerful level α test of

H: $\theta = 1$ versus K: $\theta > 1$.

Solution. H says $f(x) = e^{-x}$ while K says

$$f(x) = x^{\theta-1} e^{-x} / \Gamma(\theta).$$

The monotone likelihood ratio property holds for $\prod x_i$ since then

$$\frac{f_n(\boldsymbol{x}, \theta_2)}{f_n(\boldsymbol{x}, \theta_1)} = \frac{(\prod_{i=1}^n x_i)^{\theta_2-1}(\Gamma(\theta_1))^n}{(\prod_{i=1}^n x_i)^{\theta_1-1}(\Gamma(\theta_2))^n} = (\frac{\Gamma(\theta_1)}{\Gamma(\theta_2)})^n (\prod_{i=1}^n x_i)^{\theta_2-\theta_1}$$

which increases as $\prod_{i=1}^n x_i$ increases if $\theta_2 > \theta_1$. Hence the level α UMP test rejects H if

$$\prod_{i=1}^n X_i > c$$

where

$$P_H(\prod_{i=1}^n X_i > c) = P_H(\sum \log(X_i) > \log(c)) = \alpha.$$

7.11. (Jan 2001 Qual, see Aug 2013 Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables from a $N(\mu, \sigma^2)$ distribution where the variance σ^2 is known. We want to test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$.

a) Derive the likelihood ratio test.

b) Let λ be the likelihood ratio. Show that $-2 \log \lambda$ is a function of $(\overline{X} - \mu_0)$.

c) Assuming that H_0 is true, find $P(-2 \log \lambda > 3.84)$.

Solution. a) The likelihood function

$$L(\mu) = (2\pi\sigma^2)^{-n/2} \exp[\frac{-1}{2\sigma^2} \sum (x_i - \mu)^2]$$

and the MLE for μ is $\hat{\mu} = \overline{x}$. Thus the numerator of the likelihood ratio test statistic is $L(\mu_0)$ and the denominator is $L(\overline{x})$. So the test is reject H_0 if $\lambda(\boldsymbol{x}) = L(\mu_0)/L(\overline{x}) \leq c$ where $\alpha = P_{\mu_0}(\lambda(\boldsymbol{X}) \leq c)$.

b) As a statistic, $\log \lambda = \log L(\mu_0) - \log L(\overline{X}) = -\frac{1}{2\sigma^2}[\sum(X_i - \mu_0)^2 - \sum(X_i - \overline{X})^2] = \frac{-n}{2\sigma^2}[\overline{X} - \mu_0]^2$ since $\sum(X_i - \mu_0)^2 = \sum(X_i - \overline{X} + \overline{X} - \mu_0)^2 = \sum(X_i - \overline{X})^2 + n(\overline{X} - \mu_0)^2$. So $-2 \log \lambda = \frac{n}{\sigma^2}[\overline{X} - \mu_0]^2$.

c) $-2 \log \lambda \sim \chi^2_1$ and from a chi–square table, $P(-2 \log \lambda > 3.84) = 0.05$.

7.12. (Aug. 2001 Qual): Let $X_1, ..., X_n$ be iid from a distribution with pdf

$$f(x) = \frac{2x}{\lambda} \exp(-x^2/\lambda)$$

where λ and x are both positive. Find the level α UMP test for $H_o : \lambda = 1$ vs $H_1 : \lambda > 1$.

7.13. (Jan. 2003 Qual): Let $X_1, ..., X_n$ be iid from a distribution with pdf

$$f(x|\theta) = \frac{(\log \theta)\theta^x}{\theta - 1}$$

where $0 < x < 1$ and $\theta > 1$. Find the UMP (uniformly most powerful) level α test of $H_o : \theta = 2$ vs. $H_1 : \theta = 4$.

Solution. Let $\theta_1 = 4$. By Neyman Pearson lemma, reject Ho if

$$\frac{f(\boldsymbol{x}|\theta_1)}{f(\boldsymbol{x}|2)} = \left(\frac{\log(\theta_1)}{\theta_1 - 1}\right)^n \theta_1^{\sum x_i} \left(\frac{1}{\log(2)}\right)^n \frac{1}{2^{\sum x_i}} > k$$

iff

$$\left(\frac{\log(\theta_1)}{(\theta_1 - 1)\log(2)}\right)^n \left(\frac{\theta_1}{2}\right)^{\sum x_i} > k$$

iff

$$\left(\frac{\theta_1}{2}\right)^{\sum x_i} > k'$$

iff

$$\sum x_i \log(\theta_1/2) > c'.$$

So reject Ho iff $\sum X_i > c$ where $P_{\theta=2}(\sum X_i > c) = \alpha$.

7.14. (Aug. 2003 Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables from a distribution with pdf

$$f(x) = \frac{x^2 \exp\left(\frac{-x^2}{2\sigma^2}\right)}{\sigma^3 \sqrt{2}\ \Gamma(3/2)}$$

where $\sigma > 0$ and $x \geq 0$.

a) What is the UMP (uniformly most powerful) level α test for $H_o : \sigma = 1$ vs. $H_1 : \sigma = 2$?

b) If possible, find the UMP level α test for $H_o : \sigma = 1$ vs. $H_1 : \sigma > 1$.

Solution. a) By NP lemma reject Ho if

$$\frac{f(\boldsymbol{x}|\sigma = 2)}{f(\boldsymbol{x}|\sigma = 1)} > k'.$$

The LHS =

$$\frac{\frac{1}{2^{3n}} \exp[\frac{-1}{8} \sum x_i^2]}{\exp[\frac{-1}{2} \sum x_i^2]}$$

So reject Ho if

$$\frac{1}{2^{3n}} \exp[\sum x_i^2 (\frac{1}{2} - \frac{1}{8})] > k'$$

or if $\sum x_i^2 > k$ where $P_{Ho}(\sum x_i^2 > k) = \alpha$.

b) In the above argument, with any $\sigma_1 > 1$, get

$$\sum x_i^2 (\frac{1}{2} - \frac{1}{2\sigma_1^2})$$

and

$$\frac{1}{2} - \frac{1}{2\sigma_1^2} > 0$$

for any $\sigma_1^2 > 1$. Hence the UMP test is the same as in a).

7.15. (Jan. 2004 Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables from a distribution with pdf

$$f(x) = \frac{2}{\sigma\sqrt{2\pi}} \frac{1}{x} \exp\left(\frac{-[\log(x)]^2}{2\sigma^2}\right)$$

where $\sigma > 0$ and $x \geq 1$.

a) What is the UMP (uniformly most powerful) level α test for $H_o : \sigma = 1$ vs. $H_1 : \sigma = 2$?

b) If possible, find the UMP level α test for $H_o : \sigma = 1$ vs. $H_1 : \sigma > 1$.

Solution. a) By NP lemma reject Ho if

$$\frac{f(\boldsymbol{x}|\sigma = 2)}{f(\boldsymbol{x}|\sigma = 1)} > k'.$$

The LHS =

$$\frac{\frac{1}{2^n} \exp[\frac{-1}{8} \sum [\log(x_i)]^2]}{\exp[\frac{-1}{2} \sum [\log(x_i)]^2]}$$

So reject Ho if

$$\frac{1}{2^n} \exp[\sum [\log(x_i)]^2 (\frac{1}{2} - \frac{1}{8})] > k'$$

or if $\sum [\log(X_i)]^2 > k$ where $P_{Ho}(\sum [\log(X_i)]^2 > k) = \alpha$.

b) In the above argument, with any $\sigma_1 > 1$, get

$$\sum [\log(x_i)]^2 (\frac{1}{2} - \frac{1}{2\sigma_1^2})$$

and

$$\frac{1}{2} - \frac{1}{2\sigma_1^2} > 0$$

for any $\sigma_1^2 > 1$. Hence the UMP test is the same as in a).

7.16. (Aug. 2004 Qual): Suppose X is an observable random variable with its pdf given by $f(x)$, $x \in \mathcal{R}$. Consider two functions defined as follows:

$$f_0(x) = \begin{cases} \frac{3}{64} x^2 & 0 \leq x \leq 4 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_1(x) = \begin{cases} \frac{3}{16} \sqrt{x} & 0 \leq x \leq 4 \\ 0 & \text{elsewhere.} \end{cases}$$

Determine the most powerful level α test for $H_0 : f(x) = f_0(x)$ versus $H_a : f(x) = f_1(x)$ in the simplest implementable form. Also, find the power of the test when $\alpha = 0.01$

Solution. The most powerful test will have the following form.
Reject H_0 iff $\frac{f_1(x)}{f_0(x)} > k$.
But $\frac{f_1(x)}{f_0(x)} = 4x^{-\frac{3}{2}}$ and hence we reject H_0 iff X is small, i.e. reject H_0 is $X < k$ for some constant k. This test must also have the size α, that is we require:

$\alpha = P(X < k)$ when $f(x) = f_0(x)) = \int_0^k \frac{3}{64} x^2 dx = \frac{1}{64} k^3$,
so that $k = 4\alpha^{\frac{1}{3}}$.
For the power, when $k = 4\alpha^{\frac{1}{3}}$
$P[X < k$ when $f(x) = f_1(x)] = \int_0^k \frac{3}{16}\sqrt{x} dx = \sqrt{\alpha}$.
When $\alpha = 0.01$, the power is $= 0.10$.

7.17. (Sept. 2005 Qual): Let X be one observation from the probability density function

$$f(x) = \theta x^{\theta - 1}, \;\; 0 < x < 1, \;\; \theta > 0.$$

a) Find the most powerful level α test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

b) For testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$, find the size and the power function of the test which rejects H_0 if $X > \dfrac{5}{8}$.

c) Is there a UMP test of $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$? If so, find it. If not, prove so.

7.19. (Jan. 2009 Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables from a half normal HN(μ, σ^2) distribution with pdf

$$f(x) = \frac{2}{\sigma\ \sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

where $\sigma > 0$ and $x > \mu$ and μ is real. **Assume that μ is known.**

a) What is the UMP (uniformly most powerful) level α test for
$H_0 : \sigma^2 = 1$ vs. $H_1 : \sigma^2 = 4$?

b) If possible, find the UMP level α test for $H_0 : \sigma^2 = 1$ vs. $H_1 : \sigma^2 > 1$.

Solution. a) By the NP lemma reject Ho if

$$\frac{f(\boldsymbol{x}|\sigma^2 = 4)}{f(\boldsymbol{x}|\sigma^2 = 1)} > k'.$$

The LHS =

$$\frac{\frac{1}{2^n} \exp[(\frac{-\sum(x_i-\mu)^2}{2(4)})]}{\exp[(\frac{-\sum(x_i-\mu)^2}{2})]}.$$

So reject Ho if

$$\frac{1}{2^n} \exp[\sum (x_i - \mu)^2 (\frac{-1}{8} + \frac{1}{2})] > k'$$

or if $\sum(x_i - \mu)^2 > k$ where $P_{\sigma^2=1}(\sum(X_i - \mu)^2 > k) = \alpha$.

Under Ho, $\sum(X_i - \mu)^2 \sim \chi^2_n$ so $k = \chi^2_n(1 - \alpha)$ where $P(\chi^2_n > \chi^2_n(1 - \alpha)) = \alpha$.

b) In the above argument,

$$\frac{-1}{2(4)} + 0.5 = \frac{-1}{8} + 0.5 > 0$$

but

$$\frac{-1}{2\sigma_1^2} + 0.5 > 0$$

for any $\sigma_1^2 > 1$. Hence the UMP test is the same as in a).

Alternatively, use the fact that this is an exponential family where $w(\sigma^2) = -1/(2\sigma^2)$ is an increasing function of σ^2 with $T(X_i) = (X_i - \mu)^2$. Hence the test in a) is UMP for a) and b) by Theorem 7.3.

7.20. (Aug. 2009 Qual): Suppose that the test statistic $T(X)$ for testing $H_0 : \lambda = 1$ versus $H_1 : \lambda > 1$ has an exponential$(1/\lambda_1)$ distribution if $\lambda = \lambda_1$. The test rejects H_0 if $T(X) < \log(100/95)$.

a) Find the power of the test if $\lambda_1 = 1$.

b) Find the power of the test if $\lambda_1 = 50$.

c) Find the pvalue of this test.

Solution. $E[T(X)] = 1/\lambda_1$ and the power = P(test rejects H_0) = $P_{\lambda_1}(T(X) < \log(100/95)) = F_{\lambda_1}(\log(100/95))$
$= 1 - \exp(-\lambda_1 \log(100/95)) = 1 - (95/100)^{\lambda_1}$.

a) Power $= 1 - \exp(-\log(100/95)) = 1 - \exp(\log(95/100)) = 0.05$.

b) Power $= 1 - (95/100)^{50} = 0.923055$.

c) Let T_0 be the observed value of $T(X)$. Then pvalue $= P(W \leq T_0)$ where $W \sim$ exponential(1) since under H_0, $T(X) \sim$ exponential(1). So pvalue $= 1 - \exp(-T_0)$.

7.21. (Aug. 2009 Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables from a Burr type X distribution with pdf

$$f(x) = 2\, \tau\, x\, e^{-x^2}\, (1 - e^{-x^2})^{\tau - 1}$$

where $\tau > 0$ and $x > 0$.

a) What is the UMP (uniformly most powerful) level α test for
$H_0 : \tau = 2$ versus $H_1 : \tau = 4$?

b) If possible, find the UMP level α test for $H_0 : \tau = 2$ versus $H_1 : \tau > 2$.

Solution. Note that

$$f(x) = I(x > 0)\; 2x\; e^{-x^2}\; \tau\; \exp[(\tau - 1)(\log(1 - e^{-x^2}))]$$

is a one parameter exponential family and $w(\tau) = \tau - 1$ is an increasing function of τ. Thus the UMP test rejects H_0 if $T(\boldsymbol{x}) = \sum_{i=1}^n \log(1 - e^{-x_i^2}) > k$ where $\alpha = P_{\tau=2}(T(\boldsymbol{X}) > k)$.

Or use NP lemma.

a) Reject Ho if

$$\frac{f(\boldsymbol{x}|\tau = 4)}{f(\boldsymbol{x}|\tau = 1)} > k.$$

The LHS =

$$\frac{4^n}{2^n}\; \frac{\prod_{i=1}^n (1 - e^{-x_i^2})^{4-1}}{\prod_{i=1}^n (1 - e^{-x_i^2})} = 2^n \prod_{i=1}^n (1 - e^{-x_i^2})^2.$$

So reject Ho if

$$\prod_{i=1}^{n}(1-e^{-x_i^2})^2 > k'$$

or

$$\prod_{i=1}^{n}(1-e^{-x_i^2}) > c$$

or

$$\sum_{i=1}^{n}\log(1-e^{-x_i^2}) > d$$

where

$$\alpha = P_{\tau=2}(\prod_{i=1}^{n}(1-e^{-x_i^2}) > c).$$

b) Replace $4-1$ by $\tau_1 - 1$ where $\tau_1 > 2$. Then reject H_0 if

$$\prod_{i=1}^{n}(1-e^{-x_i^2})^{\tau_1-2} > k'$$

which gives the same test as in a).

7.22. (Jan. 2010 Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables from an inverse exponential distribution with pdf

$$f(x) = \frac{\theta}{x^2}\ \exp\left(\frac{-\theta}{x}\right)$$

where $\theta > 0$ and $x > 0$.

a) What is the UMP (uniformly most powerful) level α test for $H_0 : \theta = 1$ versus $H_1 : \theta = 2$?

b) If possible, find the UMP level α test for $H_0 : \theta = 1$ versus $H_1 : \theta > 1$.

Solution. By exponential family theory, the UMP test rejects H_0 if $T(\boldsymbol{x}) = -\sum_{i=1}^{n} \frac{1}{x_i} > k$ where $P_{\theta=1}(T(\boldsymbol{X}) > k) = \alpha$.

Alternatively, use the Neyman Pearson lemma:

a) reject Ho if

$$\frac{f(\boldsymbol{x}|\theta=2)}{f(\boldsymbol{x}|\theta=1)} > k'.$$

The LHS =

$$\frac{2^n \exp(-2\sum \frac{1}{x_i})}{\exp(-\sum \frac{1}{x_i})}.$$

So reject Ho if

$$2^n \exp[(-2+1)\sum \frac{1}{x_i}] > k'$$

or if $-\sum \frac{1}{x_i} > k$ where $P_1(-\sum \frac{1}{x_i} > k) = \alpha$.

b) In the above argument, reject H_0 if

$$2^n \exp[(-\theta_1 + 1) \sum \frac{1}{x_i}] > k'$$

or if $-\sum \frac{1}{x_i} > k$ where $P_1(-\sum \frac{1}{x_i} > k) = \alpha$ for any $\theta_1 > 1$. Hence the UMP test is the same as in a).

7.23. (Sept. 2010 Qual): Suppose that X is an observable random variable with its pdf given by $f(x)$. Consider the two functions defined as follows: $f_0(x)$ is the probability density function of a Beta distribution with $\alpha = 1$ and $\beta = 2$ and and $f_1(x)$ is the pdf of a Beta distribution with $\alpha = 2$ and $\beta = 1$.

a) Determine the UMP level $\alpha = 0.10$ test for $H_0 : f(x) = f_0(x)$ versus $H_1 : f(x) = f_1(x)$. (Find the constant.)

b) Find the power of the test in a).

Solution. a) We reject H_0 iff $\frac{f_1(x)}{f_0(x)} > k$. Thus we reject H_0 iff $\frac{2x}{2(1-x)} > k$. That is $\frac{1-x}{x} < k_1$, that is $\frac{1}{x} < k_2$, that is $x > k_3$. Now $0.1 = P(X > k_3)$ when $f(x) = f_0(x)$, so $k_3 = 1 - \sqrt{0.1}$.

7.24. (Sept. 2010 Qual): The pdf of a bivariate normal distribution is $f(x, y) =$

$$\frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \exp\left(\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right)$$

where $-1 < \rho < 1$, $\sigma_1 > 0$, $\sigma_2 > 0$, while x, y, μ_1, and μ_2 are all real. Let $(X_1, Y_1), ..., (X_n, Y_n)$ be a random sample from a bivariate normal distribution.
Let $\hat{\theta}(\boldsymbol{x}, \boldsymbol{y})$ be the observed value of the MLE of θ, and let $\hat{\theta}(\boldsymbol{X}, \boldsymbol{Y})$ be the MLE as a random variable. Let the (unrestricted) MLEs be $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\sigma}_1$, $\hat{\sigma}_2$, and $\hat{\rho}$. Then

$$T_1 = \sum_{i=1}^n \left(\frac{x_i - \hat{\mu}_1}{\hat{\sigma}_1}\right)^2 = \frac{n\hat{\sigma}_1^2}{\hat{\sigma}_1^2} = n, \quad \text{and} \quad \mathrm{T}_3 = \sum_{\mathrm{i}=1}^{\mathrm{n}} \left(\frac{\mathrm{y_i} - \hat{\mu}_2}{\hat{\sigma}_2}\right)^2 = \frac{\mathrm{n}\hat{\sigma}_2^2}{\hat{\sigma}_2^2} = \mathrm{n},$$

$$\text{and} \quad \mathrm{T}_2 = \sum_{\mathrm{i}=1}^{\mathrm{n}} \left(\frac{\mathrm{x_i} - \hat{\mu}_1}{\hat{\sigma}_1}\right)\left(\frac{\mathrm{y_i} - \hat{\mu}_2}{\hat{\sigma}_2}\right) = \mathrm{n}\hat{\rho}.$$

Consider testing $H_0 : \rho = 0$ vs. $H_A : \rho \neq 0$. The (restricted) MLEs for μ_1, μ_2, σ_1 and σ_2 do not change under H_0, and hence are still equal to $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\sigma}_1$, and $\hat{\sigma}_2$.

a) Using the above information, find the likelihood ratio test for $H_0 : \rho = 0$ vs. $H_A : \rho \neq 0$. Denote the likelihood ratio test statistic by $\lambda(\boldsymbol{x}, \boldsymbol{y})$.

b) Find the large sample (asymptotic) likelihood ratio test that uses test statistic $-2\log(\lambda(\boldsymbol{x}, \boldsymbol{y}))$.

Solution. a) Let $k = [2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}]$. Then the likelihood $L(\boldsymbol{\theta}) =$

$$\frac{1}{k^n} \exp\left(\frac{-1}{2(1-\rho^2)} \sum_{i=1}^n \left[\left(\frac{x_i-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_i-\mu_1}{\sigma_1}\right)\left(\frac{y_i-\mu_2}{\sigma_2}\right) + \left(\frac{y_i-\mu_2}{\sigma_2}\right)^2\right]\right).$$

Hence

$$L(\hat{\boldsymbol{\theta}}) = \frac{1}{[2\pi\hat{\sigma}_1\hat{\sigma}_2(1-\hat{\rho}^2)^{1/2}]^n} \exp\left(\frac{-1}{2(1-\hat{\rho}^2)}[T_1 - 2\hat{\rho}T_2 + T_3]\right)$$
$$= \frac{1}{[2\pi\hat{\sigma}_1\hat{\sigma}_2(1-\hat{\rho}^2)^{1/2}]^n} \exp(-n)$$

and

$$L(\hat{\boldsymbol{\theta}}_0) = \frac{1}{[2\pi\hat{\sigma}_1\hat{\sigma}_2]^n} \exp\left(\frac{-1}{2}[T_1 + T_3]\right)$$
$$= \frac{1}{[2\pi\hat{\sigma}_1\hat{\sigma}_2]^n} \exp(-n).$$

Thus $\lambda(\boldsymbol{x}, \boldsymbol{y}) =$

$$\frac{L(\hat{\boldsymbol{\theta}}_0)}{L(\hat{\boldsymbol{\theta}})} = (1-\hat{\rho}^2)^{n/2}.$$

So reject H_0 if $\lambda(\boldsymbol{x}, \boldsymbol{y}) \leq c$ where $\alpha = \sup_{\boldsymbol{\theta} \in \Theta_o} P(\lambda(\boldsymbol{X}, \boldsymbol{Y}) \leq c)$. Here Θ_o is the set of $\boldsymbol{\theta} = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$ such that the μ_i are real, $\sigma_i > 0$ and $\rho = 0$, i.e., such that X_i and Y_i are independent.

b) Since the unrestricted MLE has one more free parameter than the restricted MLE, $-2\log(\lambda(\boldsymbol{X}, \boldsymbol{Y})) \approx \chi^2_1$, and the approximate LRT rejects H_0 if $-2\log\lambda(\boldsymbol{x}, \boldsymbol{y}) > \chi^2_{1,1-\alpha}$ where $P(\chi^2_1 > \chi^2_{1,1-\alpha}) = \alpha$.

7.26. (Aug. 2012 Qual): Let $Y_1, ..., Y_n$ be independent identically distributed random variables with pdf

$$f(y) = e^y I(y \geq 0) \ \frac{1}{\lambda} \ \exp\left[\frac{-1}{\lambda}(e^y - 1)\right]$$

where $y > 0$ and $\lambda > 0$.

a) Show that $W = e^Y - 1 \sim \dfrac{\lambda}{2}\chi^2_2$.

b) What is the UMP (uniformly most powerful) level α test for $H_0: \lambda = 2$ versus $H_1: \lambda > 2$?

c) If $n = 20$ and $\alpha = 0.05$, then find the power $\beta(3.8386)$ of the above UMP test if $\lambda = 3.8386$. Let $P(\chi^2_d \leq \chi^2_{d,\delta}) = \delta$. The tabled values below give $\chi^2_{d,\delta}$.

d	δ							
	0.01	0.05	0.1	0.25	0.75	0.9	0.95	0.99
20	8.260	10.851	12.443	15.452	23.828	28.412	31.410	37.566
30	14.953	18.493	20.599	24.478	34.800	40.256	43.773	50.892
40	22.164	26.509	29.051	33.660	45.616	51.805	55.758	63.691

Solution. b) This family is a regular one parameter exponential family where $w(\lambda) = -1/\lambda$ is increasing. Hence the level α UMP test rejects H_0 when $\sum_{i=1}^n (e^{y_i} - 1) > k$ where $\alpha = P_2(\sum_{i=1}^n (e^{Y_i} - 1) > k) = P_2(T(\boldsymbol{Y}) > k)$.

c) Since $T(\boldsymbol{Y}) \sim \dfrac{\lambda}{2}\chi^2_{2n}$, $\dfrac{2T(\boldsymbol{Y})}{\lambda} \sim \chi^2_{2n}$. Hence

$$\alpha = 0.05 = P_2(T(\boldsymbol{Y}) > k) = P(\chi^2_{40} > \chi^2_{40,1-\alpha}),$$

and $k = \chi^2_{40,1-\alpha} = 55.758$. Hence the power

$$\beta(\lambda) = P_\lambda(T(\boldsymbol{Y}) > 55.758) = P(\frac{2T(\boldsymbol{Y})}{\lambda} > \frac{2(55.758)}{\lambda}) = P(\chi^2_{40} > \frac{2(55.758)}{\lambda})$$

$$= P(\chi^2_{40} > \frac{2(55.758)}{3.8386}) = P(\chi^2_{40} > 29.051) = 1 - 0.1 = 0.9.$$

7.27. (Jan. 2013 Qual): Let $Y_1, ..., Y_n$ be independent identically distributed $N(\mu = 0, \sigma^2)$ random variables with pdf

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-y^2}{2\sigma^2}\right)$$

where y is real and $\sigma^2 > 0$.

a) Show $W = Y^2 \sim \sigma^2 \chi^2_1$.

b) What is the UMP (uniformly most powerful) level α test for $H_0 : \sigma^2 = 1$ versus $H_1 : \sigma^2 > 1$?

c) If $n = 20$ and $\alpha = 0.05$, then find the power $\beta(3.8027)$ of the above UMP test if $\sigma^2 = 3.8027$. Let $P(\chi^2_d \le \chi^2_{d,\delta}) = \delta$. The tabled values below give $\chi^2_{d,\delta}$.

d	δ							
	0.01	0.05	0.1	0.25	0.75	0.9	0.95	0.99
20	8.260	10.851	12.443	15.452	23.828	28.412	31.410	37.566
30	14.953	18.493	20.599	24.478	34.800	40.256	43.773	50.892
40	22.164	26.509	29.051	33.660	45.616	51.805	55.758	63.691

Solution. b) This family is a regular one parameter exponential family where $w(\sigma^2) = -1/(2\sigma^2)$ is increasing. Hence the level α UMP test rejects H_0 when $\sum_{i=1}^n y_i^2 > k$ where $\alpha = P_1(\sum_{i=1}^n Y_i^2) > k) = P_1(T(\boldsymbol{Y}) > k)$.

c) Since $T(\boldsymbol{Y}) \sim \sigma^2 \chi^2_n$, $\dfrac{T(\boldsymbol{Y})}{\sigma^2} \sim \chi^2_n$. Hence

$$\alpha = 0.05 = P_1(T(\boldsymbol{Y}) > k) = P(\chi^2_{20} > \chi^2_{20,1-\alpha}),$$

and $k = \chi^2_{20,1-\alpha} = 31.410$. Hence the power

$$\beta(\sigma) = P_\sigma(T(\boldsymbol{Y}) > 31.41) = P(\frac{T(\boldsymbol{Y})}{\sigma^2} > \frac{31.41}{\sigma^2}) = P(\chi^2_{20} > \frac{31.41}{3.8027})$$

$$= P(\chi^2_{20} > 8.260) = 1 - 0.01 = 0.99.$$

7.28. (Aug. 2013 Qual): Let $Y_1, ..., Y_n$ be independent identically distributed random variables with pdf

$$f(y) = \frac{y}{\sigma^2} \exp\left[-\frac{1}{2}\left(\frac{y}{\sigma}\right)^2\right]$$

where $\sigma > 0$, μ is real, and $y \ge 0$.

a) Show $W = Y^2 \sim \sigma^2 \chi^2_2$. Equivalently, show $Y^2/\sigma^2 \sim \chi^2_2$.

b) What is the UMP (uniformly most powerful) level α test for $H_0 : \sigma = 1$ versus $H_1 : \sigma > 1$?

c) If $n = 20$ and $\alpha = 0.05$, then find the power $\beta(\sqrt{1.9193})$ of the above UMP test if $\sigma = \sqrt{1.9193}$. Let $P(\chi^2_d \le \chi^2_{d,\delta}) = \delta$. The above tabled values for problem 7.27 give $\chi^2_{d,\delta}$.

Solution. a) Let $X = Y^2/\sigma^2 = t(Y)$. Then $Y = \sigma\sqrt{X} = t^{-1}(X)$. Hence

$$\frac{dt^{-1}(x)}{dx} = \frac{\sigma}{2}\frac{1}{\sqrt{x}}$$

and the pdf of X is

$$g(x) = f_Y(t^{-1}(x))\left|\frac{dt^{-1}(x)}{dx}\right| = \frac{\sigma\sqrt{x}}{\sigma^2}\exp\left[\frac{-1}{2}\left(\frac{\sigma\sqrt{x}}{\sigma}\right)^2\right]\frac{\sigma}{2\sqrt{x}} = \frac{1}{2}\exp(-x/2)$$

for $x > 0$, which is the χ^2_2 pdf.

b) This family is a regular one parameter exponential family where $w(\sigma) = -1/(2\sigma^2)$ is increasing. Hence the level α UMP test rejects H_0 when $\sum_{i=1}^n y_i^2 > k$ where $\alpha = P_1(\sum_{i=1}^n Y_i^2 > k) = P_1(T(\boldsymbol{Y}) > k)$.

c) Since $T(\boldsymbol{Y}) \sim \sigma^2\chi^2_{2n}$, $\dfrac{T(\boldsymbol{Y})}{\sigma^2} \sim \chi^2_{2n}$. Hence

$$\alpha = 0.05 = P_1(T(\boldsymbol{Y}) > k) = P(\chi^2_{40} > \chi^2_{40,1-\alpha}),$$

and $k = \chi^2_{40,1-\alpha} = 55.758$. Hence the power

$$\beta(\sigma) = P_\sigma(T(\boldsymbol{Y}) > 55.758) = P(\frac{T(\boldsymbol{Y})}{\sigma^2} > \frac{55.758}{\sigma^2}) = P(\chi^2_{40} > \frac{55.758}{\sigma^2})$$

$$= P(\chi^2_{40} > \frac{55.758}{1.9193}) = P(\chi^2_{40} > 29.051) = 1 - 0.1 = 0.9.$$

7.29. (Aug. 2012 Qual): Consider independent random variables $X_1, ..., X_n$, where $X_i \sim \mathrm{N}(\theta_i, \sigma^2)$, $1 \le i \le n$, and σ^2 is known.

a) Find the most powerful test of

$$H_0 : \theta_i = 0, \forall i, \text{ versus } H_1 : \theta_i = \theta_{i0}, \forall i,$$

where θ_{i0} are known. Derive (and simplify) the exact critical region for a level α test.

b) Find the likelihood ratio test of

$$H_0 : \theta_i = 0, \forall i, \text{ versus } H_1 : \theta_i \ne 0, \text{ for some } i.$$

Derive (and simplify) the exact critical region for a level α test.

c) Find the power of the test in (a), when $\theta_{i0} = n^{-1/3}, \forall i$. What happens to this power expression as $n \to \infty$?

Solution: a) In Neyman Pearson's lemma, let $\theta = 0$ if H_0 is true and $\theta = 1$ if H_1 is true. Then want to find $f(\boldsymbol{x}|\theta = 1)/f(\boldsymbol{x}|\theta = 0) \equiv f_1(\boldsymbol{x})/f_0(\boldsymbol{x})$. Since

$$f(\boldsymbol{x}) = \frac{1}{(\sqrt{2\pi}\ \sigma)^n}\exp[\frac{-1}{2\sigma^2}\sum_{i=1}^n (x_i - \theta_i)^2],$$

$$\frac{f_1(\boldsymbol{x})}{f_0(\boldsymbol{x})} = \frac{\exp[\frac{-1}{2\sigma^2}\sum_{i=1}^n (x_i - \theta_{i0})^2]}{\exp[\frac{-1}{2\sigma^2}\sum_{i=1}^n x_i^2]} = \exp(\frac{-1}{2\sigma^2}[\sum_{i=1}^n (x_i - \theta_{i0})^2 - \sum_{i=1}^n x_i^2]) =$$

$$\exp(\frac{-1}{2\sigma^2}[-2\sum_{i=1}^n x_i\theta_{i0} + \sum_{i=1}^n \theta_{i0}^2]) > k'$$

if $\frac{-1}{2\sigma^2}[-2\sum_{i=1}^n x_i\theta_{i0} + \sum_{i=1}^n \theta_{i0}^2]) > k$" or if $\sum_{i=1}^n x_i\theta_{i0} > k$. Under Ho, $\sum_{i=1}^n X_i\theta_{i0} \sim N(0, \sigma^2 \sum_{i=1}^n \theta_{i0}^2)$. Thus

$$\frac{\sum_{i=1}^n X_i\theta_{i0}}{\sigma\sqrt{\sum_{i=1}^n \theta_{i0}^2}} \sim N(0,1).$$

By Neyman Pearson's lemma, reject Ho if

$$\frac{\sum_{i=1}^n X_i\theta_{i0}}{\sigma\sqrt{\sum_{i=1}^n \theta_{i0}^2}} > z_{1-\alpha}$$

where $P(Z < z_{1-\alpha}) = 1 - \alpha$ when $Z \sim N(0,1)$.

b) The MLE under Ho is $\hat{\theta}_i = 0$ for $i = 1, ..., n$, while the unrestricted MLE is $\hat{\theta}_i = x_i$ for $i = 1, ..., n$ since $\overline{x_i} = x_i$ when the sample size is 1. Hence

$$\lambda(\boldsymbol{x}) = \frac{L(\hat{\theta}_i = 0)}{L(\hat{\theta}_i = x_i)} = \frac{\exp[\frac{-1}{2\sigma^2}\sum_{i=1}^n x_i^2]}{\exp[\frac{-1}{2\sigma^2}\sum_{i=1}^n (x_i - x_i)^2]} = \exp[\frac{-1}{2\sigma^2}\sum_{i=1}^n x_i^2] \le c'$$

if $\frac{-1}{2\sigma^2}\sum_{i=1}^n x_i^2] \le c$", or if $\sum_{i=1}^n x_i^2 \ge c$. Under Ho, $X_i \sim N(0, \sigma^2)$, $X_i/\sigma \sim N(0,1)$, and $\sum_{i=1}^n X_i^2/\sigma^2 \sim \chi^2_n$. So the LRT is reject Ho if $\sum_{i=1}^n X_i^2/\sigma^2 \ge \chi^2_{n,1-\alpha}$ where $P(W \ge \chi^2_{n,1-\alpha}) = 1 - \alpha$ if $W \sim \chi^2_n$.

c) Power = P(reject Ho) =

$$P\left(\frac{n^{-1/3}\sum_{i=1}^n X_i}{\sigma\sqrt{n\ n^{-2/3}}} > z_{1-\alpha}\right) = P\left(\frac{n^{-1/3}\sum_{i=1}^n X_i}{\sigma\ n^{1/6}} > z_{1-\alpha}\right) =$$

$$P\left(\frac{n^{-1/2}\sum_{i=1}^n X_i}{\sigma} > z_{1-\alpha}\right) = P(\sum_{i=1}^n X_i > \sigma\ z_{1-\alpha}\ n^{1/2})$$

where $\sum_{i=1}^n X_i \sim N(n\ n^{-1/3}, n\ \sigma^2) \sim N(n^{2/3}, n\ \sigma^2)$. So

$$\frac{\sum_{i=1}^n X_i - n^{2/3}}{\sqrt{n}\ \sigma} \sim N(0,1), \text{ and power } = \mathrm{P}\left(\frac{\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{X_i}}{\sqrt{\mathrm{n}}\ \sigma} > \mathrm{z}_{1-\alpha}\right) =$$

$$P\left(\frac{\sum_{i=1}^n X_i - n^{2/3}}{\sqrt{n}\ \sigma} > z_{1-\alpha} - \frac{n^{2/3}}{\sqrt{n}\ \sigma}\right) = 1 - \Phi\left(z_{1-\alpha} - \frac{n^{2/3}}{\sqrt{n}\ \sigma}\right) =$$

$$1 - \Phi\left(z_{1-\alpha} - \frac{n^{1/6}}{\sigma}\right) \to 1 - \Phi(-\infty) = 1$$

as $n \to \infty$.

7.31. (Jan. 2014 Qual): Let $X_1, ..., X_m$ be iid from a distribution with pdf

$$f(x) = \mu x^{\mu-1},$$

for $0 < x < 1$ where $\mu > 0$. Let $Y_1, ..., Y_n$ be iid from a distribution with pdf

$$g(y) = \theta y^{\theta-1},$$

for $0 < y < 1$ where $\theta > 0$. Let

$$T_1 = \sum_{i=1}^{m} \log(X_i) \ \text{ and } \ \mathrm{T_2} = \sum_{\mathrm{j}=1}^{\mathrm{n}} \log(\mathrm{Y_i}).$$

Find the likelihood ratio test statistic for $H_0 : \mu = \theta$ versus $H_1 : \mu \neq \theta$ in terms of T_1, T_2 and the MLEs. Simplify.

Solution: $L(\mu) = \mu^m \exp[(\mu - 1) \sum \log(x_i)]$, and
$\log(L(\mu)) = m \log(\mu) + (\mu - 1) \sum \log(x_i)$. Hence

$$\frac{d \log(L(\mu))}{d\mu} = \frac{m}{\mu} + \sum \log(x_i) \stackrel{set}{=} 0.$$

Or $\mu \sum \log(x_i) = -m$ or $\hat{\mu} = -m/T_1$, unique. Now

$$\frac{d^2 \log(L(\mu))}{d\mu^2} = \frac{-m}{\mu^2} < 0.$$

Hence $\hat{\mu}$ is the MLE of μ. Similarly $\hat{\theta} = \dfrac{-n}{\sum_{j=1}^n \log(Y_j)} = \dfrac{-n}{T_2}$. Under H_0 combine the two samples into one sample of size $m + n$ with MLE

$$\hat{\mu}_0 = \frac{-(m+n)}{T_1 + T_2}.$$

Now the likelihood ratio statistic

$$\lambda = \frac{L(\hat{\mu}_0)}{L(\hat{\mu}, \hat{\theta})} = \frac{\hat{\mu}_0^{m+n} \exp[(\hat{\mu}_0 - 1)(\sum \log(X_i) + \sum \log(Y_i))]}{\hat{\mu}^m \hat{\theta}^n \exp[(\hat{\mu} - 1) \sum \log(X_i) + (\hat{\theta} - 1) \sum \log(Y_i)]}$$

$$= \frac{\hat{\mu}_0^{m+n} \exp[(\hat{\mu}_0 - 1)(T_1 + T_2)]}{\hat{\mu}^m \hat{\theta}^n \exp[(\hat{\mu} - 1)T_1 + (\hat{\theta} - 1)T_2]} = \frac{\hat{\mu}_0^{m+n} \exp[-(m+n)] \exp[-(T_1 + T_2)]}{\hat{\mu}^m \hat{\theta}^n \exp(-m) \exp(-n) \exp[-(T_1 + T_2)]}$$

$$= \frac{\hat{\mu}_0^{m+n}}{\hat{\mu}^m \hat{\theta}^n} = \frac{\left(\frac{-(m+n)}{T_1+T_2}\right)^{m+n}}{\left(\frac{-m}{T_1}\right)^m \left(\frac{-n}{T_2}\right)^n}.$$

7.32. (Aug. 2014 Qual): If Z has a half normal distribution, $Z \sim HN(0, \sigma^2)$, then the pdf of Z is

$$f(z) = \frac{2}{\sqrt{2\pi}\ \sigma} \exp\left(\frac{-z^2}{2\sigma^2}\right)$$

where $\sigma > 0$ and $z \geq 0$. Let $X_1, ..., X_n$ be iid HN$(0, \sigma_1^2)$ random variables and let $Y_1, ..., Y_m$ be iid HN$(0, \sigma_2^2)$ random variables that are independent of the X's.

a) Find the α level likelihood ratio test for $H_0 : \sigma_1^2 = \sigma_2^2$ vs. $H_1 : \sigma_1^2 \neq \sigma_2^2$. Simplify the test statistic.

b) What happens if $m = n$?

Solution: a)

$$(\hat{\sigma}_1^2, \hat{\sigma}_2^2) = \left(\frac{\sum_{i=1}^n X_i^2}{n}, \frac{\sum_{i=1}^m Y_i^2}{m}\right)$$

is the MLE of (σ_1^2, σ_2^2), and that under the restriction $\sigma_1^2 = \sigma_2^2 = \sigma_0^2$, say, then the restricted MLE

$$\hat{\sigma}_0^2 = \frac{\sum_{i=1}^n X_i^2 + \sum_{i=1}^m Y_i^2}{n+m}.$$

Now the likelihood ratio statistic

$$\lambda = \frac{L(\hat{\sigma}_0^2)}{L(\hat{\sigma}_1^2, \hat{\sigma}_2^2)} = \frac{\frac{1}{\hat{\sigma}_0^{m+n}} \exp\left[\frac{-1}{2\hat{\sigma}_0^2}(\sum_{i=1}^n x_i^2 + \sum_{i=1}^m y_i^2)\right]}{\frac{1}{\hat{\sigma}_1^n} \exp\left[\frac{-1}{2\hat{\sigma}_1^2} \sum_{i=1}^n x_i^2\right] \frac{1}{\hat{\sigma}_2^m} \exp\left[\frac{-1}{2\hat{\sigma}_2^2} \sum_{i=1}^m y_i^2)\right]}$$

$$= \frac{\frac{1}{\hat{\sigma}_0^{m+n}} \exp\left[-(\frac{n+m}{2})\right]}{\frac{1}{\hat{\sigma}_1^n} \exp(-n/2) \frac{1}{\hat{\sigma}_2^m} \exp(-m/2)} = \frac{\hat{\sigma}_1^n \hat{\sigma}_2^m}{\hat{\sigma}_0^{m+n}}.$$

So reject H_0 if $\lambda(\boldsymbol{x}, \boldsymbol{y}) \leq c$ where $\alpha = \sup_{\sigma^2 \in \Theta_o} P(\lambda(\boldsymbol{X}, \boldsymbol{Y}) \leq c)$. Here Θ_o is the set of $\sigma_1^2 = \sigma_2^2 \equiv \sigma^2$ such that the X_i and Y_i are iid.

b) Then

$$\lambda(\boldsymbol{x}, \boldsymbol{y}) = \frac{\hat{\sigma}_1^n \hat{\sigma}_2^n}{\hat{\sigma}_0^{2n}} \leq c$$

is equivalent to

$$\frac{\hat{\sigma}_1 \hat{\sigma}_2}{\hat{\sigma}_0^2} \leq k.$$

7.33. (Aug. 2016 QUAL): Let $\theta > 0$ be **known**. Let $X_1, ..., X_n$ be independent, identically distributed random variables from a distribution with a pdf

$$f(x) = \frac{\lambda \theta^\lambda}{x^{\lambda+1}}$$

for $x > \theta$ where $\lambda > 0$. Note that $f(x) = 0$ for $x \leq \theta$.

a) Find the UMP (uniformly most powerful) level α test for
$H_0 : \lambda = 1$ vs. $H_1 : \lambda = 2$.

b) If possible, find the UMP level α test for $H_0 : \lambda = 1$ vs. $H_1 : \lambda > 1$.

Solution: ab)

$$f(x) = \frac{I(x > \theta)}{x} \lambda \theta^{\lambda} \exp(\lambda[-\log(x)])$$

is a 1PREF where $w(\lambda) = \lambda$ is increasing. Hence the UMP level α test rejects H_0 if $T(\boldsymbol{x}) = -\sum_{i=1}^n x_i > c$ where $\alpha = P_1(-\sum_{i=1}^n X_i > c)$.

7.34. (Aug. 2016 QUAL): Let $X_1, X_2, \ldots, X_{15}$ denote a random sample from the density function

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} 4x^3 e^{-x^4/\theta} & x > 0; \\ 0 & x \le 0, \end{cases}$$

where $\theta > 0$ is an unknown parameter.

a) Find the rejection region for the most powerful (MP) test of $H_0 : \theta = 2$ against $H_A : \theta = \theta_1$, $\theta_1 > 2$, at $\alpha = .05$. (Hint: $X_i^4/\theta \sim$ Exponential(1).)

b) If you observe $\sum_{i=1}^{15} x_i^4 = 46.98$, what is the p-value?

c) Suppose we decide to reject H_0 at level $\alpha = 0.05$, then what is your decision based on part b)?

d) What is the approximate power of your MP test at $\theta_1 = 5$?

e) Is your MP test also a uniformly most powerful (UMP) test for testing $H_0 : \theta = 2$ versus $H_A : \theta > 2$? Give reasons.

d	δ								
	0.034	0.05	0.1	0.25	0.75	0.9	0.95	0.975	0.99
15	6.68	7.26	8.55	11.04	19.31	22.31	25.00	27.49	30.58
30	17.51	18.49	20.60	24.48	34.80	40.26	43.77	46.98	50.89
40	25.31	26.51	29.05	33.66	47.27	51.81	55.76	59.34	63.69

Solution:

An easier way to do much of this problem is to not that the distribution is a 1PREF with $w(\theta) = -1/\theta$ an increasing function of θ and $t(x) = x^4$. Hence reject H_0 if $\sum_{i=1}^n X_i^4 > c$.

a) We can use the Neyman-Pearson Lemma for specifying the rejection region. Let R represents the rejection region.

$$X \in R \quad \text{if} \quad \frac{f(\mathbf{x}|\theta_1)}{f(\mathbf{x}|\theta_0)} > k \Rightarrow$$

$$\frac{f(\mathbf{x}|\theta_1)}{f(\mathbf{x}|\theta = 2)} = \frac{2^n \prod_{i=1}^{15} 4x_i^3 e^{-x_i^4/\theta_1}}{\theta_1^n \prod_{i=1}^{15} 4x_i^3 e^{-x_i^4/2}} = \left(\frac{2}{\theta_1}\right)^n \exp\left\{\frac{\theta_1 - 2}{2\theta_1} \sum_{i=1}^{15} x_i^4\right\}$$

$$\text{Since } \theta_1 > 2, \text{ then} \quad \left(\frac{2}{\theta_1}\right)^n \exp\left\{\frac{\theta_1 - 2}{2\theta_1} \sum_{i=1}^{15} x_i^4\right\} > k \Leftrightarrow \sum_{i=1}^{15} x_i^4 > c$$

So

$$R = \left\{X : \sum_{i=1}^{15} X_i^4 > c\right\}$$

We should determine the c in such way that the size of the test be equal $\alpha = .05$. i.e.,

$$P_{\theta_0}(X \in R) = \alpha \qquad \Rightarrow P_{\theta_0}(\sum_{i=1}^{15} X_i^4 > c) = .05$$

From the hint, we have $\sum_{i=1}^{15} \frac{X_i^4}{\theta_0} \sim Gamma(15, 1)$ or $2\sum_{i=1}^{15} \frac{X_i^4}{\theta_0} \sim \chi^2_{(30)}$, and since $\theta_0 = 2$, therefore $\sum_{i=1}^{15} X_i^4 \sim \chi^2_{(30)}$; hence we can conclude that $c = \chi^2_{(30,\ 0.05)}$=43.77. Or

$$R = \big\{X : \sum_{i=1}^{15} X_i^4 > 43.77\big\}.$$

b)

$$\text{p-value} = \sup_{\theta \in \Theta_0} P_\theta(\sum_{i=1}^{15} X_i^4 > 46.98) = P_{\theta=2}(\sum_{i=1}^{15} X_i^4 > 46.98) = 0.0249$$

c)

Since $p - value < \alpha$, we reject the null hypothesis in favor of H_A. Or $\sum_{i=1}^{15} X_i^4 = 46.98 > 43.77$, so reject H_0 by a).

d)

Note that the power function is given by

$$\begin{aligned}\beta(\theta) = P_\theta(X \in R) = P_\theta(\sum_{i=1}^{15} X_i^4 > 43.77) &= P(2\sum_{i=1}^{15} \frac{X_i^4}{\theta} > 2\frac{43.77}{\theta}) \\ &= P(W > 2\frac{43.77}{\theta})\end{aligned}$$

where W has chi-square distribution with 30 degree of freedom. Then, we can compute the power of the test as follows

$$\beta(\theta = 5) = P(W > 2\frac{43.77}{5}) = P(W > 17.5) = 0.966$$

e)

Let $T = \sum_{i=1}^{15} X_i^4$, then from the hint, it can be shown that $T \sim Gamma(15, \theta)$, with density function

$$f_T(t|\theta) = \frac{1}{\Gamma(15)} \frac{1}{\theta^{15}} t^{14} e^{-t/\theta}$$

Now, let $\theta_2 > \theta_1$, then it can be shown that the family of pdf $\{f(t|\theta), \theta \in \Theta\}$ has a MLR. That is, the ratio

$$\frac{f_T(t|\theta_2)}{f_T(t|\theta_1)} = \big(\frac{\theta_1}{\theta_2}\big)^{15} e^{t(\frac{1}{\theta_1} - \frac{1}{\theta_2})}$$

is a increasing function of t. Then, by applying the Karlin-Rubin Theorem we can conclude that the MP test given in part a) is UMP test for $H_0 : \theta = 2$ versus $H_A : \theta > 2$.

7.35. (Jan. 2018 Qual): As in Problem 5.55, let $X_1, ..., X_n$ be iid from a Kumaraswamy distribution with pdf

$$f(x) = \theta x^{\theta-1}\beta(1-x^\theta)^{\beta-1}$$

where $\theta > 0$ is **known**, $\beta > 0$, and $0 < x < 1$. Then $Y = -\log(1-X^\theta) \sim EXP(1/\beta)$ with $E(Y) = 1/\beta$, and if $\beta = 0.5$, then $-\sum_{i=1}^n \log(1-X_i^\theta) \sim \chi^2_{2n}$. Find the uniformly most powerful level α test for $H_0 : \beta = 0.5$ versus $H_1 : \beta > 0.5$.

Solution: As in Problem 5.55a), this family is a 1PREF with $w(\beta) = \beta - 1$ increasing and $t(x) = \log(1-x^\theta)$. Hence the UMP level α test rejects H_0 if $\sum_{i=1}^n \log(1-X_i^\theta) > k$ where $\alpha = P_{\beta=0.5}(\sum_{i=1}^n \log(1-X_i^\theta) > k) = P(\chi^2_{2n} < -k)$ with $-k = \chi^2_{2n,\alpha}$ where $P(\chi^2_{2n} < \chi^2_{2n,\alpha}) = \alpha$.

7.36. (Jan. 2019 Qual): Let $X \sim$ binomial $(2, \theta)$. Consider tests for

$$H_0 : \theta = \frac{1}{2} \text{ versus } H_1 : \theta = \frac{3}{4}.$$

Consider the most powerful (MP) test using the Neyman-Pearson Lemma when

(a) $k = .2, .5, 1, 2.2$; find the size of the MP test for each value of k.

(b) Consider $k = .75$; how do you find size of the MP test for this case?

Solution: Using the Neyman Pearson lemma gives reject H_0 if the ratio $\dfrac{3^x}{4} > k$. Take $\gamma = 0$ since we are finding the level $\alpha = P_{1/2}(\text{reject } H_0)$ which depends on k.

Table 1:

x	0	1	2
$P_{1/2}(X = x)$	1/4	2/4	1/4
ratio $\frac{3^x}{4}$	1/4	3/4	9/4

a) Refer to Table 12.1: $k = 0.2 : \alpha = P_{1/2}(X = 0, 1, 2) = 1$ (always reject H_0)
$k = 0.5 : \alpha = P_{1/2}(X = 1, 2) = 3/4$
$k = 1 : \alpha = P_{1/2}(X = 2) = 1/4$
$k = 2.2 : \alpha = P_{1/2}(X = 2) = 1/4$

($k = 2.5 : \alpha = 0$ (never reject H_0))

b) $k = 0.75 : \alpha = P_{1/2}(X = 2) = 1/4$

7.37. (Jan. 2019 Qual): a) Suppose the likelihood function $L(\lambda) = A\lambda^m \prod_{i=1}^m [(x_i - \mu)\exp[-\lambda(R_i + 1)(x_i - \mu)^2]]$ where A, m, μ and the R_i are **known**, $\lambda > 0$, and $x_i > \mu$ for $i = 1, ..., m$. Find the maximum likelihood estimator of λ.

b) Let $W_m = \sum_{i=1}^m (R_i + 1)(x_i - \mu)^2$. It can be shown that $2\lambda W_m \sim \chi^2_{2m}$. Consider a level $\alpha = 0.1$ test of $H_0 : \lambda = 5$ versus $H_A : \lambda \neq 5$. If $m = 20$ and $W_m = 5.7$ would you fail to reject H_0? Explain.

Let $P(\chi^2_d \leq \chi^2_{d,\delta}) = \delta$. The tabled values below give $\chi^2_{d,\delta}$.

d	δ							
	0.01	0.05	0.1	0.25	0.75	0.9	0.95	0.99
20	8.260	10.851	12.443	15.452	23.828	28.412	31.410	37.566
40	22.164	26.509	29.051	33.660	45.616	51.805	55.758	63.691

Solution: a) Show that the MLE is $\hat{\lambda} = m/W_m$ where W_m is defined in b).

b) Fail to reject H_0 if $2\lambda W_m \in [26.509, 55.768]$. Since $2(5)5.7 = 57$, reject H_0.

Note that a χ^2_{40} random variable is approximately symmetric (approximately normal), and to get a rejection region with $\alpha = 0.1$ from the table, you need to use $\delta = 0.05, 0.95$. So reject H_0 if $2\lambda W_m < 26.509$ or if $2\lambda W_m > 55.768$.

7.38. (Jan. 2019 Qual): Suppose $Y_1, ..., Y_n$ are iid from a distribution with probability mass function $f(y) = \dfrac{1}{\theta}$ for $y = 1, ..., \theta$ where θ is a nonnegative integer. Then the likelihood function

$$L(\theta) = \frac{1}{\theta^n} I(\theta \geq Y_{(n)}) I(\theta \in \mathbb{Z})$$

where $Y_{(n)} = \max(Y_1, ..., Y_n)$, $\mathbb{Z}$ is the set of integers, the indicator function $I(\theta \in A) = 1$ if $\theta \in A$, and $I(\theta \in A) = 0$ if $\theta \notin A$. Consider the likelihood ratio test (LRT) for $H_0 : \theta \leq \theta_0$ versus $H_A : \theta > \theta_0$ where θ_0 is a known positive integer.

a) Find the likelihood ratio test statistic if $\theta_0 \geq Y_{(n)}$.

b) Find the likelihood ratio test statistic if $\theta_0 < Y_{(n)}$. Do you reject H_0 or fail to reject H_0 in this case? Explain.

Solution: Note that $L(\theta) > 0$ for $\theta = Y_{(n)}, Y_{(n)} + 1, Y_{(n)} + 2, ...$ Hence the MLE $\hat{\theta} = Y_{(n)}$. If $\theta_0 \geq Y_{(n)}$, then under H_0, $L(\theta) > 0$ for $\theta = Y_{(n)}, Y_{(n)} + 1, Y_{(n)} + 2, ..., L(\theta_0)$. Hence $\hat{\theta}_0 = Y_{(n)}$ if $\theta_0 \geq Y_{(n)}$. If $Y_{(n)} > \theta_0$, then H_0 is not true, but $L(0) = L(1) = \cdots = L(\theta_0) = 0$. Hence $\hat{\theta}_0 = j$ for any $j = 1, ..., \theta_0$ if $Y_{(n)} > \theta_0$. We will take $\hat{\theta}_0 = \theta_0$ if $Y_{(n)} > \theta_0$ since θ_0 will be the j closest to $\hat{\theta}$.

a) If $\theta_0 \geq Y_{(n)}$, then $\lambda(\boldsymbol{y}) = \dfrac{L(\hat{\theta}_0)}{L(Y_{(n)})} = \dfrac{L(Y_{(n)})}{L(Y_{(n)})} = 1.$

b) If $\theta_0 < Y_{(n)}$, then $\lambda(\boldsymbol{y}) = \dfrac{L(\hat{\theta}_0)}{L(Y_{(n)})} = \dfrac{sup_{\theta \in \Theta_0} L(\theta)}{L(Y_{(n)})} = \dfrac{L(\theta_0)}{L(Y_{(n)})} = 0$, and reject H_0.

Note: if H_0 is true, then $P(Y_{(n)} = \theta_0) \to 1$ fast as $n \to \infty$. Under a) fail to reject H_0 and under b) reject H_0 since $\lambda(\boldsymbol{y}) \in [0, 1]$. No error is made under b), and the probability of an error goes to 0 fast as $n \to \infty$ under a).

7.39. (Sept. 2022 Qual): Let (X_i, Y_i) be independent identically distributed random variables with pdf

$$f_{X,Y}(x, y) = \exp\left[-\left(x\theta + \frac{y}{\theta}\right)\right]$$

for $i = 1, ..., n$ where the constant $\theta > 0$, $y > 0$ and $x > 0$.

a) Find the maximum likelihood estimator (MLE) of θ.

b) Give the level α likelihood ratio test (LRT) for the null hypothesis $H_0 : \theta = 1$ versus an alternative hypothesis $H_1 : \theta \neq 1$. Do not find the distribution on the LRT test statistic.

Solution: a)

$$L(\theta) = \prod_{i=1}^n f_{X,Y}(x_i, y_i) = \exp(-\theta \sum_{i=1}^n x_i - \frac{1}{\theta} \sum_{i=1}^n y_i)$$

$$\log(L(\theta)) = -\theta \sum_{i=1}^n x_i - \frac{1}{\theta} \sum_{i=1}^n y_i$$

$$\frac{d\log(L(\theta))}{d\theta} = -\sum_{i=1}^n x_i + \frac{1}{\theta^2} \sum_{i=1}^n y_i \stackrel{set}{=} 0, \text{ or } \theta^2 \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_i} = \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{y_i}$$

Hence

$$\hat{\theta}^2 = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i}, \text{ and } \hat{\theta} = \sqrt{\frac{\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{Y_i}}{\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{X_i}}}, \quad \text{unique.}$$

$$\frac{d^2 \log(L(\theta))}{d\theta^2} = \frac{-2}{\theta^3} \sum_{i=1}^n y_i < 0$$

Thus $\hat{\theta}$ is the MLE.

b) Since $\hat{\theta}_0 = 1$,

$$\lambda = \frac{L(1)}{L(\hat{\theta})} = \frac{\exp(-\sum_{i=1}^n x_i - \sum_{i=1}^n y_i)}{\exp(-\sqrt{\frac{\sum_{i=1}^n y_i}{\sum_{i=1}^n x_i}} \sum_{i=1}^n x_i - \sqrt{\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n y_i}} \sum_{i=1}^n y_i)} =$$

$$\frac{\exp(-\sum_{i=1}^n x_i - \sum_{i=1}^n y_i)}{\exp(-2\sqrt{\sum_{i=1}^n x_i \sum_{i=1}^n y_i})}.$$

Then the α level LRT rejects H_0 if $\lambda \leq c$ where $\alpha = P_1(\lambda \leq c)$.

7.40. (Feb. 2023 Qual): Let $Y_1, ..., Y_n$ be iid from a distribution with probability mass function

$$f(y) = \frac{e^{-\theta}\ \theta^y}{(1 - e^{-\theta})\ y!}$$

for $y = 1, 2, 3, ...$ where $\theta > 0$. Find the uniformly most powerful level α test for $H_0 : \theta = 1$ versus $H_1 : \theta > 1$.

Solution:

$$f(y) = \frac{1}{y!} I[y \in \{1, ...\}]\ \frac{e^{-\theta}}{1 - e^{-\theta}}\ \exp[\log(\theta) y]$$

is a **1P–REF**. Thus $\Theta = (0, \infty)$, $\eta = \log(\theta)$ and $\Omega = (-\infty, \infty)$. Since $w(\theta) = \log(\theta)$ is increasing, the UMP level α test rejects H_0 if $T(\boldsymbol{y}) > k$ and rejects H_0 with probability γ if $T(\boldsymbol{y}) = k$ where $\alpha = P_{\theta_0}(T(\boldsymbol{Y}) > k) + \gamma P_{\theta_0}(T(\boldsymbol{Y}) = k)$ where $\theta_0 = 1$ and $T(\boldsymbol{Y}) = \sum_{i=1}^n Y_i$.

7.41. (Jan. 2024 Qual): Suppose $X_1, ..., X_n$ are iid Uniform $(\theta_1 = \alpha - \beta, \theta_2 = \alpha + \beta)$ random variables with pdf $f(x) = 1/(2\beta)$ if $\alpha - \beta \leq x \leq \alpha + \beta$ where $\theta_1 < \theta_2$, $\beta > 0$, and α is a real number. Then the likelihood $L(\alpha, \beta) = \dfrac{1}{(2\beta)^n} I(\alpha - \beta \leq x_{(1)} \leq x_{(n)} \leq \alpha + \beta)$. Thus $x_{(n)} - x_{(1)} \leq \tau = 2\beta$ and the maximum likelihood estimator of τ is $\hat{\tau} = X_{(n)} - X_{(1)}$. Hence $\hat{\beta} = \hat{\tau}/2$ regardless of the value of α. Then the profile likelihood

$$L_P(\alpha) = L(\alpha, \hat{\beta}) = cI\left(\alpha - \frac{x_{(n)} - x_{(1)}}{2} \leq x_{(1)} \leq x_{(n)} \leq \alpha + \frac{x_{(n)} - x_{(1)}}{2}\right),$$

and it can be shown that the MLE $\hat{\alpha} = (X_{(1)} + X_{(n)})/2$. Then $L(\hat{\alpha}, \hat{\beta}) = 1/[X_{(n)} - X_{(1)}]^n$ since the indicator for $L(\hat{\alpha}, \hat{\beta})$ is equal to one.

Consider the likelihood ratio test (LRT) for the null hypothesis $H_0 : \alpha = 0$ versus an alternative hypothesis $H_1 : \alpha \neq 0$. If H_0 is true, then the X_i are iid $U(-\beta, \beta)$. Then $\hat{\alpha}_0 = 0$ and it can be shown that $\hat{\beta}_0 = Z = \max(-X_{(1)}, X_{(n)})$ with likelihood $L_0(\hat{\alpha}_0, \hat{\beta}_0) = 1/(2Z)^n$ since the indicator is equal to one. Let $\lambda(\boldsymbol{x})$ be the likelihood ratio test statistic. It can be shown that $-2\log\lambda(\boldsymbol{x}) \xrightarrow{D} \chi^2_2$. Assume this approximation is good for $n = 6$. Let the critical value for the following $\alpha = 0.05$ test be $\chi^2_2(0.05) = 5.99$.

Suppose the ordered data from the uniform distribution are $-0.4, -0.3, -0.2, 0.3, 0.4, 0.5$ with $n = 6$. Compute $-2\log\lambda(\boldsymbol{x})$. Does the LRT reject H_0 or fail to reject H_0? Explain.

Solution.

$$\lambda(\boldsymbol{x}) = \frac{L_0(\hat{\alpha}_0, \hat{\beta}_0)}{L(\hat{\alpha}, \hat{\beta})} = \frac{1/(2Z)^n}{1/[X_{(n)} - X_{(1)}]^n} = \left[\frac{X_{(n)} - X_{(1)}}{2Z}\right]^n = \left[\frac{0.5 - (-0.4)}{2\max(0.4, 0.5)}\right]^6$$

$= (0.9)^6 = 0.5314$. Hence $-2\log\lambda(\boldsymbol{x}) = -2\log(0.5314) = 1.2643 < 5.99$. Fail to reject H_0.

7.42. (Aug. 2024 Qual.) Let $Y_1, ..., Y_n$ be iid exponential (λ) random variables where $\lambda > 0$.

a) Find the α level likelihood ratio test (LRT) for $H_0 : \lambda = 1$ vs. $H_1 : \lambda \neq 1$.

b) If $\lambda(\boldsymbol{y})$ is the LRT test statistic of the above test, use the approximation

$$-2\log\lambda(\boldsymbol{y}) \approx \chi^2_d$$

for the appropriate degrees of freedom d to find the rejection region of the test **in useful form** if $\alpha = 0.05$. Use the table shown below.

Let $P(\chi^2_d > \chi^2_{d,\delta}) = \delta$. The tabled values below give $\chi^2_{d,\delta}$ (the upper tail cutoff).

d	δ					
	0.01	0.025	0.05	0.1	0.15	0.25
1	6.63	5.02	3.84	2.71	2.07	1.32
2	9.21	7.38	5.99	4.61	3.79	2.77
3	11.34	9.35	7.81	6.25	5.32	4.11

Solution: a) $\hat{\lambda} = \overline{Y}$, $\hat{\lambda}_0 = 1$, $L(\lambda) = (1/\lambda^n)e^{-\sum y_i/\lambda}$, and

$$\lambda(\boldsymbol{y}) = \frac{L(\hat{\lambda}_0)}{L(\hat{\lambda})} = \frac{e^{-\sum y_i}}{\frac{1}{[\overline{y}]^n}e^{-\sum y_i/\overline{y}}} = \frac{(\overline{y})^n e^{-\sum y_i}}{e^{-n}} = (\overline{y})^n e^{n - n\overline{y}} = (\overline{y})^n e^{n - \sum y_i}.$$

Reject H_0 if $\lambda(\boldsymbol{y}) \leq c$ where $\alpha = P_1(\lambda(\boldsymbol{y}) \leq c)$.

b) $d = 1$, reject H_0 if $-2\log(\lambda(\boldsymbol{y})) > \chi^2_{1,0.05} = 3.84$ where $P(\chi^2_1 > \chi^2_{1,0.05}) = 0.05$.

7.43. (Jan. 2025 Qual): Suppose $X_1, ..., X_n$ are iid EXP(λ) and $Y_1, ..., Y_m$ are iid EXP(2μ) where the X_i and Y_i are independent.

a) If $\mu = \lambda$, then the likelihood

$$L(\lambda) = \frac{1}{\lambda^n}\frac{1}{(2\lambda)^m}e^{-\sum_{i=1}^n X_i/\lambda}e^{-\sum_{j=1}^m Y_j/(2\lambda)} = \frac{1}{2^m}\frac{1}{\lambda^{n+m}}\exp\left[\frac{-1}{2\lambda}\left(2\sum_{i=1}^n X_i + \sum_{j=1}^m Y_j\right)\right].$$

Find the MLE of λ.

b) Find the likelihood ratio test statistic for testing $H_0 : \mu = \lambda$ versus $H_1 : \mu \neq \lambda$. Simplify the test statistic.

Solution. a) $\log(L(\lambda)) = -(n+m)\log(\lambda) - \frac{1}{2\lambda}\left(2\sum_{i=1}^n X_i + \sum_{j=1}^m Y_j\right)$. Thus

$$\frac{d\log(\lambda)}{d\lambda} = \frac{-(n+m)}{\lambda} + \frac{1}{2\lambda^2}\left(2\sum_{i=1}^n X_i + \sum_{j=1}^m Y_j\right) \stackrel{set}{=} 0.$$

So $2\sum_{i=1}^n X_i + \sum_{j=1}^m Y_j = 2(n+m)\lambda$ or $\hat{\lambda} = \dfrac{2\sum_{i=1}^n X_i + \sum_{j=1}^m Y_j}{2(n+m)}$, unique.

$$\frac{d^2\log(\lambda)}{d\lambda^2}\Big|_{\hat{\lambda}} = \frac{n+m}{\hat{\lambda}^2} - \frac{2(n+m)}{\hat{\lambda}^2} = \frac{-(n+m)}{\hat{\lambda}^2} < 0.$$

b)

$$L(\lambda,\mu|\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\lambda^n}\exp(-\sum_i X_i/\lambda)\frac{1}{(2\mu)^m}\exp(-\sum_j Y_j/(2\mu)).$$

Since $\hat{\lambda} = \overline{X}$ and $\widehat{2\mu} = \overline{Y}$, we have $\hat{\mu} = \overline{Y}/2$. Thus

$$L(\hat{\lambda},\hat{\mu}|\boldsymbol{x},\boldsymbol{y}) = \frac{1}{(\overline{X})^n}\exp(-\sum_i X_i/\overline{X})\frac{1}{(2\overline{Y}/2)^m}\exp(-\sum_j Y_j/\overline{Y}) =$$

$$\frac{1}{(\overline{X})^n}\frac{1}{(\overline{Y})^m}e^{-(n+m)}.$$

Now let $\hat{\lambda}_0$ be equal to the $\hat{\lambda}$ given by a).

$$L(\hat{\lambda}_0) = L(\hat{\lambda}_0|\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2^m}\frac{1}{\hat{\lambda}_0^{n+m}}\exp\left[\frac{-1}{2\hat{\lambda}_0}(2\sum_i X_i + \sum_j Y_j)\right] = \frac{1}{2^m}\frac{1}{\hat{\lambda}_0^{n+m}}e^{-(n+m)}.$$

Thus the likelihood ratio test statistic

$$\lambda(\boldsymbol{x},\boldsymbol{y}) = \frac{L(\hat{\lambda}_0|\boldsymbol{x},\boldsymbol{y})}{L(\hat{\lambda},\hat{\mu}|\boldsymbol{x},\boldsymbol{y})} = \frac{1}{2^m}\frac{1}{\hat{\lambda}_0^{n+m}}(\overline{X})^n(\overline{Y})^m.$$

8.3. (Aug. 2003 Qual): Let $X_1, ..., X_n$ be a random sample from a population with pdf

$$f(x) = \begin{cases} \frac{\theta x^{\theta-1}}{3^\theta} & 0 < x < 3 \\ 0 & \text{elsewhere} \end{cases}$$

The method of moments estimator for θ is $T_n = \dfrac{\overline{X}}{3-\overline{X}}$.

a) Find the limiting distribution of $\sqrt{n}(T_n - \theta)$ as $n \to \infty$.

b) Is T_n asymptotically efficient? Why?

c) Find a consistent estimator for θ and show that it is consistent.

Solution. a) $E(X) = \frac{3\theta}{\theta+1}$, thus
$\sqrt{n}(\overline{X} - E(X)) \xrightarrow{D} N(0, V(X))$, where
$V(X) = \frac{9\theta}{(\theta+2)(\theta+1)^2}$. Let $g(y) = \frac{y}{3-y}$, thus $g'(y) = \frac{3}{(3-y)^2}$. Using the delta method,
$\sqrt{n}(T_n - \theta) \xrightarrow{D} N(0, \frac{\theta(\theta+1)^2}{\theta+2})$.

b) It is asymptotically efficient if $\sqrt{n}(T_n - \theta) \xrightarrow{D} N(0, \nu(\theta))$, where

$$\nu(\theta) = \frac{\frac{d}{d\theta}(\theta)}{-E(\frac{d^2}{d\theta^2} lnf(x|\theta))}.$$

But, $E((\frac{d^2}{d\theta^2} lnf(x|\theta)) = \frac{1}{\theta^2}$. Thus $\nu(\theta) = \theta^2 \neq \frac{\theta(\theta+1)^2}{\theta+2}$.

c) $\overline{X} \to \frac{3\theta}{\theta+1}$ in probability. Thus $T_n \to \theta$ in probability.

8.8. (Sept. 2005 Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables with probability density function

$$f(x) = \theta x^{\theta-1}, \ \ 0 < x < 1, \ \ \theta > 0.$$

a) Find the MLE of $\dfrac{1}{\theta}$. Is it unbiased? Does it achieve the information inequality lower bound?

b) Find the asymptotic distribution of the MLE of $\dfrac{1}{\theta}$.

c) Show that $\overline{X}_n$ is unbiased for $\dfrac{\theta}{\theta+1}$. Does $\overline{X}_n$ achieve the information inequality lower bound?

d) Find an estimator of $\dfrac{1}{\theta}$ from part (c) above using $\overline{X}_n$ which is different from the MLE in (a). Find the asymptotic distribution of your estimator using the delta method.

e) Find the asymptotic relative efficiency of your estimator in (d) with respect to the MLE in (b).

8.14. (Sept. 2022, Aug. 2018, Aug. 2015 Quals): Let $X_1, ..., X_n$ be iid with cdf $F(x) = P(X \leq x)$. Let $Y_i =$
$I(X_i \leq x)$ where the indicator equals 1 if $X_i \leq x$ and 0, otherwise.

a) Find $E(Y_i)$.

b) Find VAR(Y_i).

c) Let $\hat{F}_n(x) = \dfrac{1}{n}\sum_{i=1}^{n} I(X_i \leq x)$ for some fixed real number x. Find the limiting distribution of $\sqrt{n}\left(\hat{F}_n(x) - c_x\right)$ for an appropriate constant c_x.

Solution: $Y_i \sim bin(n=1, F(x))$ since an indicator random variable Y_i takes on values 0 and 1, so $Y_i \sim bin(n=1,p)$ with $p =$ to the probability of the indicator event: $p = P(X_i \leq x) = F(x)$.

a) $E(Y_i) = np = 1F(x) = F(X)$

b) $V(Y_i) = p(1-p) = F(x)[1-F(x)]$

c) Then

$$\sqrt{n}\left(\hat{F}_n(x) - F(x)\right) \xrightarrow{D} N(0, F(x)[1-F(x)])$$

by the CLT since $\hat{F}_n(x) = \overline{Y}_n$ and the Y_i are iid.

8.17. (Jan. 2025 Qual): Suppose that $Y_1, ..., Y_n$ are iid with $E(Y) = (1-\rho)/\rho$ and $\text{VAR}(Y) = (1-\rho)/\rho^2$ where $0 < \rho < 1$. a) Find the limiting distribution of $\sqrt{n}\left(\overline{Y}_n - \dfrac{1-\rho}{\rho}\right)$.

b) Find the limiting distribution of $\sqrt{n}\ \left[\ g(\overline{Y}_n) - \ \rho\right]$ for appropriate function g. Hint: want $g(\mu)$ such that $g((1-\rho)/\rho) = \rho$. So set $\mu = (1-\rho)/\rho$ and solve for $\rho = g(\mu)$.

Solution.

a) $\xrightarrow{D} N\left(0, \dfrac{1-\rho}{\rho^2}\right)$ by the CLT.

b) $\mu = \dfrac{1-\rho}{\rho} = \dfrac{1}{\rho} - 1$ so $\rho = \dfrac{1}{1+\mu} = g(\mu)$. By the delta method, $\sqrt{n}(g(\overline{Y}) - g(\mu)) \xrightarrow{D} N(0, [g'(\mu)]^2\tau^2)$ where $\tau^2 = (1-\rho)/\rho^2$. Now $g'(\mu) = \dfrac{d}{d\mu}(1+\mu)^{-1} = \dfrac{-1}{(1+\mu)^2}$. Thus

$$\left[g'\left(\frac{1-\rho}{\rho}\right)\right]^2 = \left[\frac{1}{1+\frac{1-\rho}{\rho}}\right]^4 = \frac{1}{(1+\frac{1}{\rho}-1)^4} = \rho^4.$$

Since $g(\mu) = \rho$, $\sqrt{n}\ \left[\ g(\overline{Y}_n) - \ \rho\right] \xrightarrow{D} N\left(0, \dfrac{\rho^4}{\rho^2}(1-\rho)\right) \sim N(0, \rho^2(1-\rho))$.

8.27. (Sept. 2005 Qual): Let $X \sim \text{Binomial}(n,p)$ where the positive integer n is large and $0 < p < 1$.

a) Find the limiting distribution of $\sqrt{n}\ \left(\ \dfrac{X}{n} - \ p\ \right)$.

b) Find the limiting distribution of $\sqrt{n}\ \left[\ \left(\dfrac{X}{n}\right)^2 - \ p^2\ \right]$.

c) Show how to find the limiting distribution of $\left[\ \left(\dfrac{X}{n}\right)^3 - \dfrac{X}{n}\ \right]$ when $p = \dfrac{1}{\sqrt{3}}$.

(Actually want the limiting distribution of

$$n\ \left(\left[\ \left(\frac{X}{n}\right)^3 - \frac{X}{n}\ \right] - g(p)\right)$$

where $g(\theta) = \theta^3 - \theta$.)

Solution. a) $X \stackrel{D}{=} \sum_{i=1}^n Y_i$ where $Y_1, ..., Y_n$ are iid Ber(ρ). Hence

$$\sqrt{n}(\frac{X}{n} - \rho) \stackrel{D}{=} \sqrt{n}(\overline{Y}_n - \rho) \xrightarrow{D} N(0, \rho(1-\rho)).$$

b) Let $g(p) = p^2$. Then $g'(p) = 2p$ and by the delta method and a),

$$\sqrt{n} \left[\left(\frac{X}{n}\right)^2 - p^2 \right] = \sqrt{n} \left(g(\frac{X}{n}) - g(p) \right) \xrightarrow{D}$$

$$N(0, p(1-p)(g'(p))^2) = N(0, p(1-p)4p^2) = N(0, 4p^3(1-p)).$$

c) Refer to a) and Theorem 8.30. Let $\theta = p$. Then $g'(\theta) = 3\theta^2 - 1$ and $g''(\theta) = 6\theta$. Notice that

$$g(1/\sqrt{3}) = (1/\sqrt{3})^3 - 1/\sqrt{3} = (1/\sqrt{3})(\frac{1}{3} - 1) = \frac{-2}{3\sqrt{3}} = c.$$

Also $g'(1/\sqrt{3}) = 0$ and $g''(1/\sqrt{3}) = 6/\sqrt{3}$. Since $\tau^2(p) = p(1-p)$,

$$\tau^2(1/\sqrt{3}) = \frac{1}{\sqrt{3}}(1 - \frac{1}{\sqrt{3}}).$$

Hence

$$n \left[g\left(\frac{X_n}{n}\right) - \left(\frac{-2}{3\sqrt{3}}\right) \right] \xrightarrow{D} \frac{1}{2}\frac{1}{\sqrt{3}}(1 - \frac{1}{\sqrt{3}})\frac{6}{\sqrt{3}}\ \chi_1^2 = (1 - \frac{1}{\sqrt{3}})\ \chi_1^2.$$

8.28. (Aug. 2004 Qual): Let $X_1, ..., X_n$ be independent and identically distributed (iid) from a Poisson(λ) distribution.

a) Find the limiting distribution of $\sqrt{n}\ (\ \overline{X} - \lambda\ )$.

b) Find the limiting distribution of $\sqrt{n}\ [\ (\overline{X})^3 - (\lambda)^3\ ]$.

Solution. a) By the CLT, $\sqrt{n}(\overline{X} - \lambda)/\sqrt{\lambda} \xrightarrow{D} N(0, 1)$. Hence $\sqrt{n}(\overline{X} - \lambda) \xrightarrow{D} N(0, \lambda)$.

b) Let $g(\lambda) = \lambda^3$ so that $g'(\lambda) = 3\lambda^2$ then $\sqrt{n}[(\overline{X})^3 - (\lambda)^3] \xrightarrow{D} N(0, \lambda[g'(\lambda)]^2) = N(0, 9\lambda^5)$.

8.29. (Jan. 2004 Qual): Let $X_1, ..., X_n$ be iid from a normal distribution with unknown mean μ and known variance σ^2. Let $\overline{X} = \frac{\sum_{i=1}^n X_i}{n}$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X})^2$.

a) Show that $\overline{X}$ and S^2 are independent.

b) Find the limiting distribution of $\sqrt{n}((\overline{X})^3 - c)$ for an appropriate constant c.

Solution. a) $\overline{X}$ is a complete sufficient statistic. Also, we have $\frac{(n-1)S^2}{\sigma^2}$ has a chi square distribution with $df = n - 1$, thus since σ^2 is known the distribution of S^2 does not depend on μ, so S^2 is ancillary. Thus, by Basu's Theorem $\overline{X}$ and S^2 are independent.

b) by CLT (n is large) $\sqrt{n}(\overline{X} - \mu)$ has approximately normal distribution with mean 0 and variance σ^2. Let $g(x) = x^3$, thus, $g^{'}(x) = 3x^2$. Using delta method $\sqrt{n}(g(\overline{X}) - g(\mu))$ goes in distribution to $N(0, \sigma^2(g^{'}(\mu))^2)$ or $\sqrt{n}(\overline{X}^3 - \mu^3)$ goes in distribution to $N(0, \sigma^2(3\mu^2)^2)$.

8.34. (abc Jan. 2010 Qual): Let $Y_1, ..., Y_n$ be independent and identically distributed (iid) from a distribution with probability mass function $f(y) = \rho(1-\rho)^y$ for $y = 0, 1, 2, ...$ and $0 < \rho < 1$. Then $E(Y) = (1-\rho)/\rho$ and $\text{VAR}(Y) = (1-\rho)/\rho^2$.

a) Find the limiting distribution of $\sqrt{n}\ \left(\overline{Y} - \dfrac{1-\rho}{\rho} \right)$.

b) Show how to find the limiting distribution of $g(\overline{Y}) = \frac{1}{1+\overline{Y}}$. Deduce it completely. (This bad notation means find the limiting distribution of $\sqrt{n}(g(\overline{Y}) - c)$ for some constant c.)

c) Find the method of moments estimator of ρ.

d) Find the limiting distribution of $\sqrt{n}\ \left((1 + \overline{Y}) - \ d \right)$ for appropriate constant d.

e) Note that $1 + E(Y) = 1/\rho$. Find the method of moments estimator of $1/\rho$.

Solution. a)

$$\sqrt{n}\ \left(\overline{Y} - \frac{1-\rho}{\rho} \right) \xrightarrow{D} N\left(0, \frac{1-\rho}{\rho^2}\right)$$

by the CLT.

c) The method of moments estimator of ρ is $\hat{\rho} = \frac{1}{1+\overline{Y}}$.

d) Let $g(\theta) = 1 + \theta$ so $g'(\theta) = 1$. Then by the delta method,

$$\sqrt{n}\ \left(g(\overline{Y}) - \ g(\frac{1-\rho}{\rho}) \right) \xrightarrow{D} N\left(0, \frac{1-\rho}{\rho^2}1^2\right)$$

or

$$\sqrt{n}\ \left((1 + \overline{Y}) - \ \frac{1}{\rho} \right) \xrightarrow{D} N\left(0, \frac{1-\rho}{\rho^2}\right).$$

This result could also be found with algebra since $1 + \overline{Y} - \frac{1}{\rho} = \overline{Y} + 1 - \frac{1}{\rho} = \overline{Y} + \frac{\rho - 1}{\rho} = \overline{Y} - \frac{1-\rho}{\rho}$.

e) $\overline{Y}$ is the method of moments estimator of $E(Y) = (1-\rho)/\rho$, so $1 + \overline{Y}$ is the method of moments estimator of $1 + E(Y) = 1/\rho$.

8.35. (Sept. 2010 Qual): Let $X_1, ..., X_n$ be independent identically distributed random variables from a normal distribution with mean μ and variance σ^2.

a) Find the approximate distribution of $1/\bar{X}$. Is this valid for all values of μ?

b) Show that $1/\bar{X}$ is asymptotically efficient for $1/\mu$, provided $\mu \neq \mu^*$. Identify μ^*.

Solution. a) $\sqrt{n}(\bar{X}-\mu)$ is approximately $N(0,\sigma^2)$ Define $g(x)=\frac{1}{x}$, $g'(x)=\frac{-1}{x^2}$. Using delta method $\sqrt{n}(\frac{1}{\bar{X}}-\frac{1}{\mu})$ has approximately $N(0,\frac{\sigma^2}{\mu^4})$. Thus $1/\overline{X}$ is approximately $N(\frac{1}{\mu},\frac{\sigma^2}{n\mu^4})$, provided $\mu \neq 0$.

b) Using part a) $\frac{1}{\bar{X}}$ is asymptotically efficient for $\frac{1}{\mu}$ if

$$\frac{\sigma^2}{\mu^4} = \left[\frac{\left(\tau'(\mu)\right)^2}{E_\mu\left(\frac{\partial}{\partial\mu}\ln f(X/\mu)\right)^2}\right]$$

$\tau(\mu)=\frac{1}{\mu}$
$\tau'(\mu)=\frac{-1}{\mu^2}$
$\ln f(x|\mu)=\frac{-1}{2}\ln 2\pi\sigma^2-\frac{(x-\mu)^2}{2\sigma^2}$

$$E\left[\frac{\partial}{\partial\mu}\ln f(X/\mu)\right]^2 = \frac{E(X-\mu)^2}{\sigma^4}=\frac{1}{\sigma^2}$$

Thus

$$\frac{\left(\tau'(\mu)\right)^2}{E_\mu\left[\frac{\partial}{\partial\mu}\ln f(X/\mu)\right]^2}=\frac{\sigma^2}{\mu^4}.$$

8.36. (Jan. 2011 Qual): Let $Y_1,...,Y_n$ be independent and identically distributed (iid) from a distribution with probability density function

$$f(y)=\frac{2y}{\theta^2}$$

for $0<y\le\theta$ and $f(y)=0$, otherwise.

a) Find the limiting distribution of $\sqrt{n}\ \left(\ \overline{Y}-c\ \right)$ for appropriate constant c.

b) Find the limiting distribution of $\sqrt{n}\ \left(\ \log(\ \overline{Y}\ )-\ d\ \right)$ for appropriate constant d.

c) Find the method of moments estimator of θ^k.

Solution. a) $E(Y^k)=2\theta^k/(k+2)$ so $E(Y)=2\theta/3$, $E(Y^2)=\theta^2/2$ and $V(Y)=\theta^2/18$. So $\sqrt{n}\ \left(\ \overline{Y}-\frac{2\theta}{3}\ \right)\xrightarrow{D} N\left(0,\frac{\theta^2}{18}\right)$ by the CLT.

b) Let $g(\tau)=\log(\tau)$ so $[g'(\tau)]^2=1/\tau^2$ where $\tau=2\theta/3$. Then by the delta method, $\sqrt{n}\ \left(\ \log(\overline{Y})-\log\left(\frac{2\theta}{3}\right)\ \right)\xrightarrow{D} N\left(0,\frac{1}{8}\right)$.

c) $\hat{\theta}^k=\frac{k+2}{2n}\sum Y_i^k$.

8.37. (Jan. 2013 Qual): Let $Y_1,...,Y_n$ be independent identically distributed discrete random variables with probability mass function

$$f(y)=P(Y=y)=\binom{r+y-1}{y}\rho^r(1-\rho)^y$$

for $y = 0, 1, \ldots$ where positive integer r **is known** and $0 < \rho < 1$. Then $E(Y) = r(1-\rho)/\rho$, and $V(Y) = r(1-\rho)/\rho^2$.

a) Find the limiting distribution of $\sqrt{n}\ \left(\ \overline{Y} - \ \frac{r(1-\rho)}{\rho}\ \right)$.

b) Let $g(\overline{Y}) = \frac{r}{r + \overline{Y}}$. Find the limiting distribution of $\sqrt{n}\ \left(\ g(\overline{Y}) - \ c\ \right)$ for appropriate constant c.

c) Find the method of moments estimator of ρ.

Solution: a) $\sqrt{n}\ \left(\ \overline{Y} - \ \frac{r(1-\rho)}{\rho}\ \right) \xrightarrow{D} N\left(0, \frac{r(1-\rho)}{\rho^2}\right)$ by the CLT.

b) Let $\theta = r(1-\rho)/\rho$. Then

$$g(\theta) = \frac{r}{r + \frac{r(1-\rho)}{\rho}} = \frac{r\rho}{r\rho + r(1-\rho)} = \rho = c.$$

Now

$$g'(\theta) = \frac{-r}{(r+\theta)^2} = \frac{-r}{(r + \frac{r(1-\rho)}{\rho})^2} = \frac{-r\rho^2}{r^2}.$$

So

$$[g'(\theta)]^2 = \frac{r^2\rho^4}{r^4} = \frac{\rho^4}{r^2}.$$

Hence by the delta method

$$\sqrt{n}\ (\ g(\overline{Y}) - \rho\ ) \xrightarrow{D} N\left(0, \frac{r(1-\rho)}{\rho^2}\frac{\rho^4}{r^2}\right) = N\left(0, \frac{\rho^2(1-\rho)}{r}\right).$$

c) $\overline{Y} \stackrel{\text{set}}{=} r(1-\rho)/\rho$ or $\rho\overline{Y} = r - r\rho$ or $\rho\overline{Y} + r\rho = r$ or $\hat{\rho} = r/(r + \overline{Y})$.

8.38. (Aug. 2013 Qual): Let $X_1, ..., X_n$ be independent identically distributed uniform $(0, \theta)$ random variables where $\theta > 0$.

a) Find the limiting distribution of $\sqrt{n}(\overline{X} - c_\theta)$ for an appropriate constant c_θ that may depend on θ.

b) Find the limiting distribution of $\sqrt{n}[(\overline{X})^2 - k_\theta]$ for an appropriate constant k_θ that may depend on θ.

Solution: a) By the CLT,

$$\sqrt{n}\ \left(\ \overline{X} - \frac{\theta}{2}\ \right) \xrightarrow{D} N\left(0, \frac{\theta^2}{12}\right).$$

b) Let $g(y) = y^2$. Then $g'(y) = 2y$ and by the delta method,

$$\sqrt{n}\ \left(\overline{X}^2 - (\frac{\theta}{2})^2\ \right) = \sqrt{n}\ \left(\overline{X}^2 - \frac{\theta^2}{4}\ \right) = \sqrt{n}\ \left(\ g(\overline{X}) - g(\frac{\theta}{2})\ \right) \xrightarrow{D}$$

$$N\left(0, \frac{\theta^2}{12}\ [g'(\frac{\theta}{2})]^2\right) = N\left(0, \frac{\theta^2}{12}\ \frac{4\theta^2}{4}\right) = N\left(0, \frac{\theta^4}{12}\right).$$

8.39. (Aug. 2014 Qual): Let $X_1, ..., X_n$ be independent identically distributed (iid) beta(β, β) random variables.

a) Find the limiting distribution of $\sqrt{n}(\ \overline{X}_n - \theta\ )$, for appropriate constant θ.

b) Find the limiting distribution of $\sqrt{n}(\ \log(\overline{X}_n) - d\ )$, for appropriate constant d.

Solution. a) $E(X_i) = \beta/(\beta + \beta) = 1/2$ and $V(X_i) = \dfrac{\beta^2}{(2\beta)^2(2\beta+1)} = \dfrac{1}{4(2\beta+1)} = \dfrac{1}{8\beta+4}$. So

$$\sqrt{n}\left(\ \overline{X}_n - \frac{1}{2}\ \right) \xrightarrow{D} N\left(0, \frac{1}{8\beta+4}\right)$$

by the CLT.

b) Let $g(x) = \log(x)$. So $d = g(1/2) = \log(1/2)$. Now $g'(x) = 1/x$ and $(g'(x))^2 = 1/x^2$. So $(g'(1/2))^2 = 4$. So

$$\sqrt{n}(\ \log(\overline{X}_n) - \log(1/2)\ ) \xrightarrow{D} N\left(0, \frac{1}{8\beta+4}\ 4\right) = N\left(0, \frac{1}{2\beta+1}\right)$$

by the delta method.

8.40. (Jan. 2018 Qual): Let $X_1, ..., X_n$ be a random sample of size n from $U(\theta, 2\theta)$.

a) Find a minimal sufficient statistic for θ.

b) Is the minimal sufficient statistic found in part (a) complete? Please justify your answer.

c) Find the limiting distribution of $\sqrt{n}(\overline{X} - c)$ for an appropriate constant c.

d) Find the limiting distribution of $\sqrt{n}(\log(\overline{X}) - d)$ for an appropriate constant d.

e) Let $T = a\overline{X}$ be an estimator of θ where a is a constant. Find the value a such that minimizes the mean square error (MSE). Show that your answer is the minimizer.

Solution: a) Suppose statistic $T(\mathbf{X})$ is a minimal sufficient statistics, then the ratio $\frac{f(\mathbf{X}|\theta)}{f(\mathbf{Y}|\theta)}$ does not depend on θ if and only if $T(\mathbf{X}) = T(\mathbf{Y})$. Here, the ratio is given as

$$\frac{f(\mathbf{X}|\theta)}{f(\mathbf{Y}|\theta)} = \frac{(1/\theta)^n I_{\{X_{(n)}<2\theta\}} I_{\{X_{(1)}>\theta\}}}{(1/\theta)^n I_{\{Y_{(n)}<2\theta\}} I_{\{Y_{(1)}>\theta\}}}.$$

This ratio does not depend on θ if and only if $X_{(1)} = Y_{(1)}$ and $X_{(n)} = Y_{(n)}$. Therefore, $T(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is a minimal sufficient statistics.

b) $T(\mathbf{X}) = (X_{(1)}, X_{(n)})$ obtained in part (a) is not complete, since

$$E(\frac{n+1}{n+2}X_{(1)} - \frac{n+1}{2n+1}X_{(n)}) = 0$$

but

$$P(\frac{n+1}{n+2}X_{(1)} - \frac{n+1}{2n+1}X_{(n)} = 0) \neq 1.$$

c) We have $E[X] = \frac{3\theta}{2}$ and $Var(X) = \frac{\theta^2}{12}$.

Therefore $\sqrt{n}(\overline{X} - \frac{3\theta}{2}) \xrightarrow{D} N(0, \frac{\theta^2}{12})$ by the CLT.

d) Let $g(x) = \log(x)$ so $(g'(x))^2 = 1/x^2$ where $x = \frac{3\theta}{2}$. Then by using the delta method we have

$$\sqrt{n}\big(\log(\overline{X}) - \log(\frac{3\theta}{2})\big) \xrightarrow{D} N(0, \frac{1}{27}).$$

e) Note that

$$\begin{aligned} MSE(T) &= Var(T) + (E[T] - \theta)^2 \\ &= Var(a\overline{X}) + (E[a\overline{X}] - \theta)^2 \\ &= a^2\frac{\theta^2}{12n} + (\frac{3a}{2}\theta - \theta)^2 = \frac{a^2\theta^2}{12n} + \frac{(3a-2)^2}{4}\theta^2. \end{aligned}$$

then, we can minimizes MSE with respect to a as follows

$$\frac{dMSE(T)}{da} = \frac{2a\theta^2}{12n} + \frac{6(3a-2)}{4}\theta^2$$

by setting the derivative equal to zero and solving, we get $a = \frac{18n}{1+27n}$. This is a global minimum, because

$$\frac{d^2MSE(T)}{da^2} = \frac{2\theta^2}{12n} + \frac{18}{4}\theta^2 > 0.$$

8.41. Let $Y_n \sim$ Poisson(n).

a) Find the limiting distribution of $\sqrt{n}\ \left(\frac{Y_n}{n} - 1\right)$.

b) Find the limiting distribution of $\sqrt{n}\ \left[\left(\frac{Y_n}{n}\right)^2 - 1\right]$.

Solution. a) Let $Y_n \stackrel{D}{=} \sum\limits_{i=1}^{n} X_i$, where X_i are iid Poisson(1), then by central limit theorem, we have

$$\sqrt{n}\ \left(\frac{Y_n}{n} - 1\right) \xrightarrow{D} N(0,1).$$

b) Let $g(t) = t^2, g'(t) = 2t \neq 0$. Using the Delta method, we have

$\sqrt{n}\ \left[\left(\frac{Y_n}{n}\right)^2 - 1\right] \xrightarrow{D} N(0, 1(2\cdot 1)^2) \sim N(0,4)$.

8.42. (Sept. 2022 Qual): Let $Y_1, ..., Y_n$ be iid uniform $U(\theta, 2\theta)$ for $\theta > 0$ and iid $U(2\theta, \theta)$ for $\theta < 0$.

a) Find the limiting distribution of $\sqrt{n}[\ \overline{Y} - c]$ for appropriate constant c.

b) Find the limiting distribution of $\sqrt{n}[\ (\overline{Y})^2 - d]$ for appropriate constant d.

Solution. $E(Y) = 3\theta/2$ and $V(Y) = \theta^2/12$.

a) $\sqrt{n}(\overline{Y} - 3\theta/2) \xrightarrow{D} N(0, \theta^2/12)$ by the CLT

b) Let $g(\mu) = \mu^2$, $g'(\mu) = 2\mu$, and $g'(3\theta/2) = 3\theta$. Then by the delta method,

$$\sqrt{n}[\ (\overline{Y})^2 - g(3\theta/2)] \xrightarrow{D} N(0, [g'(3\theta/2)]^2\theta^2/12), \text{ or}$$

$$\sqrt{n}\left[\ (\overline{Y})^2 - \frac{9\theta^2}{4}\right] \xrightarrow{D} N\left(0, \frac{9\theta^2\theta^2}{12}\right) \sim N\left(0, \frac{9\theta^4}{12}\right) \sim N\left(0, \frac{3\theta^4}{4}\right).$$

8.43. (Jan. 2024 Qual): Let $\boldsymbol{x}_1, ..., \boldsymbol{x}_k$ be iid with $E(\boldsymbol{x}) = \boldsymbol{\mu}$ where $\boldsymbol{x}$ is $p \times 1$. Let $n = \text{floor(k/2)} = \lfloor \text{k/2} \rfloor$ be the integer part of $k/2$. So floor(100/2) = floor(101/2) = 50. Let the iid random variables $W_i = \boldsymbol{x}_{2i-1}^T \boldsymbol{x}_{2i}$ for $i = 1, ..., n$. Hence $W_1, W_2, ..., W_n = \boldsymbol{x}_1^T \boldsymbol{x}_2, \boldsymbol{x}_3^T \boldsymbol{x}_4, ..., \boldsymbol{x}_{2n-1}^T \boldsymbol{x}_{2n}$. Then $E(W_i) = \boldsymbol{\mu}^T \boldsymbol{\mu} = \theta \geq 0$ and $V(W_i) = \sigma_W^2$.

a) Find the limiting distribution of $\sqrt{n}(\overline{W} - \theta)$.

b) Find the limiting distribution of $\sqrt{n}(\sqrt{\overline{W}} - \sqrt{\theta})$.

Solution.

a) $\sqrt{n}(\overline{W} - \theta) \xrightarrow{D} N(0, \sigma_W^2)$ by the CLT.

b) Let $g(\theta) = \sqrt{\theta}$ with $g'(\theta) = 0.5\theta^{-0.5}$. Then $\sqrt{n}(\sqrt{\overline{W}} - \sqrt{\theta}) \xrightarrow{D} N(0, \sigma_W^2[g'(\theta)]^2) \sim N(0, 0.25\sigma_W^2/\theta)$ by the delta method provided $\theta > 0$.

8.44. (Aug. 2024 Qual): Let $W_1, \ldots, W_n$ be iid random variables with probability density function (pdf)

$$f(w|\lambda) = \frac{3w^2}{\lambda} e^{-w^3/\lambda} \text{ if } w > 0,$$

and $f(w|\lambda) = 0$, elsewhere, where $\lambda > 0$. Use $E(W^3) = \lambda$,

$$\mu = E(W) = \frac{1}{3}\Gamma(1/3)\ \lambda^{1/3}, \text{ and } \sigma^2 = V(W) = \left(\frac{2}{3}\Gamma(2/3) - \frac{1}{9}[\Gamma(1/3)]^2\right)\lambda^{2/3}.$$

a) Find the method of moments estimator $\hat{\lambda}_{MM}$ of λ based on $W_1, \ldots, W_n$.

b) Give the asymptotic behavior of $\sqrt{n}(\hat{\lambda}_{MM} - \lambda)$ as $n \to \infty$. Derive the answer which is given in simplified form in e). Hint: use the Delta Method with

$$g(\mu) = \left[\frac{3\mu}{\Gamma(1/3)}\right]^3 = \lambda.$$

c) Find the maximum likelihood estimator $\hat{\lambda}_{MLE}$ of λ based on $W_1, \ldots, W_n$.

d) Give the asymptotic behavior of $\sqrt{n}(\hat{\lambda}_{MLE} - \lambda)$ as $n \to \infty$.

e) Find the asymptotic relative efficiency of $\hat{\lambda}_{MM}$ relative to $\hat{\lambda}_{MLE}$. (Assume $\sqrt{n}(\hat{\lambda}_{MM} - \lambda) \xrightarrow{D} N(0, 1.18884\lambda^2)$.)

Solution. a)

$$\overline{W} \stackrel{set}{=} \frac{1}{3}\Gamma(1,3)\lambda^{1/3}$$

gives

$$\hat{\lambda}_{MM} = \left[\frac{3\overline{W}}{\Gamma(1/3)}\right]^3.$$

b) By the CLT, $\sqrt{n}(\overline{W} - \mu) \xrightarrow{D} N(0, \sigma^2)$. By the Delta Method, $\sqrt{n}(g(\overline{W}) - g(\mu)) = \sqrt{n}(\hat{\lambda}_{MM} - \lambda) \xrightarrow{D} N(0, [g'(\mu)]^2\sigma^2)$. Now

$$g'(\mu) = 3\left[\frac{3\mu}{\Gamma(1/3)}\right]^2 \frac{3}{\Gamma(1/3)} = \frac{81\mu^2}{[\Gamma(1/3)]^3} = \frac{81}{[\Gamma(1/3)]^3}\left[\frac{1}{3}\Gamma(1/3)\lambda^{1/3}\right]^2 = \frac{9}{\Gamma(1/3)}\lambda^{2/3}.$$

Thus

$$[g'(\mu)]^2\sigma^2 = \left[\frac{9}{\Gamma(1/3)}\lambda^{2/3}\right]^2 \sigma^2 = \frac{81}{[\Gamma(1/3)]^2}\lambda^{4/3}\left(\frac{2}{3}\Gamma(2/3) - \frac{1}{9}[\Gamma(1/3)]^2\right)\lambda^{2/3} =$$

$$\frac{81}{[\Gamma(1/3)]^2}\left(\frac{2}{3}\Gamma(2/3) - \frac{1}{9}[\Gamma(1/3)]^2\right)\lambda^2 = c\lambda^2.$$

Thus $\sqrt{n}(\hat{\lambda}_{MM} - \lambda) \xrightarrow{D} N(0, c\lambda^2)$.

c) Note that $L(\lambda) = (a/\lambda^n)e^{-\sum w_i^3/\lambda}$, and $\log(L(\lambda)) = d - n\log(\lambda) - \sum w_i^3/\lambda$. Thus

$$\frac{d\ \log(L(\lambda))}{d\lambda} = -n/\lambda + \sum w_i^3/\lambda^2 \stackrel{set}{=} 0,$$

or $\hat{\lambda}_{MLE} = \frac{1}{n}\sum_{i=1}^n W_i^3$, unique. Now

$$\frac{d^2\ \log(L(\lambda))}{d\lambda^2} = n/\lambda^2 - 2\sum w_i^3/\lambda^3 \ \ |_{\hat{\lambda}} = n/\hat{\lambda}^2 - 2n\hat{\lambda}/\hat{\lambda}^3 = -n/\hat{\lambda}^2 < 0.$$

d) The family is a 1PREF. Now $\log(f(w|\lambda)) = \log(3w^2) - \log(\lambda) - w^3/\lambda$, and

$$\frac{d\ \log(f(w|\lambda))}{d\lambda} = \frac{-1}{\lambda} + \frac{w^3}{\lambda^2}.$$

So

$$\frac{d^2\ \log(f(w|\lambda))}{d\lambda^2} = \frac{1}{\lambda^2} - \frac{2w^3}{\lambda^3},$$

and

$$I_1(\lambda) = \frac{-1}{\lambda^2} + \frac{2E[W^3]}{\lambda^3} = \frac{-1}{\lambda^2} + \frac{2\lambda}{\lambda^3} = \frac{1}{\lambda^2}.$$

Thus $\sqrt{n}(\hat{\lambda}_{MLE} - \lambda) \xrightarrow{D} N(0, 1/I_1(\lambda)) \sim N(0, \lambda^2)$.

e) $ARE(\hat{\lambda}_{MM}, \hat{\lambda}_{MLE}) = \lambda^2/c\lambda^2 = 1/c = 1/1.18884 = 0.8412 < 1$

9.1. (Aug. 2003 Qual): Suppose that $X_1, ..., X_n$ are iid with the Weibull distribution, that is the common pdf is

$$f(x) = \begin{cases} \frac{b}{a}x^{b-1}e^{-\frac{x^b}{a}} & 0 < x \\ 0 & \text{elsewhere} \end{cases}$$

where a is the unknown parameter, but $b(> 0)$ is assumed known.

a) Find a minimal sufficient statistic for a.

b) Assume $n = 10$. Use the Chi-Square Table and the minimal sufficient statistic to find a 95% two sided confidence interval for a.

Solution. a) $\sum_{i=1}^n X_i^b$ is minimal sufficient for a.

b) It can be shown that $\frac{X^b}{a}$ has an exponential distribution with mean 1. Thus, $\frac{2\sum_{i=1}^n X_i{}^b}{a}$ is distributed χ^2_{2n}. Let $\chi^2_{2n,\alpha/2}$ be the upper $100(\frac{1}{2}\alpha)\%$ point of the chi-square distribution with $2n$ degrees of freedom. Thus, we can write

$$1-\alpha = P(\chi^2_{2n,1-\alpha/2} < \frac{2\sum_{i=1}^n X_i^b}{a} < \chi^2_{2n,\alpha/2})$$

which translates into

$$\left(\frac{2\sum_{i=1}^n X_i^b}{\chi^2_{2n,\alpha/2}}, \frac{2\sum_{i=1}^n X_i^b}{\chi^2_{2n,1-\alpha/2}}\right)$$

as a two sided $(1-\alpha)$ confidence interval for a. For $\alpha = 0.05$ and $n = 20$, we have $\chi^2_{2n,\alpha/2} = 34.1696$ and $\chi^2_{2n,1-\alpha/2} = 9.59083$. Thus the confidence interval for a is

$$\left(\frac{\sum_{i=1}^n X_i^b}{17.0848}, \frac{\sum_{i=1}^n X_i^b}{4.795415}\right).$$

9.12. (Aug. 2009 qual): Let $X_1, ..., X_n$ be a random sample from a uniform$(0,\theta)$ distribution. Let $Y = \max(X_1, X_2, ..., X_n)$.

a) Find the pdf of Y/θ.

b) To find a confidence interval for θ, can Y/θ be used as a pivot?

c) Find the shortest $(1-\alpha)\%$ confidence interval for θ.

Solution. a) Let $W_i \sim U(0,1)$ for $i = 1, ..., n$ and let $T_n = Y/\theta$. Then

$$P(\frac{Y}{\theta} \le t) = P(\max(W_1, ..., W_n) \le t) =$$

P(all $W_i \le t) = [F_{W_i}(t)]^n = t^n$ for $0 < t < 1$. So the pdf of T_n is

$$f_{T_n}(t) = \frac{d}{dt} t^n = n t^{n-1}$$

for $0 < t < 1$.

b) Yes, the distribution of $T_n = Y/\theta$ does not depend on θ by a).

c) Not sure this is shortest. Let $W_i = X_i/\theta \sim U(0,1)$ which has cdf $F_Z(t) = t$ for $0 < t < 1$. Let $W_{(n)} = X_{(n)}/\theta = \max(W_1, ..., W_n)$. Then

$$F_{W_{(n)}}(t) = P(\frac{X_{(n)}}{\theta} \le t) = t^n$$

for $0 < t < 1$ by a).

Want c_n so that

$$P(c_n \le \frac{X_{(n)}}{\theta} \le 1) = 1-\alpha$$

for $0 < \alpha < 1$. So

$$1 - F_{W_{(n)}}(c_n) = 1-\alpha \;\text{ or }\; 1 - \mathrm{c}_\mathrm{n}^\mathrm{n} = 1-\alpha$$

or

$$c_n = \alpha^{1/n}.$$

Then

$$\left(X_{(n)},\ \frac{X_{(n)}}{\alpha^{1/n}} \right)$$

is an exact $100(1-\alpha)\%$ CI for θ.

9.14. (Aug. 2016 Qual, continuation of Problem 6.44):

a) Find the distribution function of $\frac{\widehat{\theta}}{\theta}$, and use it to explain why this is a pivotal quantity.

b) Using this pivotal quantity, derive a statistics $\widehat{\theta}_L$ such that $P(\widehat{\theta}_L < \theta) = 0.9$.

c) Find the method of moments estimator of θ. Is this estimator unbiased?

d) Is the method of moments estimator consistent? Fully justify your answer.

e) How do you think the variance of $a_n\widehat{\theta}$ compares to that of the method of moments estimator, and why?

f) What is the MLE of θ? Is the MLE consistent?

Solution:

a) Refer to 6.44 c). Let $Y = \frac{\widehat{\theta}}{\theta}$, then we have

$$\begin{aligned} F_y(y) = P(Y \le y) = P(\frac{\widehat{\theta}}{\theta} \le y) = P(\widehat{\theta} \le \theta y) = P(X_{(1)} \le \theta y) = F_{X_{(1)}}(\theta y) \\ = 1 - \big(\frac{\theta}{\theta y}\big)^{4n} = 1 - \big(\frac{1}{y}\big)^{4n} \end{aligned}$$

As it can been seen, the distribution of $\frac{\widehat{\theta}}{\theta}$ is independent of any parameter, and that is the definition of a pivotal quantity.

b)

First, let us to find a b value such that

$$P(\frac{\widehat{\theta}}{\theta} \le b) = 0.9 \Rightarrow 1 - \big(\frac{1}{b}\big)^{4n} = 0.9 \Rightarrow b = 10^{1/4n}$$

or

$$P(\frac{\widehat{\theta}}{\theta} \le 10^{1/4n}) = 0.9 \Rightarrow P(\frac{\widehat{\theta}}{10^{1/4n}} \le \theta) = 0.9$$

Therefore, $\widehat{\theta}_L = \frac{\widehat{\theta}}{10^{1/4n}} = \frac{X_{(1)}}{10^{1/4n}}$.

c) MME:

$$\mu_1' = E[X] = \frac{4}{3}\theta, \qquad\qquad m_1' = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$$

$$\mu_1' = m_1' \Rightarrow \quad \widetilde{\theta} = \frac{3}{4}\bar{X}$$

Check the unbiasedness of MME:

$$E[\widetilde{\theta}] = E[\frac{3}{4}\bar{X}] = \frac{3}{4}E[\bar{X}] = \frac{3}{4} * (\frac{4}{3}\theta) = \theta$$

hence, $\widetilde{\theta}$ is an unbiased estimator.

d) One way to check the consistency of an estimator is to check the limits of its MSE, it goes to zero, then the estimator is consistent. We have,

$$\begin{aligned} MSE(\widetilde{\theta}) &= Var(\widetilde{\theta}) + \text{Bias}(\widetilde{\theta}) \\ &= \frac{9}{16} Var(\bar{X}) + 0 = \frac{9}{16}\frac{4\theta^2}{18n} = \frac{\theta^2}{8n} \end{aligned}$$

then,

$$\lim_{n\to\infty} MSE(\widetilde{\theta}) = \lim_{n\to\infty} \frac{\theta^2}{8n} = 0,$$

and this proves the consistency of the MM estimator.

An easier way is $\overline{X} \xrightarrow{P} E(X)$ by the WLLN, so $0.75\overline{X} \xrightarrow{P} 0.75E(X) = \theta$.

e) The variance of $a_n\widehat{\theta}$ should be less than the variance of any other unbiased estimator, because we proved that the first one is the UMVUE. Also note that

$$\begin{aligned} Var(a_n\widehat{\theta}) &= a_n^2 Var(X_{(1)}) = \big(\frac{4n-1}{4n}\big)^2 \frac{4n\theta^2}{(4n-1)^2(4n-2)} = \frac{\theta^2}{4n(4n-2)}, \\ Var(\widetilde{\theta}) &= \frac{\theta^2}{8n}, \end{aligned}$$

f) From part a) of Problem 6.44, we have

$$L(\theta|x_1,\ldots,x_n) = \prod_{i=1}^n f_\theta(x_i) = 4^n\theta^{4n} I_{[\theta,\infty)}(x_{(1)}) \prod_{i=1}^n x_i^{-5}.$$

The indicator can be written as $I(0 < \theta \le x_{(1)})$, so $L(\theta) > 0$ on $(0, x_{(1)}]$, and $L(\theta) \propto \theta^{4n}$ is an increasing function on $(0, x_{(1)}]$. (Make a sketch of $L(\theta)$.) Hence $X_{(1)}$ is the MLE.

Alternatively, taking derivative with respect to θ (without considering the indicator function) from the likelihood function we have

$$\frac{d}{d\theta} L(\theta|x_1,\ldots,x_n) = 4^n(4n)\theta^{4n-1} \prod_{i=1}^n x_i^{-5} > 0$$

Therefore, the likelihood function is a increasing function of θ on $(0, x_{(1)}]$. Therefore $X_{(1)}$ is the MLE of θ.

Following the argument in part d), and recall the MSE of $X_{(1)}$ from part d) of Problem 6.44, we have

$$\begin{aligned} \lim_{n\to\infty} MSE(X_{(1)}) &= \lim_{n\to\infty} Var(X_{(1)}) + \lim_{n\to\infty} \text{Bias}(X_{(1)})^2 \\ &= \lim_{n\to\infty} \frac{4n\theta^2}{(4n-1)^2(4n-2)} + \lim_{n\to\infty} \frac{\theta^2}{(4n-1)^2} = 0, \end{aligned}$$

which shows that the MLE is also consistent.

9.15. (Jan. 2018 Qual): Let $X_1, ..., X_n$ be a random sample from the following density function

$$f(x,\theta) = \frac{2}{\sqrt{\pi\theta}} \exp\{-\frac{x^2}{\theta}\}, \qquad x > 0, \quad \theta > 0.$$

Hint: note that $\frac{2X_1^2}{\theta} \sim \chi_1^2$.

a) Find the likelihood ratio test of size $\alpha \in (0,1)$ for $H_0 : \theta = \theta_0$ vs. $H_A : \theta \neq \theta_0$

b) Use the likelihood ratio test obtained in part (a) to obtain a $100(1-\alpha)\%$ confidence interval for θ. Justify your answer.

Solution: a) The likelihood function is given as

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) = \prod_{i=1}^{n} \frac{2}{\sqrt{\pi\theta}} \exp\{-\frac{X_i^2}{\theta}\} = 2^n(\pi\theta)^{-n/2} \exp\{-\frac{\sum_{i=1}^n X_i^2}{\theta}\},$$

and it is easy to show that the MLE of θ is $\widehat{\theta} = \frac{2\sum_{i=1}^n X_i^2}{n}$.

Then the likelihood ratio test statistic for $H_0 : \theta = \theta_0$ vs. $H_A : \theta \neq \theta_0$ is given by

$$\lambda(\mathbf{x}) = \frac{L(\theta_0)}{L(\widehat{\theta})} = \frac{2^n(\pi\theta_0)^{-n/2} \exp\{-\frac{\sum_{i=1}^n X_i^2}{\theta_0}\}}{2^n(\pi\widehat{\theta})^{-n/2} \exp\{-\frac{\sum_{i=1}^n X_i^2}{\widehat{\theta}}\}} = \left(\frac{2\sum_{i=1}^n X_i^2}{n\theta_0}\right)^{n/2} \exp\left\{\frac{n}{2} - \frac{\sum_{i=1}^n X_i^2}{\theta_0}\right\}.$$

Now let $x = \frac{2\sum_{i=1}^n X_i^2}{\theta_0}$. Then the likelihood statistic λ above can be written as $g(x) = \left(\frac{x}{n}\right)^{n/2} \exp\{\frac{n-x}{2}\}$, $x > 0$. It can be shown that the $g(x)$ is a concave down ($\lim_{x\to 0^+} g(x) = \lim_{x\to\infty} g(x) = 0$), and takes its maximum at $x = n$ with maximum value $g(n) = 1$. That is, $g(x)$ is increasing on $(0, n]$ and decreasing on $[n, \infty]$. Then, the test reject H_0 if $\lambda(\mathbf{x}) < c$, if and only if $x < c_1$ or $x > c_2$, where $0 < c_1 < n < c_2$ are constants satisfying $g(c_1) = g(c_2) = c$, such that

$$\alpha = P_{\theta_0}(x < c_1 \text{ or } \ x > c_2) = P_{\theta_0}(\frac{2\sum_{i=1}^n X_i^2}{\theta_0} < c_1 \text{ or } \frac{2\sum_{i=1}^n X_i^2}{\theta_0} > c_2)$$

which is equivalent to

$$\alpha = 1 - P_{\theta_0}(c_1 < \frac{2\sum_{i=1}^n X_i^2}{\theta_0} < c_2)$$

Under $H_0 : \theta = \theta_0$, we have $\frac{2\sum_{i=1}^n X_i^2}{\theta_0} \sim \chi_n^2$. So we choose $c_1 = \chi^2_{n,1-\alpha/2}$ and $c_2 = \chi^2_{n,\alpha/2}$.

b) The acceptance region for the likelihood ratio test of size α for $H_0 : \theta = \theta_0$ is given as

$$A(\theta_0) = \left\{(X_1, ..., X_n) : c_1 < \frac{2\sum_{i=1}^n X_i^2}{\theta_0} < c_2\right\}$$

By inverting the region, the $100(1-\alpha)\%$ confidence region of θ is given $\{\theta > 0 : (X_1, ..., X_n) \in A(\theta)\}$ which can be written as follows

$$\left\{\theta > 0 : c_1 < \frac{2\sum_{i=1}^n X_i^2}{\theta} < c_2\right\} = \left[\frac{2\sum_{i=1}^n X_i^2}{c_2}, \frac{2\sum_{i=1}^n X_i^2}{c_1}\right].$$