Math 403 Exam 1 is Wed. Sept. 20. **You are allowed 7 sheets of notes and a calculator.** The exam covers HW1-3, and Q1-3. Numbers refer to types of problems on exam. In this class $\log(t) = \ln(t) = \log_e(t)$ while $\exp(t) = e^t$. See 9) for $e_X(d) = E(Y^P)$.

0) Get familiar with the following distributions. For continuous distributions, assume formulas are given on the support, and the support is $x > 0$, unless told otherwise.

a) Exponential(θ)= Gamma($\alpha = 1, \theta$): $f(x) = \frac{1}{\theta} e^{-x/\theta}$ where $x, \theta > 0$.

$F(x) = 1 - e^{-x/\theta}$, $E(X) = \theta$, $V(X) = \theta^2$, $E[X \wedge x] = \theta(1 - e^{-x/\theta})$, $e_X(d) = \theta$.

$E(X^k) = \theta^k \Gamma(k+1)$ for $k > -1$. If k is a positive integer, $E(X^k) = \theta^k k!$.

$M(t) = (1-\theta t)^{-1}$, $t < 1/\theta$. $VaR_p(X) = -\theta \ln(1-p)$. $TVaR_p(X) = -\theta \ln(1-p) + \theta$.

b) Gamma(α, θ): $f(x) = \frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\theta}$ where α, θ, and x are positive.

$E(X) = \alpha\theta$, $V(X) = \alpha\theta^2$, $E(X^k) = \frac{\theta^k \Gamma(\alpha + k)}{\Gamma(\alpha)}$ for $k > -\alpha$.

$M(t) = (1 - \theta t)^{-\alpha}$ for $t < 1/\theta$.

c) (two parameter) Pareto(α, θ): $f(x) = \frac{\alpha\theta^\alpha}{(\theta + x)^{\alpha+1}}$ where α, θ, and x are positive.

$F(x) = 1 - \left(\frac{\theta}{x + \theta}\right)^\alpha$, $E(X) = \frac{\theta}{\alpha - 1}$ for $\alpha > 1$, $V(X) = \frac{\theta^2 \alpha}{(\alpha-1)^2(\alpha-2)}$ for $\alpha > 2$.

$e_X(d) = \frac{\theta + d}{\alpha - 1}$, $E(X^k) = \frac{\theta^k \Gamma(k+1)\Gamma(\alpha - k)}{\Gamma(\alpha)}$ for $-1 < k < \alpha$.

If $k < \alpha$ is a positive integer, $E(X^k) = \frac{\theta^k k!}{(\alpha-1)\cdots(\alpha-k)}$.

$E[X \wedge x] = \frac{\theta}{\alpha - 1}\left[1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\right]$, for $\alpha \neq 1$, and $E[X \wedge x] = -\theta \ln\left(\frac{\theta}{x+\theta}\right)$ for $\alpha = 1$.

$VaR_p(X) = \theta[(1-p)^{-1/\alpha} - 1]$, $TVaR_p(X) = VaR_p(X) + \frac{\theta(1-p)^{-1/\alpha}}{\alpha - 1}$ for $\alpha > 1$.

d) If $X \sim$ single parameter Pareto(α, θ): $f(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}} I(x > \theta)$ where $\alpha > 0$ and θ is real. Note the **support** is $x > \theta$. $F(x) = 1 - \left(\frac{\theta}{x}\right)^\alpha$ for $x > \theta$. $E(X) = \frac{\alpha\theta}{\alpha - 1}$ for $\alpha > 1$. $V(X) = \frac{\alpha\theta^2}{\alpha - 2} - \left(\frac{\alpha\theta}{\alpha - 1}\right)^2$ for $\alpha > 2$. $E(X^k) = \frac{\alpha\theta^k}{\alpha - k}$ for $k < \alpha$. $E(X \wedge x) = \frac{\alpha\theta}{\alpha - 1} - \frac{\theta^\alpha}{(\alpha-1)x^{\alpha-1}}$ for $x \geq \theta$. $E(X \wedge x) = x$ for $x < \theta$. **Use** $\theta \geq 0$ for loss models.

$VaR_p(X) = \theta[(1-p)^{-1/\alpha}]$, $TVaR_p(X) = \frac{\alpha\theta(1-p)^{-1/\alpha}}{\alpha - 1} = VaR_p(X) + \frac{1}{\alpha - 1} VaR_p(X)$ for $\alpha > 1$.

e) Uniform(a, b). This distribution has **support** on $a \leq x \leq b$, $f(x) = \frac{1}{b-a}$, $F(x) = (x-a)/(b-a)$, $E(X) = (a+b)/2$, $V(X) = (b-a)^2/12$, $e_X(d) = \frac{b-d}{2}, 0 \leq a \leq d \leq b$.

f) Weibull(θ, τ): $f(x) = \frac{\tau (x/\theta)^\tau e^{-(x/\theta)^\tau}}{x}$ where $\theta > 0$ and $\tau > 0$.

$F(x) = 1 - e^{-(x/\theta)^\tau}$, $E(X^k) = \theta^k \Gamma(1 + k/\tau)$ for $k > -\tau$. Here $\theta, \tau > 0$ and the Weibull$(\theta, \tau = 1)$ RV is the Exponential(θ) RV. $VaR_p(X) = \theta[-\ln(1-p)]^{1/\tau}$.

g) Inverse Weibull(θ, τ): $f(x) = \dfrac{\tau(\theta/x)^\tau e^{-(\theta/x)^\tau}}{x}$.

$F(x) = e^{-(\theta/x)^\tau}$, $E(X^k) = \theta^k \Gamma(1 - k/\tau)$ for $k < \tau$. Here $\theta, \tau > 0$ and the Inverse Weibull$(\theta, \tau = 1)$ RV is the Inverse Exponential(θ) RV. $VaR_p(X) = \theta[-\ln(p)]^{-1/\tau}$.

h) normal(μ, σ): $E(X) = \mu$, $V(X) = \sigma^2$. The **support** is $(-\infty, \infty)$. If $Z \sim N(0,1)$, then the cdf of Z is $\Phi(x)$ and the pdf of Z is $\phi(x)$. If $X \sim N(\mu, \sigma^2)$, then the cdf of X is $F(x) = \Phi\left(\dfrac{x-\mu}{\sigma}\right)$. If $X \sim N(\mu, \sigma^2)$, then the cdf $F_X(x) = \Phi\left(\dfrac{x-\mu}{\sigma}\right)$, and the pdf $f_X(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left[\dfrac{-1}{2\sigma^2}(x-\mu)^2\right]$. $TVaR_p(X) = \mu + \sigma \dfrac{\phi(z_p)}{1-p}$ where $P(Z \le z_p) = p$ if $Z \sim N(0,1)$. $VaR_p(X) = \mu + \sigma z_p$. Here $\sigma > 0$ and μ is real.

i) lognormal(μ, σ): $E(X) = \exp(\mu + \frac{1}{2}\sigma^2)$, $V(X) = \exp(\sigma^2)(\exp(\sigma^2) - 1)\exp(2\mu)$, $F(x) = \Phi\left(\dfrac{\ln(x) - \mu}{\sigma}\right)$, $E(X \wedge x) = \exp(\mu + \dfrac{1}{2}\sigma^2)\Phi\left(\dfrac{\ln x - \mu - \sigma^2}{\sigma}\right) + x[1 - \Phi(\dfrac{\ln x - \mu}{\sigma})]$. If $X \sim LN(\mu, \sigma)$, then $\ln(X) \sim N(\mu, \sigma^2)$. Here $x > 0$, $\sigma > 0$ and μ is real.
$VaR_p(X) = \exp(\mu + z_p\sigma)$. For $a > 0$, $aX \sim LN(\mu + \ln(a), \sigma)$.

j) beta(a, b): The **support** is [0,1]. The pdf $f(x) = \dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$ where $a > 0$ and $b > 0$. $E(X) = \dfrac{a}{a+b}$. $V(X) = \dfrac{ab}{(a+b)^2(a+b+1)}$.

The following are discrete distributions. These are used to count the number of claims, so the random variable X is often denoted by N. Note: $p_k = P(X = k) = p(k)$.

k) binomial(q, m): m is a (usually known) positive integer

$$p_k = \binom{m}{k} q^k (1-q)^{m-k} \quad \text{for} \quad \mathrm{k = 0, 1, \ldots, m} \quad \text{where} \quad 0 < \mathrm{q} < 1.$$

$E(N) = mq$, $V(N) = mq(1-q)$, $P(z) = [1 + q(z-1)]^m$.

l) Poisson(λ): $p_k = \dfrac{e^{-\lambda}\lambda^k}{k!}$ for $k = 0, 1, \ldots$, where $\lambda > 0$. $E(N) = \lambda = V(N)$, $P(z) = e^{\lambda(z-1)}$.

m) Negative Binomial(β, r): $\beta, r > 0$ and $p_0 = (1+\beta)^{-r}$. For $k = 1, 2, ...,$

$$p_k = \frac{r(r+1)\cdots(r+k-1)\beta^k}{k!(1+\beta)^{r+k}} \quad \text{and} \quad \mathrm{p_k} = \frac{(\mathrm{k+r-1})!\beta^{\mathrm{k}}}{\mathrm{k!(r-1)!}(1+\beta)^{\mathrm{r+k}}} \text{ for integer r.}$$

$E(N) = r\beta$, $V(N) = r\beta(1+\beta)$, $P(z) = [1 - \beta(z-1)]^{-r}$. The Geometric$(\beta)$ is the special case with $r = 1$ and $p_k = \dfrac{\beta^k}{(1+\beta)^{k+1}}$ for $k = 0, 1, \ldots$.

Some properties of the gamma function follow.

i) $\Gamma(k) = (k-1)!$ for integer $k \ge 1$.

ii) $\Gamma(x+1) = x\,\Gamma(x)$ for $x > 0$.

iii) $\Gamma(x) = (x-1)\,\Gamma(x-1)$ for $x > 1$.

iv) $\Gamma(0.5) = \sqrt{\pi}$.

Let $X \geq 0$ be a nonnegative random variable.

Then the **cumulative distribution function** (cdf) $F(x) = P(X \leq x)$. Since $X \geq 0$, $F(0) = 0, F(\infty) = 1$, and $F(x)$ is nondecreasing.

The probability density function (**pdf**) $f(x) = F'(x)$.

The **survival function** $S(x) = P(X > x)$. $S(0) = 1, S(\infty) = 0$ and $S(x)$ is nonincreasing.

The **hazard rate function** = *force of mortality* $= \mu(x) = h(x) = \dfrac{f(x)}{1 - F(x)} = \dfrac{f(x)}{S(x)}$ for $x > 0$ and $F(x) < 1$. Note that $h(x) \geq 0$ if $F(x) < 1$.

The **cumulative hazard function** $H(x) = \int_0^x h(t)dt$ for $x > 0$. It is true that $H(0) = 0, H(\infty) = \infty$, and $H(x)$ is nondecreasing.

Assume $X \geq 0$ unless told otherwise.

1) Given one of $F(x), f(x), S(x), h(x)$, or $H(x)$, be able to find the other 4 quantities for $x > 0$. See HW1.

A) $F(x) = \int_0^x f(t)dt = 1 - S(x) = 1 - \exp[-H(x)] = 1 - \exp[-\int_0^x h(t)dt]$.

B) $f(x) = F'(x) = -S'(x) = h(x)[1 - F(x)] = h(x)S(x) = h(x)\exp[-H(x)] = H'(x)\exp[-H(x)]$.

C) $S(x) = 1 - F(x) = 1 - \int_0^x f(t)dt = \int_x^\infty f(t)dt = \exp[-H(x)] = \exp[-\int_0^x h(t)dt]$.

D) $h(x) = \dfrac{f(x)}{1 - F(x)} = \dfrac{f(x)}{S(x)} = \dfrac{F'(x)}{1 - F(x)} = \dfrac{-S'(x)}{S(x)} = -\dfrac{d}{dx}\ln[S(x)] = H'(x)$.

E) $H(x) = \int_0^x h(t)dt = -\ln[S(x)] = -\ln[1 - F(x)]$.

Tip: if $F(x) = 1 - \exp[G(x)]$ for $x > 0$, then $H(x) = -G(x)$ and $S(x) = \exp[G(x)]$.

Tip: For $S(x) > 0$, note that $S(x) = \exp[\ln(S(x))] = \exp[-H(x)]$. Finding $\exp[\ln(S(x))]$ and setting $H(x) = -\ln[S(x)]$ is easier than integrating $h(x)$.

2) **Know:** Except for the inverse Gaussian distribution, the continuous distributions in Appendix A with parameter θ are scale families with scale parameter θ if any other parameters $\boldsymbol{\tau}$ are fixed, written $X \sim SF(\theta|\boldsymbol{\tau})$. Let $a > 0$. Then $Y = aX \sim SF(a\theta|\boldsymbol{\tau})$. See 31). If $X \sim LN(\mu, \sigma)$, then $Y = aX \sim LN(\mu + \ln(a), \sigma)$. Often $a = 1 + r$.

3) Let $X \geq 0$ be continuous. If $\lim_{x\to\infty} xS(x) = 0$, then $E(X) = \int_0^\infty xf(x)dx = \int_0^\infty S(x)dx = \int_0^\infty [1 - F(x)]dx = \mu =$ mean. The kth raw moment $= \mu'_k = E(X^k) = \displaystyle\int_0^\infty x^k f(x)dx$. If $\lim_{x\to\infty} x^k S(x) = 0$ and $k \geq 1$, then $E(X^k) = \displaystyle\int_0^\infty kx^{k-1}S(x)dx$.

If X is discrete, $= E(X^k) = \displaystyle\sum_k x^k P(X = x)$.

4) The kth central moment $\mu_k = E[(X - \mu)^k]$. The variance uses $k = 2$ and the short cut formula for the variance is $V(X) = E[(X - \mu)^2] = \sigma^2 = E(X^2) - [E(X)]^2$ where $\mu = E(X)$. Note: $\mu_3 = \mu'_3 - 3\mu'_2\mu + 2\mu^3$ and $\mu_4 = \mu'_4 - 4\mu'_3\mu + 6\mu'_2\mu^2 - 3\mu^4$.

The standard deviation $SD(X) = \sqrt{V(X)} = \sigma$.

5) Suppose $X \geq 0$. Then $E[g(X)] = \int_0^\infty g(x)f(x)dx$ for X continuous and $E[g(X)] = \displaystyle\sum_{x:p(x)>0} g(x)\ p(x)$ for X discrete.

6) The coefficient of variation $= CV = \dfrac{\sigma}{\mu}$, skewness $= \gamma_1 = \dfrac{\mu_3}{\sigma^3}$, and kurtosis $= \gamma_2 = \dfrac{\mu_4}{\sigma^4}$. For a statistic T, $CV(T) = SD(T)/E(T)$.

7) $X \wedge d = \min(X, d)$ is the limited loss RV. This RV is right censored. The limited expected value $E[X \wedge d] = \int_0^d x f(x) dx + dS(d) = \int_0^d S(x) dx$. The expected loss (per loss) for a policy holder with deductible d is $E[X \wedge d]$.

8) The **per loss** RV $Y^L = (X - d)_+ = 0$ if $X \leq d$, $Y^L = (X - d)_+ = X - d$ if $X > d$. The RV is left censored since values $X \leq d$ are not ignored but are set to d. So values of $X - d < 0$ are set to 0. Note that $(X - d)_+$ is the positive part of $X - d$, and represents payment for **insurance with a deductible**. The superscript L represents the "payment," possibly 0, made per loss. $E[(X-d)_+] = e_X(d)[1-F(d)] = e_X(d)S(d) = \int_d^\infty (x-d) f(x) dx = \int_d^\infty S(x) dx = E(Y^L) = E(Y^P)S(d)$.

9) For a given value of $d > 0$ with $P(X > d) > 0$, the excess loss variable or **per payment** RV $Y^P = (X - d)|X > d$. This is a left truncated and shifted RV. The mean excess loss function $e_X(d) = E(Y^P) = E[(X-d)|X > d] = \dfrac{\int_d^\infty (x-d)f(x)dx}{1 - F(d)} = \dfrac{\int_d^\infty S(x)dx}{S(d)} = \dfrac{E(Y^L)}{S(d)}$. The superscript P represents "payment" per payment > 0 actually made (so the loss $> d$).

10) Since insurance with a limit d plus insurance with a deductible d equals full coverage insurance: $X \wedge d \ + \ (X-d)_+ = X$, we get $E[X \wedge d] \ + \ E[(X-d)_+] = E[X]$, and $E[X \wedge d] = E[X] - E[(X-d)_+]$. So $E(Y^L) = E(X) - E(X \wedge d)$.

11) $E[(d - X)_+] = d - E[X \wedge d]$

12) The 100pth percentile $VaR_p(X) = \pi_p$ satisfies $F(\pi_p) = P(X \leq \pi_p) = p$ if X is a continuous RV with increasing $F(x)$. Then to find π_p, let $\pi = \pi_p$ and solve $F(\pi) \overset{\text{set}}{=} p$ for π.

For a general RV X, π_p satisfies $F(\pi_p-) = P(X < \pi_p) \leq p \leq F(\pi_p) = P(X \leq \pi_p)$. So $F(\pi_p-) \leq p$ and $F(\pi_p) \geq \alpha$. Then graphing $F(x)$ can be useful for finding π_p.

13) Assume all relevant expectations exist. Let $S_n = \sum_{i=1}^n X_i$. Then $E(S_n) = E(\sum_{i=1}^n X_i) = \sum_{i=1}^n E(X_i)$. If the X_i are independent, $V(S_n) = V(\sum_{i=1}^n X_i) = \sum_{i=1}^n V(X_i)$.

14) **Central Limit Theorem (CLT).** Let $X_1, ..., X_n$ be iid with $E(X) = \mu$ and $V(X) = \sigma^2$. Let the sample mean $\overline{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$. Then

$$\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{D} N(0, \sigma^2).$$

Hence $\sqrt{n}\left(\dfrac{\overline{X}_n - \mu}{\sigma}\right) = \sqrt{n}\left(\dfrac{\sum_{i=1}^n X_i - n\mu}{n\sigma}\right) = \sqrt{n}\left(\dfrac{S_n - n\mu}{n\sigma}\right) \xrightarrow{D} N(0,1)$.

15) The notation $Y_n \xrightarrow{D} X$ means that for large n we can approximate the cdf of Y_n by the cdf of X. The distribution of $X$is the limiting distribution or asymptotic distribution of Y_n, and the limiting distribution does not depend on n.

16) The notation

$$\overline{X}_n \approx N(\mu, \sigma^2/n),$$

also written as $\overline{X}_n \sim AN(\mu, \sigma^2/n)$, means approximate the cdf of $\overline{X}_n$ as if $\overline{X}_n \sim N(\mu, \sigma^2/n)$. Note that the approximate distribution, unlike the limiting distribution, does depend on n. By the CLT, $\overline{X}_n \sim AN(\mu, \sigma^2/n)$ and $S_n = \sum_{i=1}^n X_i \sim AN(n\mu, n\sigma^2)$.

17) The **moment generating function** (mgf) of a random variable X is $M_X(t) = E[e^{tX}]$. If X is discrete, then $M_X(t) = \sum_x e^{tx} p(x)$, and if X is continuous, then $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$. If the mgf $M_X(t)$ exists for $|t| < \delta$ for some constant $\delta > 0$, find the kth derivative $M_X^{(k)}(t)$. Then $E[X^k] = M_X^{(k)}(0)$. In particular, $E[X] = M_X'(0)$ and $E[X^2] = M_X''(0)$.

18) The **probability generating function** (pgf) of a random variable X is $P_X(z) = E[z^X]$. If X is discrete, then $P_X(z) = \sum_x z^x p(x)$, and if X is continuous, then $P_X(z) = \int_{-\infty}^{\infty} z^x f(x) dx$. If the pgf $P_X(z)$ exists for $z \in (1-\epsilon, 1+\epsilon)$ for some constant $\epsilon > 0$, find the kth derivative $P_X^{(k)}(z)$. Then $E[X(X-1)\cdots(X-k+1)] = P_X^{(k)}(1)$ where the product has k terms. In particular, $E[X] = P_X'(1)$ and $E[X^2 - X] = E(X^2) - E(X) = P_X''(1)$.

19) $M_X(t) = P_X(e^t)$ and $P_X(z) = M_X(\ln(z))$.

20) Let $S_n = \sum_{i=1}^n X_i$ where the X_i are independent with mgf $M_{X_i}(t)$ and pgf $P_{X_i}(z)$. The mgf of S_n is $M_{S_n}(t) = \displaystyle\prod_{i=1}^n M_{X_i}(t) = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_n}(t)$. The pgf of S_n is $P_{S_n}(z) = \displaystyle\prod_{i=1}^n P_{X_i}(z) = P_{X_1}(z) P_{X_2}(z) \cdots P_{X_n}(z)$.

Tips: a) in the product, anything that does not depend on the product index i is treated as a constant. b) $\exp(a) = e^a$ and $\log(y) = \ln(y) = \log_e(y)$ is the **natural logarithm**. c) $\displaystyle\prod_{i=1}^n a^{b\theta_i} = a^{\sum_{i=1}^n b\theta_i}$. In particular, $\displaystyle\prod_{i=1}^n \exp(b\theta_i) = \exp(\sum_{i=1}^n b\theta_i)$. d) $\sum_{i=1}^n b = nb$. e) $\prod_{i=1}^n a = a^n$.

21) Assume the X_i are independent.

a) If $X_i \sim N(\mu_i, \sigma_i^2)$, with support $(-\infty, \infty)$, then $\sum_{i=1}^n X_i \sim N(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$, and $\sum_{i=1}^n (a_i X_i + b_i) \sim N(\sum_{i=1}^n (a_i \mu_i + b_i), \sum_{i=1}^n a_i^2 \sigma_i^2)$.

b) If $X_i \sim G(\alpha_i, \theta)$, then $\sum_{i=1}^n X_i \sim G(\sum_{i=1}^n \alpha_i, \theta)$. Note that the X_i have the same θ, and if $\alpha_i \equiv \alpha$, then $\sum_{i=1}^n \alpha = n\alpha$. G stands for Gamma.

c) If $X_i \sim EXP(\theta) \sim G(1, \theta)$, then $\sum_{i=1}^n X_i \sim G(n, \theta)$.

d) If $X_i \sim$ Poisson(λ_i) then $\sum_{i=1}^n X_i \sim$ Poisson($\sum_{i=1}^n \lambda_i$). Note that if $\lambda_i \equiv \lambda$, then $\sum_{i=1}^n \lambda = n\lambda$.

e) If $X_i \sim \chi^2_{p_i} \sim G\left(\dfrac{p_i}{2}, 2\right)$, then $\displaystyle\sum_{i=1}^n X_i \sim \chi^2_{\sum_{i=1}^n p_i}$.

f) If $X_i \sim BIN(q, m_i)$, then $\sum_{i=1}^n X_i \sim BIN(q, \sum_{i=1}^n m_i)$. Note that the X_i have the same q, and if $m_i \equiv m$, then $\sum_{i=1}^n m = nm$.

g) Let NB stand for negative binomial. If $X_i \sim NB(\beta, r_i)$, then $\sum_{i=1}^n X_i \sim NB(\beta, \sum_{i=1}^n r_i)$. Note that the X_i have the same β, and if $r_i \equiv r$, then $\sum_{i=1}^n r = nr$.

h) Let $X_i \sim geom(\beta) \sim NB(\beta, 1)$. Then $\sum_{i=1}^n X_i \sim NB(\beta, n)$.

22) X has a heavier tail than Y is one of the following holds:

i) X has fewer moments than Y: $E(X^k)$ does not exist but $E(Y^k)$ does exist for some positive integer k.

ii) $\displaystyle\lim_{x\to\infty} \frac{S_X(x)}{S_Y(x)} = \lim_{x\to\infty} \frac{S_X'(x)}{S_Y'(x)} = \lim_{x\to\infty} \frac{f_X(x)}{f_Y(x)} = \infty$.

23) Let decreasing = nonincreasing and increasing = nondecreasing.
i) X has a "heavy tail" if $h(x)$ decreases (often to 0), and a light tail if $h(x)$ increases as $x \to \infty$.
ii) X has a "light tail" if $E(X^k)$ exists for all positive integers k, and has a heavy tail otherwise. Existence of an mgf implies a light tail, but the converse is false: the lognormal distribution has all moments but not an mgf.
iii) X has a "heavy tail" if $e_X(x)$ increases, and a light tail if $e_X(x)$ decreases as $x \to \infty$.

24) $\lim_{x\to\infty} e_X(x) = \lim_{x\to\infty} \frac{1}{h(x)}$ if the limit exists. $\lim_{x\to\infty} h(x) = -\lim_{x\to\infty} \frac{f'(x)}{f(x)} = -\lim_{x\to\infty} \frac{d}{dx}\ln(f(x))$.

25) The **Value at Risk** of X at the $100p\%$ level = $100p$th percentile $VaR_p(X) = \pi_p$ satisfies $F(\pi_p) = P(X \le \pi_p) = p$ if X is a continuous RV with increasing $F(x)$. Then to find π_p, let $\pi = \pi_p$ and solve $F(\pi) \overset{\text{set}}{=} p$ for π.

For a general RV X, π_p satisfies $F(\pi_p-) = P(X < \pi_p) \le p \le F(\pi_p) = P(X \le \pi_p)$. So $F(\pi_p-) \le p$ and $F(\pi_p) \ge \alpha$. Then graphing $F(x)$ can be useful for finding π_p.

26) Let $c > 0$ be a constant. A risk measure $\rho(X)$ is **coherent** if it satisfies
i) subadditivity: $\rho(X+Y) \le \rho(X) + \rho(Y)$,
ii) monotonicity: if $X < Y$, in that $P(X > Y) = 0$, then $\rho(X) \le \rho(Y)$,
iii) positive homogeneity: for any $c > 0$, $\rho(cX) = c\rho(X)$,
iv) translation invariance: for any $c > 0$, $\rho(X + c) = \rho(X) + c$.

27) $VaR_p(X) = \pi_p$ does not satisfy subadditivity, and so in not a coherent risk measure.

28) If $X \sim AN(\mu, \sigma^2)$, then $VaR_p(X) = \pi_p \approx \mu + \sigma z_p$ where $P(Z \le z_p) = p$ when $Z \sim N(0,1)$.

29) The *tail value at risk* of X at $100p\%$ security level is
$TVaR_p(X) = E(X|X > \pi_p) = \frac{\int_{\pi_p}^\infty x f(x)dx}{1 - F(\pi_p)} = \frac{\int_p^1 \pi_u du}{1-p} = VaR_p(X) + e_X(\pi_p) =$
$\pi_p + \frac{\int_{\pi_p}^\infty (x - \pi_p) f(x)dx}{1-p} = \pi_p + \frac{E(X) - E(X \wedge \pi_p)}{1-p}$. $TVaR_p(X) \ge VaR_P(X)$, and $TVaR$ is a coherent risk measure.

30) If $X \sim AN(\mu, \sigma^2)$ then $TVaR_p(X) \approx \mu + \sigma\frac{\phi(z_p)}{1-p}$ where $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)$ is the N(0,1) pdf.

31) Let $f_Z(x)$ be the pdf of RV Z. Then the family of pdfs $f_X(x) = \frac{1}{\theta} f_Z\left(\frac{x}{\theta}\right)$ indexed by a *scale parameter* $\theta > 0$ is the *scale family* for the RV $X = \theta Z$ with standard pdf $f_Z(x)$. If the expected values exist, then $E(X) = \theta E(Z)$ and $V(X) = \theta^2 V(Z)$. See 2).

32) X is a loss RV from a scale family with scale parameter θ if $Y = cX$ is from the same scale family with scale parameter $c\theta$ for any constant $c > 0$.

33) Let $f_Z(x)$ be the pdf of RV Z. Then the family of pdfs $f_X(x) = \frac{1}{\theta} f_Z\left(\frac{x-\mu}{\theta}\right)$ indexed by a *scale parameter* $\theta > 0$ and real *location parameter* μ is the *location scale family* for the RV $X = \mu + \theta Z$ with standard pdf $f_Z(x)$. If second moments exist, then $E(X) = \mu + \theta E(Z)$ and $V(X) = \theta^2 V(Z)$.

34) Let $f_Z(x)$ be the pdf of RV Z. Then the family of pdfs $f_X(x) = f_Z(x - \mu)$ indexed by a real *location parameter* μ is the *location family* for the RV $X = \mu + Z$ with standard

pdf $f_Z(x)$. If the variance exists, then $E(X) = \mu + E(Z)$ and $V(X) = V(Z)$.

35) Let Y be a random variable with cdf $F(y)$. Let h be a function such that the *expected value* $E[h(Y)]$ exists. $E[h(Y)] = \int_{-\infty}^{\infty} h(y)dF(y)$.

36) Assume all expectations exist. a) If Y is a discrete random variable that has a pmf $p(y)$ with support $\mathcal{Y}$, then

$$E[h(Y)] = \int_{-\infty}^{\infty} h(y)dF(y) = \sum_{y \in \mathcal{Y}} h(y)p(y).$$

b) If Y is a continuous random variable that has a pdf $f(y)$, then

$$E[h(Y)] = \int_{-\infty}^{\infty} h(y)dF(y) = \int_{-\infty}^{\infty} h(y)f(y)dy.$$

c) If Y is a random variable that has a **mixture distribution** with cdf $F_Y(y) = \sum_{i=1}^{k} \alpha_i F_{W_i}(y)$, then

$$E[h(Y)] = \int_{-\infty}^{\infty} h(y)dF(y) = \sum_{i=1}^{k} \alpha_i E_{W_i}[h(W_i)]$$

where $E_{W_i}[h(W_i)] = \int_{-\infty}^{\infty} h(y)dF_{W_i}(y)$.

37) If the cdf of X is $F_X(x) = (1-\epsilon)F_Z(x) + \epsilon F_W(x)$ where $0 \leq \epsilon \leq 1$ and F_Z and F_W are cdfs, then $E[g(X)] = (1-\epsilon)E[g(Z)] + \epsilon E[g(W)]$. In particular, $E(X^2) = (1-\epsilon)E[Z^2] + \epsilon E[W^2] = (1-\epsilon)[V(Z) + (E[Z])^2] + \epsilon[V(W) + (E[W])^2]$.

38) If $P(A), P(B) > 0$, then $P(A \cap B) = P(AB) = P(A)P(B|A) = P(B)P(A|B)$, and $P(A|B) = P(AB)/P(B)$.

39) If the region of integration Ω is bounded on top by the function $y = \phi_T(x)$, on the bottom by the function $y = \phi_B(x)$ and to the left and right by the lines $x = a$ and $x = b$ then $\int\int_\Omega f(x,y)dxdy = \int\int_\Omega f(x,y)dydx =$

$$\int_a^b \left[\int_{\phi_B(x)}^{\phi_T(x)} f(x,y)dy \right] dx.$$

Within the inner integral, treat y as the variable, anything else, including x, is treated as a constant. If the region of integration Ω is bounded on the left by the function $x = \psi_L(y)$, on the right by the function $x = \psi_R(y)$ and to the top and bottom by the lines $y = c$ and $y = d$ then $\int\int_\Omega f(x,y)dxdy = \int\int_\Omega f(x,y)dydx =$

$$\int_c^d \left[\int_{\psi_L(y)}^{\psi_R(y)} f(x,y)dx \right] dy.$$

Within the inner integral, treat x as the variable, anything else, including y, is treated as a constant.

40) In particular, if $X|\Lambda = \lambda$ has conditional cdf $F_{X|\Lambda=\lambda}(x|\lambda)$ and pdf $f_{X|\Lambda=\lambda}(x|\lambda)$, then the unconditional (marginal) pdf of X is $f_X(x) = \int f_{X|\Lambda}(x|\lambda)f_\Lambda(\lambda)d\lambda$, and the cdf of X is $F_X(x) = \int F_{X|\Lambda}(x|\lambda)f_\Lambda(\lambda)d\lambda$.

41) The conditional pdf $f_{Y|X=x}(y|X = x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$ where the marginal pdf $f_X(x) = \int_{\phi_B(x)}^{\phi_T(x)} f_{X,Y}(x,y)dy$. Note that $f_Y(y) = \int_{\psi_L(y)}^{\psi_R(y)} f_{X,Y}(x,y)dx$.

42) If $E(X|Y = y) = m(y)$, then the random variable $E(X|Y) = m(Y)$. Similarly if $V(X|Y = y) = v(y)$, then the random variable $V(X|Y) = v(Y) = E(X^2|Y) - [E(X|Y)]^2$.

43) Assume all relevant expectations exist. Then iterated expectations or the conditional mean formula is $E(X) = E[E(X|Y)] = E_Y[E_{X|Y}(X|Y)]$. The conditional variance formula is $V(X) = E[V(X|Y)] + V[E(X|Y)]$. Also, $E(X^k) = E[E(X^k|Y)]$.

44) A counting RV N is a discrete RV with support $\subseteq \{0, 1, 2, ...\}$. Let $P(z)$ denote the pgf and $p(k) = p_k = P(N = k)$ denote the pmf.

45) If $N \sim$ Poisson(λ), then $E(N) = \lambda = V(N)$. If $N \sim NB(\beta, r)$, then $E(N) = r\beta < V(N) = r\beta(1 + \beta)$.

46) If $N \sim$ geom(β), then $N \sim NB(\beta, r)$, $P(N \geq n) = \left(\dfrac{\beta}{1+\beta}\right)^n$. Also the geometric RV has the memoryless property: $P(N \geq n + k|N \geq n) = P(N \geq k)$ where $n, k \geq 0$ are integers. (This is the discrete analog of the memoryless property of $X \sim EXP(\theta)$ where $P(X > t + d|X > d) = P(X > t)$ for any $t, d > 0$.)

47) If $N \sim Bin(q, m)$ then the support is 0, 1, .., m, and $E(N) = mq < V(N) = mq(1 - q)$.

48) The Poisson, binomial, NB and Geometric distributions are members of the $(a, b, 0)$ class. X is a member of this class if $\dfrac{p_k}{p_{k-1}} = a + \dfrac{b}{k}$ for $k = 1, 2, ...$ Hence $\dfrac{k\ p_k}{p_{k-1}} = a\ k + b$ for $k = 1, 2, ...$ except the recursion goes up to $k = m$ for the binomial. In a sample (e.g. 0,1,1,5,0,3,7,0,5,2,1,1,1,4,2,2), let n_k = number in sample equal to k and $n = \sum_k n_k$. Plot k versus $\dfrac{k\ \hat{p}_k}{\hat{p}_{k-1}} = \dfrac{k\ n_k}{n_{k-1}}$ where k is omitted if $n_k = 0$. If the n_k are large, the plot should follow a straight line with slope a. Here the slope is zero $a = 0$ for the Poisson, the slope is negative $a < 0$ for the binomial, and the slope is positive $a > 0$ for the NB and Geometric = NB($\beta, r = 1$) distributions.

49) Let N be a counting RV with support $\subseteq \{0, 1, 2, ...\}$. Let $N \perp\!\!\!\perp X_i$ where the X_i are independent, $E(X_i) = E(X)$ and $V(X_i) = V(X)$. Let $S_N = X_1 + X_2 + \cdots + X_N = \sum_{i=1}^N X_i$. Then $E(S_N) = E(N)E(X)$ and $V(S_N) = E(N)V(X) + [E(X)]^2 V(N)$. If $N = 0$, then $S_N = 0$.

50) Let the $X_i = M_i$ be iid from a discrete distribution where the M_i and N are as in 49). Suppose that the pdf of the M_i is $P_M(z)$ and the pgf of N is $P_N(Z)$. Then the pgf of $S_N = \sum_{i=1}^N M_i$ is $P_N(P_M(z))$. Then S_N is called a compound distribution where N is the primary distribution and M is the secondary distribution.

51) *Bernoulli trick or shortcut:* Suppose X takes on two values a and b with $q = P(X = a) = 1 - P(X = b)$. Then $V(X) = (b - a)^2 P(X = a)P(X = b)$.