Math 402 Exam 3 is Wed. April 27. **You are allowed 12 sheets of notes and a calculator.** The exam emphasizes HW8-10 and Q8-10.

Chapter 7

A policy value or NLP terminal reserve ${}_tV = E({}_tL) = E(Z_{x+t} - PY_{x+t}) = A_{x+t} - Pa_{x+t}$ for insurance. So the terminal reserve = (APV of insurance or annuity at time $x+t$) − (APV of premiums yet to be paid at time $x+t$) = value of policy at time t. ${}_0L = L$ from ch. 6. **If the insurance benefit is B instead of 1, multiply the unit benefit formula ${}_tV$ by B.** Formulas for ${}_tV$ are for unit benefit except 151) and 152).

139) Given $X > x+t$, $T_{x+t} = T_x - t$ for $t > 0$ while $K_{x+t} = K_x - t$ for integer t.

140) Continuous funding, continuous payment whole life insurance: Given $T_x > t$, ${}_t\overline{L}(\overline{A}_x) = v^{T_x - t} - [\overline{P}(\overline{A}_x)]\ \overline{a}_{\overline{T_x - t|}} = \overline{Z}_{x+t} - [\overline{P}(\overline{A}_x)]\ \overline{Y}_{x+t}$. The NLP terminal reserve ${}_t\overline{V}(\overline{A}_x) = E[\ {}_t\overline{L}(\overline{A}_x)] = \overline{A}_{x+t} - [\overline{P}(\overline{A}_x)]\ \overline{a}_{x+t} = 1 - \dfrac{\overline{a}_{x+t}}{\overline{a}_x} = [\overline{P}(\overline{A}_x)][\overline{s}_{x:\overline{t}|}] - {}_t\overline{k}_x = 1 - [\overline{P}(\overline{A}_x) + \delta]\ \overline{a}_{x+t} = [\overline{P}(\overline{A}_{x+t}) - \overline{P}(\overline{A}_x)]\ \overline{a}_{x+t}$.

$$\mathrm{Var}[\ {}_t\overline{\mathrm{L}}(\overline{\mathrm{A}}_{\mathrm{x}})] = \mathrm{V}[\ {}_t\overline{\mathrm{L}}(\overline{\mathrm{A}}_{\mathrm{x}})] = \left(\frac{1}{\delta\ \overline{\mathrm{a}}_{\mathrm{x}}}\right)^2\ [\ {}^2\overline{\mathrm{A}}_{\mathrm{x+t}} - (\overline{\mathrm{A}}_{\mathrm{x+t}})^2] = \frac{{}^2\overline{\mathrm{A}}_{\mathrm{x+t}} - (\overline{\mathrm{A}}_{\mathrm{x+t}})^2}{(1-\overline{\mathrm{A}}_{\mathrm{x+t}})^2}.$$

141) Discrete whole life insurance with annual premiums. Given $K_x \geq t$, i) whole life: ${}_tL_x = v^{K_x+1-t} - P_x\ \ddot{a}_{\overline{K_x+1-t|}} = Z_{x+t} - P_x\ \ddot{Y}_{x+t}$. Then ${}_0L_x = L_x$. The NLP terminal reserve ${}_tV_x = E(\ {}_tL_x) = A_{x+t} - P_x\ \ddot{a}_{x+t} = 1 - \dfrac{\ddot{a}_{x+t}}{\ddot{a}_x} = \dfrac{P_{x+t} - P_x}{P_{x+t} + d} = [1 - \dfrac{P_x}{P_{x+t}}]\ A_{x+t} = 1 - (P_x + d)\ddot{a}_{x+t}$.

$$\mathrm{Var}(\ {}_t\mathrm{L}_{\mathrm{x}}) = \mathrm{V}(\ {}_t\mathrm{L}_{\mathrm{x}}) = \left(1 + \frac{\mathrm{P}_{\mathrm{x}}}{\mathrm{d}}\right)^2\ [\ {}^2\mathrm{A}_{\mathrm{x+t}} - (\mathrm{A}_{\mathrm{x+t}})^2] = \left(\frac{1}{\mathrm{d}\ \ddot{\mathrm{a}}_{\mathrm{x}}}\right)^2\ [\ {}^2\mathrm{A}_{\mathrm{x+t}} - (\mathrm{A}_{\mathrm{x+t}})^2]$$
$$= \mathrm{Var}[\ {}_t\mathrm{L}_{\mathrm{x}}] = \frac{{}^2\mathrm{A}_{\mathrm{x+t}} - (\mathrm{A}_{\mathrm{x+t}})^2}{(1-\mathrm{A}_{\mathrm{x}})^2}.$$

ii) n year term: ${}_tV^1_{x:\overline{n}|} = A^1_{x+t:\overline{n-t}|} - P^1_{x:\overline{n}|}\ \ddot{a}_{x+t:\overline{n-t}|}$ for $t < n$.

iii) n year pure endowment: ${}_tV_{x:\overline{n}|}^{\ \ 1} = A_{x+t:\overline{n-t}|}^{\ \ \ \ 1} - P_{x:\overline{n}|}^{\ \ 1}\ \ddot{a}_{x+t:\overline{n-t}|}$ for $t < n$.

iv) n year endowment: ${}_tV_{x:\overline{n}|} = A_{x+t:\overline{n-t}|} - P_{x:\overline{n}|}\ \ddot{a}_{x+t:\overline{n-t}|}$ for $t < n$.

v) h-pay whole life insurance (h premiums): ${}^h_tV_x = A_{x+t} - {}_hP_x\ \ddot{a}_{x+t:\overline{h-t}|}$ for $t < h$, and ${}^h_tV_x = A_{x+t}$ for $t \geq h$.

vi) n-year deferred insurance with n premiums: ${}^n_tV(\ {}_n|A_x) = {}_{n-t}|A_{x+t} - {}_nP(\ {}_n|A_x)\ddot{a}_{x+t:\overline{n-t}|}$ for $t < n$, and ${}^n_tV(\ {}_n|A_x) = A_{x+t}$ for $t \geq n$.

vii) n-year deferred annuity with n premiums: ${}_tV(\ {}_n|\ddot{a}_x) = {}_{n-t}|\ddot{a}_{x+t} - P(\ {}_n|\ddot{a}_x)\ddot{a}_{x+t:\overline{n-t}|}$.

These formulas are for integer t.

142) Fully continuous insurance (or annuity) has continuous payment and continuous funding with a continuous insurance (or annuity). Formulas are valid for real $t > 0$ and the formulas tend to be the same as those in 141) after barring V, A, and a. Assume $T_x > t$.

i) whole life: see 140). ${}_t\overline{V}(\overline{A}_x) = 1 - \dfrac{\overline{a}_{x+t}}{\overline{a}_x}$ and $\mathrm{Var}[\ {}_t\overline{\mathrm{L}}(\mathrm{A}_{\mathrm{x}})] = \dfrac{{}^2\overline{\mathrm{A}}_{\mathrm{x+t}} - (\overline{\mathrm{A}}_{\mathrm{x+t}})^2}{(1-\overline{\mathrm{A}}_{\mathrm{x+t}})^2}$.

ii) n year term: ${}_t\overline{V}(\overline{A}^1_{x:\overline{n}|}) = \overline{A}^1_{x+t:\overline{n-t}|} - \overline{P}(\overline{A}^1_{x:\overline{n}|})\ \overline{a}_{x+t:\overline{n-t}|}$ for $t < n$.

iii) n year pure endowment: ${}_t\overline{V}_{\substack{1\\ x:\overline{n}|}} = A_{\substack{1\\ x+t:\overline{n-t}|}} - \overline{P}_{\substack{1\\ x:\overline{n}|}}\ \overline{a}_{x+t:\overline{n-t}|}$ for $t < n$. Note that there is no bar over the A.

iv) n year endowment: ${}_t\overline{V}(\overline{A}_{x:\overline{n}|}) = \overline{A}_{x+t:\overline{n-t}|} - \overline{P}(\overline{A}_{x:\overline{n}|})\ \overline{a}_{x+t:\overline{n-t}|}$ for $t < n$.

v) h-pay whole life insurance: ${}^h_t\overline{V}(\overline{A}_x) = \overline{A}_{x+t} - {}_h\overline{P}(\overline{A}_x)\ \overline{a}_{x+t:\overline{h-t}|}$ for $t < h$, and ${}^h_t\overline{V}(\overline{A}_x) = \overline{A}_{x+t}$ for $t \geq h$.

vi) n-year deferred annuity: ${}_t\overline{V}({}_n|\overline{a}_x) = {}_{n-t}|\overline{a}_{x+t} - \overline{P}({}_n|\overline{a}_x)\overline{a}_{x+t:\overline{n-t}|}$.

143) Note that fully continuous insurance and annuities tend to have the insurance or annuity in parentheses for the reserve ${}_tV$ and the premium P. For discrete insurance and annuities, the parentheses are dropped but the subscripts are used for the reserve ${}_tV$ and the premium P. An exception is discrete deferred insurance and annuities which do use parentheses.

144) **Know:** Suppose the equivalence principle is used to determine premiums.

i) fully continuous whole life: $\mathrm{Var}[{}_t\mathrm{\overline{L}}(\mathrm{\overline{A}_x})] = \dfrac{{}^2\mathrm{\overline{A}_{x+t}} - (\mathrm{\overline{A}_{x+t}})^2}{(1-\mathrm{\overline{A}_x})^2}$. See 140).

ii) fully continuous n-year endowment insurance $t < n$:

$\mathrm{Var}[{}_t\mathrm{\overline{L}}(\mathrm{\overline{A}_{x:\overline{n}|}})] = \dfrac{{}^2\mathrm{\overline{A}_{x+t:\overline{n-t}|}} - (\mathrm{\overline{A}_{x+t:\overline{n-t}|}})^2}{(1-\mathrm{\overline{A}_{x:\overline{n}|}})^2}$.

iii) discrete whole life: $\mathrm{Var}[{}_t\mathrm{L_x}] = \dfrac{{}^2\mathrm{A_{x+t}} - (\mathrm{A_{x+t}})^2}{(1-\mathrm{A_x})^2}$. See 138) i).

iv) discrete n-year endowment insurance with integral $t < n$:

$\mathrm{Var}[{}_t\mathrm{L_{x:\overline{n}|}}] = \dfrac{{}^2\mathrm{A_{x+t:\overline{n-t}|}} - (\mathrm{A_{x+t:\overline{n-t}|}})^2}{(1-\mathrm{A_{x:\overline{n}|}})^2}$.

Take $t = 0$ to get $\mathrm{Var}[{}_0\mathrm{L}] = \mathrm{Var}[\mathrm{L}]$ for chapter 6.

145) Continuous payment, continuous funding, discrete insurance puts bars over V, P, and a. So ${}_t\overline{V}_x = A_{x+t} - \overline{P}_x\overline{a}_{x+t}$.

146) Policy values = NLP terminal reserves for continuous payment insurance with annual premiums put a bar over A.

i) whole life: ${}_tV(\overline{A}_x) = \overline{A}_{x+t} - P(\overline{A}_x)\ \ddot{a}_{x+t}$.

ii) n year term: ${}_tV(\overline{A}^1_{x:\overline{n}|}) = \overline{A}^1_{x+t:\overline{n-t}|} - P(\overline{A}^1_{x:\overline{n}|})\ \ddot{a}_{x+t:\overline{n-t}|}$ for $t < n$.

iii) n year endowment: ${}_tV(\overline{A}_{x:\overline{n}|}) = \overline{A}_{x+t:\overline{n-t}|} - P(\overline{A}_{x:\overline{n}|})\ \ddot{a}_{x+t:\overline{n-t}|}$ for $t < n$.

iv) h-pay n-year term (h premiums): ${}^h_tV(\overline{A}^1_{x:\overline{n}|}) = \overline{A}^1_{x+t:\overline{n-t}|} - {}_hP(\overline{A}^1_{x:\overline{n}|})\ \ddot{a}_{x+t:\overline{h-t}|}$ for $t < h < n$, and ${}^h_tV(\overline{A}^1_{x:\overline{n}|}) = \overline{A}^1_{x+t:\overline{n-t}|}$ for $h < t < n$.

147) At time t, let ${}_tV$ be the policy value = NLP terminal reserve, A_{x+t} be the APV of the insurance, a_{x+t} be the APV of the remaining unit premiums, and P_{x+t} be the premium for $(x+t)$.

i) The prospective formula is ${}_tV = A_{x+t} - P_x a_{x+t}$.

ii) The premium difference formula is ${}_tV = a_{x+t}(P_{x+t} - P_x)$.

iii) The paid up insurance formula is ${}_tV = A_{x+t}\left(1 - \dfrac{P_x}{P_{x+t}}\right)$.

148) For example, i) discrete whole life: ${}_tV_x = A_{x+t} - P_x\ \ddot{a}_{x+t} = \ddot{a}_{x+t}(P_{x+t} - P_x) = A_{x+t}\ [1 - \frac{P_x}{P_{x+t}}] = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}$.

ii) fully continuous whole life: ${}_t\overline{V}(\overline{A}_x) = \overline{A}_{x+t} - [\overline{P}(\overline{A}_x)]\ \overline{a}_{x+t} =$
$\overline{a}_{x+t}[\overline{P}(\overline{A}_{x+t}) - \overline{P}(\overline{A}_x)] = \overline{A}_{x+t}\ [1 - \dfrac{\overline{P}(\overline{A}_x)}{\overline{P}(\overline{A}_{x+t})}] = 1 - \dfrac{\overline{a}_{x+t}}{\overline{a}_x}$.

149) Using the same notation as in 147), for whole life and endowment insurance,

i) the annuity ratio formula is ${}_tV = \left(1 - \dfrac{a_{x+t}}{a_x}\right)$.

ii) The insurance ratio formula is ${}_tV = \dfrac{A_{x+t} - A_x}{1 - A_x}$.

iii) The premium ratio formula is ${}_tV = \dfrac{P_{x+t} - P_x}{P_{x+t} + d}$, but replace d by δ for fully continuous insurance.

Know how to use i) and ii) to calculate a reserve for discrete whole life insurance when mortality follows the illustrative life table where ${}_tV_x = \left(1 - \dfrac{\ddot{a}_{x+t}}{\ddot{a}_x}\right) = \dfrac{A_{x+t} - A_x}{1 - A_x}$.

For fully continuous whole life insurance, ${}_t\overline{V}(\overline{A}_x) = 1 - \dfrac{\overline{a}_{x+t}}{\overline{a}_x} = \dfrac{\overline{A}_{x+t} - \overline{A}_x}{1 - \overline{A}_x} = \dfrac{\overline{P}(\overline{A}_{x+t}) - \overline{P}(\overline{A}_x)}{\overline{P}(\overline{A}_{x+t}) + \delta}$.

150) If $T_x \sim EXP(\mu)$, then ${}_t\overline{V}(\overline{A}_x) = 0$ since $\overline{a}_x = \overline{a}_{x+t} = \dfrac{1}{\mu + \delta}$ and $\overline{A}_x = \overline{A}_{x+t} = \dfrac{\mu}{\mu + \delta}$ do not depend on t. See 28). Also, $Var(\ {}_t\overline{L}(\overline{A}_x) = {}^2\overline{A}_x = \dfrac{\mu}{\mu + 2\delta}$.

151) A recursive formula is ${}_{t+1}V = (\ {}_tV + P)(1 + i) - q_{x+t}(B - \ {}_{t+1}V)$ where B is the benefit paid, often $B = 1$ unit.

152) Let π_t be the premium paid at time $t = 0, 1, ...$ and let b_k be the benefit paid at time k if the death occurs in the kth year of the policy for $k = 1, 2, ...$ Then
$(\ {}_{k-1}V + \pi_{k-1})(1 + i) = q_{x+k-1}(b_k - \ {}_kV) + \ {}_kV$.

153) ${}_t\overline{L}(\overline{A}_x) = v^{T_{x+t}} - \overline{P}(\overline{A}_x)\overline{a}_{\overline{T_{x+t}}|} = \overline{Z}_{x+t} - \overline{P}(\overline{A}_x)\overline{Y}_{x+t}$. where $T_{x+t} = T_x - t$. Taking expectations gives ${}_t\overline{V}(\overline{A}_x) = \overline{A}_{x+t} - \overline{P}(\overline{A}_x)\overline{a}_{x+t}$.

154) A *gross premium* G takes into account expenses. Then an expense–augmented reserve = *gross premium reserve* ${}_tV^E$ = (APV of future benefits and expenses) − (APV of future gross premiums).

155) Suppose premiums P are paid every year including at time t. Let $0 \leq s \leq 1$, then ${}_{t+s}V \approx (\ {}_tV + P)(1 - s) + (\ {}_{t+1}V)s$. Note that $(\ {}_tV + P)$ is the terminal reserve at time $t+$ just after the premium P has been paid. Also ${}_0V = 0$ under the equivalence principle.

156) Consider a fully continuous whole life model with payment benefit b_r at time of death r and benefit payment at rate $\overline{P}(r)$ at time r. Provided $T_x > t$, the NLP terminal reserve is ${}_tV = \int_0^\infty b_{t+s}\ e^{-\delta s}\ {}_sp_{x+t}\ \ \mu_{x+t+s}\ ds - \int_0^\infty \overline{P}(t + s)\ e^{-\delta s}\ {}_sp_{x+t}\ ds$.

157) **Know** In 156), suppose $\overline{P}(t + s) = \pi_0 e^{\gamma(t+s)}$ for $s, t \geq 0$, $\mu_{x+t} \equiv \mu$ for $t > 0$ so $T_{x+t} \sim EXP(\mu)$ for any $t \geq 0$, $b_{t+s} = Je^{\theta(t+s)}$ for $s, t \geq 0$. Then ${}_sp_{x+t} = e^{-\mu s}$, for $s, t \geq 0$. Then for $T_x > t$, $\pi_0 = \dfrac{J\mu(\mu + \delta - \gamma)}{\mu + \delta - \theta}$, and ${}_t\overline{V} = \dfrac{J\mu\ e^{\theta t}}{\mu + \delta - \theta} - \dfrac{\pi_0\ e^{\gamma t}}{\mu + \delta - \gamma}$.

158) **Know** In 157), $\overline{P}(t+s) = \pi_0 e^{\gamma(t+s)}$ is often written as *the annual premium rate is* $\pi_0 e^{\gamma t}$ for all t (or $t \geq 0$). The benefit $b_{t+s} = Je^{\theta(t+s)}$ is often written as *the benefit is* $Je^{\theta t}$ *if death occurs at time* t. If $\gamma = 0$, then $\overline{P}(t+s) \equiv \pi_0 = \dfrac{J\mu(\mu+\delta)}{\mu+\delta-\theta}$ for $s, t \geq 0$. If $\theta = 0$, then $b_{t+s} \equiv J$ for $s, t \geq 0$.

More Topics from Ch. 7, 8, 9, 10

ch. 9

159) **Know:** Assume time and cause of decrement are independent. Suppose there are m decrements and a continuous whole life insurance pays benefit $b_t^{(j)}$ if decrement j occurs at time t. Let $\overline{Z}$ be the benefit random variable for the insurance. Then the **single benefit premium** (buy the insurance at time 0 with 1 payment, also called the net single premium) is

$$\text{APV} = \overline{A} = E[\overline{Z}] = \sum_{j=1}^{m} \int_0^\infty b_t^{(j)} e^{-\delta t} \; {}_tp_x^{(\tau)} \mu_{x+t}^{(j)} \, dt = \sum_{j=1}^{m} \overline{A}^{(j)}, \text{ and}$$

$$E[\overline{Z}^2] = \sum_{j=1}^{m} \int_0^\infty [b_t^{(j)}]^2 e^{-2\delta t} \; {}_tp_x^{(\tau)} \mu_{x+t}^{(j)} \, dt = \sum_{j=1}^{m} {}^2\overline{A}^{(j)}.$$

If $\mu_{x+t}^{(j)} \equiv \mu^{(j)} \equiv \mu_j$, and $b_t^{(j)} \equiv b_j$ are free of $t \geq 0$, then $\mu^{(\tau)} = \sum_{j=1}^m \mu^{(j)}$, the single benefit premium $= E[\overline{Z}] = \displaystyle\sum_{j=1}^{m} \frac{b_j \mu^{(j)}}{\mu^{(\tau)} + \delta}$, and $E[\overline{Z}^2] = \displaystyle\sum_{j=1}^{m} \frac{(b_j)^2 \; \mu^{(j)}}{\mu^{(\tau)} + 2\delta}$.

160) For discrete whole life insurance as in 159) except the benefit b_j is paid at the end of the year $k = 1, 2, \ldots$, if decrement j occurs in the kth year, $\overline{A}^{(j)} = b_j \sum_{k=0}^\infty v^{k+1} \; {}_kp_x^{(\tau)} \; q_{x+k}^{(j)}$, and APV = single net premium $= \sum_{j=1}^m \overline{A}^{(j)}$.

161) If a contract is taken out that pays b_j if decrement j occurs but 0 otherwise, then the APV of the contract is APV = single net premium $= \overline{A}^{(j)}$. (Just set $b_k = 0$ for the other decrements.)

ch. 10

162) Premiums for joint life status or last survivor status are calculated in the usual way under the equivalence principle. Let (w) = (xy) or (w) = $(\overline{xy})$. Then APV insurance = APV premiums so $P = \dfrac{A_w}{\ddot{a}_w}$.

163) **Know:** For fully continuous whole life insurance (with premiums paid continuously until first death for (xy) and until second death for $(\overline{xy})$), $\overline{P}(\overline{A}_{xy}) = \dfrac{\overline{A}_{xy}}{\overline{a}_{xy}}$ for the joint life status, while $\overline{P}(\overline{A}_{\overline{xy}}) = \dfrac{\overline{A}_{\overline{xy}}}{\overline{a}_{\overline{xy}}} = \dfrac{\overline{A}_x + \overline{A}_y - \overline{A}_{xy}}{\overline{a}_x + \overline{a}_y - \overline{a}_{xy}}$ for the last survivor status. These formulas are for unit benefit.

164) Variant: if premiums are paid until first death but insurance is paid after last death, then $P = \dfrac{A_{\overline{xy}}}{\ddot{a}_{xy}}$. See 165).

165) **Know:** Fully continuous whole life insurance of 1 on the last survivor of (x) and (y), but premiums payable (continuously) until first death has premium $\overline{P} = \dfrac{\overline{A}_{\overline{xy}}}{\overline{a}_{xy}} =$

$\dfrac{\overline{A}_x + \overline{A}_y - \overline{A}_{xy}}{\overline{a}_{xy}}$.

166) Often have $T_x \sim EXP(\mu_1) \perp\!\!\!\perp T_y \sim EXP(\mu_2)$. Then $T_{xy} \sim EXP(\mu_1 + \mu_2)$, $\overline{A}_w \overset{E}{=} \dfrac{\mu}{\mu+\delta}$, and $\overline{a}_w \overset{E}{=} \dfrac{1}{\mu+\delta}$ where w and μ correspond to x, y, or xy.

A variant is the common shock model where $T_x \sim EXP(\mu_x = \mu_1^* + \lambda), T_y \sim EXP(\mu_y = \mu_2^* + \lambda)$, and $T_{xy} \sim EXP(\mu_1^* + \mu_2^* + \lambda)$.

167) The policy value = terminal reserve for a joint life status (w) = (xy) is like the reserve for a single life status (w), provided that the status (w) = (xy) has not yet failed at time t. Note that the subscript $w + t = x + t : y + t$.

168) The policy value = terminal reserve for a last survivor life status (w) = ($\overline{xy}$) is more complicated, but is like the reserve for a single life status (w), provided that both (x) and (y) still survive at time t. Then the subscript $w + t = \overline{x + t : y + t}$.

$\oint$ **6.4.2 gross (annual) premium** = contract (annual) premium

169) For a gross premium G, the equivalence principle says that $E({}_0L_e) = E({}_0L) =$ APV(benefits + expenses) $-$ APV(gross premiums), where ${}_0L_e = {}_0L$ is the loss random variable at issue. One type of problem uses this equation to solve for G.

170) A variant of 169) is to find (the observed value of) ${}_0L_e = {}_0L =$ APV(benefits + expenses) $-$ APV(gross premiums) for person who died in the kth year where k is small and G is given.

$\oint$ **7.2.4**

171) $${}_kAS = \frac{[\,{}_{k-1}AS + G(1 - c_{k-1}) - e_{k-1}](1+i) - b_k q^{(d)}_{x+k-1} - {}_kCV q^{(w)}_{x+k-1}}{1 - q^{(d)}_{x+k-1} - q^{(w)}_{x+k-1}}$$

172) Given all of the variables in 171) except one, usually ${}_kAS$ or i, calculate the unknown variable where ${}_kAS$ is the asset share at the end of year k,
G is the gross premium (= contract premium),
c_k is the proportion of the premium payable as an expense at time k, starting at $k = 0$,
e_k is the per policy expense at time k,
b_k is the face amount,
$q^{(d)} = q^{(1)}$ is the death probability,
$q^{(w)} = q^{(2)}$ is the withdrawal probability,
${}_kCV$ is the cash value at time k.

173) Usually assume ${}_0AS = 0$.

174) The formula is 171) is for fully discrete insurance. So premiums are paid at the beginning of the year and benefits at the end of the year.

$\oint$ **10.3**

175) A *revisionary annuity* is for two lives (x) and (y). a) If the beneficiary is (y), then provided that (y) survives (x), after (x) dies, (y) receives an annuity until (y) dies. If (y) dies first, then the insurance contract ends and no benefits are paid. b) Similarly, if the beneficiary is (x), then provided that (x) survives (y), after (y) dies, (x) receives an annuity until (x) dies. If (x) dies first, then the insurance contract ends and no benefits are paid. Suppose the insurance is as in a). i) discrete whole life $a_{x|y} = a_y - a_{xy}$. ii) discrete n year term $a_{x|y:\overline{n}|} = a_{y:\overline{n}|} - a_{xy:\overline{n}|}$. iii) continuous whole life $\overline{a}_{x|y} = \overline{a}_y - \overline{a}_{xy}$.

iv) continuous n year term term $\overline{a}_{x|y:\overline{n}|} = \overline{a}_{y:\overline{n}|} - \overline{a}_{xy:\overline{n}|}$.

176) Premiums are paid until one of (x) or (y) dies (failure of joint life status). Assume the equivalence principle is used. Consider insuarance a) in 175). i) If the fully discrete whole life insurance is funded by annual premiums, then $P(a_{x|y}) = \dfrac{a_{x|y}}{\ddot{a}_{xy}} = \dfrac{a_y - a_{xy}}{\ddot{a}_{xy}}$. ii) If the fully continuous whole life insurance is funded by a continuously paid premium, then $\overline{P}(\overline{a}_{x|y}) = \dfrac{\overline{a}_{x|y}}{\overline{a}_{xy}} = \dfrac{\overline{a}_y - \overline{a}_{xy}}{\overline{a}_{xy}}$.

177) Referring to 176), if both (x) and (y) are alive, then ${}_tV(a_{x|y}) = a_{x+t|y+t} - P(a_{x|y})\ddot{a}_{x+t:y+t}$ and ${}_t\overline{V}(\overline{a}_{x|y}) = \overline{a}_{x+t|y+t} - \overline{P}(\overline{a}_{x|y})\overline{a}_{x+t:y+t}$. If only (y) is alive then ${}_tV(a_{x|y}) = a_{y+t}$ and ${}_t\overline{V}(\overline{a}_{x|y}) = \overline{a}_{y+t}$. If only (x) is alive then the contract is expired and the benefit reserves ${}_tV(a_{x|y}) = 0$ and ${}_t\overline{V}(\overline{a}_{x|y}) = 0$.

178) For revisionary insurance b), use formulas like $\overline{a}_{y|x} = \overline{a}_x - \overline{a}_{xy}$.

∮ **8.10 Markov Chains:** Insurance, Annuities, Premiums, Reserves

179) For discrete insurance Z that pays benefit b_k if $\lfloor T_x \rfloor = k-1$ at time k with probability $p_k = P(\lfloor T_x \rfloor = k-1) = {}_{k-1}|q_x$, the possible values of Z are $b_k v^k$ which occur with probability p_k. Hence $APV = E(Z) = \sum_{k=1}^m b_k v^k p_k$ where $m = \infty$ is possible. Note that the summand is the triple product of i) the benefit b_k, ii) the discount factor v^k, and iii) the probability p_k. The APV of a set of cashflows using a Markov chain is similar. The APV is the sum of a triple product of i) the cashflow c_k, ii) the discount factor v^k, and iii) the probability p_k that the cashflow occurs, computed using the Markov chain. So for an insurance Z, the possible values are of Z are $c_k v^k$ which occur with probability p_k.

Lots of variants are possible. For example there could be states i) alive, ii) death from accident, and iii) death from non-accident. Could have benefit $b_{k,1}$ at time k if death occurred due to accident with probability $p_{k,1}$, and benefit $b_{k,2}$ at time k if death occurred due to non-accident with probability $p_{k,2}$, where the probabilities are determined using the Markov chain. Then APV $= \sum_k (b_{k,1} v^k p_{k,1} + b_{k,2} v^k p_{k,2})$, but the principle is the same, each summand is the triple product: (cashflow at time k)$(v^k)(p_k)$ where p_k is the probability that the cashflow is made.

180) An annuity with Markov chains is similar. There will be consecutive possible cash flows, say $C_1, ..., C_m$ at times 1, ..., m which are paid with probability p_k determined from the Markov chain. Then the APV $= \sum_{k=1}^m C_k v^k p_k$.

181) The premium is determined from the equivalence principle. Let the APV(benefits) be computed as in 179) or 180). The annuity due of 1 for $J+1$ periods has possible values $v^0, v^1, ..., v^J$. Assume these have probabilities p_i where $p_0 = 1$ since the first premium of 1 is certain to be paid. Then the APV(annuity due of 1) is $\sum_{i=0}^J v^i p_i$, and the premium is $P =$ APV(benefits)/(APV(annuity due of 1). Time diagrams are useful.

The following material is on the final but not on Exam 3.

Ch. 13: 182) The expected profit Pr_{t+1} in the $(t+1)$st contract year is measured at the end of the contract year, and, assuming 2 decrements, $Pr_{t+1} =$
$[\,{}_tV^G + G_{t+1}(1-r_{t+1}) - e_{t+1}](1+i_{t+1}) - [(b^{(1)}_{t+1} + s^{(1)}_{t+1})q^{(1)}_{x+t} + (b^{(2)}_{t+1} + s^{(2)}_{t+1})q^{(2)}_{x+t} + {}_{t+1}V^G p^{(\tau)}_{x+t}]$.
Here an expense of $s^{(i)}_{t+1}$ is used to settle a benefit claim due to cause i in the $(t+1)$st year. Subscript t is for the tth year, subscript $t+1$ for the $(t+1)$st year, ${}_tV^G$ is the benefit

reserve using gross premiums, G_t is the gross premium, r_t is the percent of premium expense factor, and e_t is the fixed expense.

183) If death is the only decrement,
$Pr_{t+1} = [\ {}_tV^G + G_{t+1}(1 - r_{t+1}) - e_{t+1}](1 + i_{t+1}) - [(b_{t+1} + s_{t+1})q_{x+t} +\ {}_{t+1}V^G p_{x+t}]$.

184) $Pr_t \stackrel{set}{=} 0$ was used to calculate $G \equiv G_t$ where the quantities were assumed known. Now the quantities are the ones actually observed (some are unknown).

185) **Know:** The **profit vector** $\boldsymbol{P_r} = (Pr_0, Pr_1, ..., Pr_n)$. Given the profit vector and ${}_kp_x$, $\Pi_0 = Pr_0$, $\Pi_1 = Pr_1$, and $\Pi_{t+1} = Pr_{t+1}(\ {}_tp_x)$ for $t = 1, ..., n-1$. The **profit signature** is the vector $\boldsymbol{\Pi} = (\Pi_0, \Pi_1, ..., \Pi_n) = (Pr_0, Pr_1, (\ {}_1p_x)Pr_2, (\ {}_2p_x)Pr_3, ..., (\ {}_{n-1}p_x)Pr_n)$. Note that $\Pi_t = (\ {}_{t-1}p_x)Pr_t$ for $t = 2, ..., n$. Π_{t+1} is the *expected profit* in the $(t+1)$st contract year.

186) **Know:** The rate of interest for discounting the Π_t is the *risk discount rate* or *hurdle rate* r. The interest rate to accumulate beginning of the year values is i. Usually $r > i$ to compensate the insurer for risks (such as interest rate risk).

187) **Know:** The NPV of the expected profits is
$NPV = \sum_{k=0}^n \Pi_k v_r^k = \Pi_0 + \Pi_1(1+r)^{-1} + \cdots + \Pi_n(1+r)^{-n}$.

188) The internal rate of return r_{IRR} is the value of r that makes the NPV $= 0$, but is hard to find by hand. You could be given r_{IRR} and asked to show that $NPV \approx 0$ where the approximation is due to rounding (even if r_{IRR} is given to 10 digits, the calculated NPV won't be exactly 0).

189) **Know:** The *profit margin* $= \dfrac{NPV}{APV(premiums)}$ where the APV is computed with the interest rate r used to compute the NPV. If level premiums are potentially paid (paid if the insured has not died by time n-1) at times 0, ..., $n-1$, then APV (unit premium) $= \ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v_r^k\ {}_kp_x$ where ${}_kp_x = \prod_{t=0}^{k-1}(1 - q_{x+t}) = \prod_{t=0}^{k-1} p_{x+t}$. Hence APV(premiums) $=$ P APV(unit premiums) or G APV(unit premiums) if the premium is P or G.

190) **Know:** The *partial net present value* $NPV(t) = \sum_{k=0}^t \Pi_k v_r^k$ for $t = 0, 1, ..., n-1$ where $NPV(n) = NPV$. The *discounted payback period* (DPP) (or *break even period*) is he smallest value of m such that $NPV(m) \geq 0$. The DPP represents the time until the insurer starts to make a profit on the contract. Given Π_t and r, be able to compute the DPP and NPV. The DPP does not exist if $NPV(t) < 0$ for $t = 0, 1, ..., n$.

$\oint$ 9.4

191) Assume the UDD assumption holds for all m decrements. Then the UDD assumption holds for the total decrement τ. So ${}_tq_x^{(j)} \approx (t)(q_x^{(j)})$ and ${}_tq_x^{(\tau)} \approx (t)(q_x^{(\tau)})$ for $j = 1, ..., m$ and $0 \leq t \leq 1$. Thus ${}_tp_x^{(k)} \approx 1 - (t)(q_x^{(k)})$ for $k = j, \tau$ and $0 \leq t \leq 1$.

192) If the UDD assumption holds for the jth decrement, then $q_x^{(j)} \approx\ {}_tp_x^{(\tau)}\mu_{x+t}^{(j)}$.

193) If the UDD assumption holds for the jth decrement and the total decrement τ and $0 \leq t \leq 1$, then i) $\mu_{x+t}^{(j)} \approx \dfrac{q_x^{(j)}}{{}_tp_x^{(\tau)}} \approx \dfrac{q_x^{(j)}}{1 - (t)(q_x^{(\tau)})}$. For the single decrement table probabilities, ii) ${}_tp_x^{'(j)} \approx \left[1 - (t)(q_x^{(\tau)})\right]^{q_x^{(j)}/q_x^{(\tau)}} \approx \left[\ {}_tp_x^{(\tau)}\right]^{q_x^{(j)}/q_x^{(\tau)}}$.

194) Can also assume that the m single decrement quantities satisfy the UDD assumption. So ${}_tq_x^{'(j)} \approx (t)(q_x^{'(j)})$ and ${}_tp_x^{'(j)}\mu_{x+t}^{(j)} \approx q_x^{'(j)}$ for $j = 1, ..., m$ and $0 \leq t \leq 1$.

195) In 194), assume $m = 2$. So $0 \le t \le 1$ and i) $q_x^{(1)} \approx q_x^{'(1)} \left(1 - \frac{q_x^{'(2)}}{2}\right)$

ii) ${}_tq_x^{(1)} \approx q_x^{'(1)} \left(t - \frac{t^2 q_x^{'(2)}}{2}\right)$, iii) ${}_{t|s}q_x^{(1)} \approx (s)(q_x^{'(1)}) \left[1 - \left(t + \frac{s}{2}\right) q_x^{'(2)}\right]$
where $0 < s + t \le 1$.

196) In 194), assume $m = 3$. Then $q_x^{(1)} \approx q_x^{'(1)} \left[1 - \frac{1}{2}(q_x^{'(2)} + q_x^{'(3)}) + \frac{1}{3}(q_x^{'(2)})(q_x^{'(3)})\right]$.

197) Recall $\delta = \log(1 + i)$. Under the UDD assumption, i) $\overline{A}_x \approx \frac{i}{\delta} A_x$,

ii) ${}^2\overline{A}_x \approx \left(\frac{2i + i^2}{2\delta}\right) \ ({}^2A_x)$, iii) $A_x^{(m)} \approx \frac{i}{i^{(m)}} A_x$,

iv) $\overline{a}_x = \frac{1 - \overline{A}_x}{\delta} \approx \frac{\frac{i}{\delta} A_x}{\delta} \approx \frac{1}{\delta} - \frac{i}{\delta^2} A_x$.
The illustrative life table gives 1000 A_x and 1000 $({}^2A_x)$.

198) Suppose the UDD assumption holds for all m decrements. Then the UDD assumption holds for the total decrement τ. Hence ${}_tq_x^{(k)} \approx (t)(q_x^{(k)})$ and ${}_tp_x^{(k)} \approx 1 - (t)(q_x^{(k)})$ for $k = \tau$ and $k = j = 1, ..., m$, and $0 \le t \le 1$.

199) If the UDD assumption holds for the jth decrement, then ${}_tp_x^{(\tau)} \mu_{x+t}^{(j)} \approx q_x^{(j)}$ for $0 \le t \le 1$.

200) If the UDD assumption holds for the jth decrement and the total decrement τ, let $0 \le t \le 1$. Then i) $\mu_{x+t}^{(j)} \approx \frac{q_x^{(j)}}{{}_tp_x^{(\tau)}} \approx \frac{q_x^{(j)}}{1 - (t)(q_x^{(\tau)})}$.
ii) For single decrement probabilities,
${}_tp_x^{'(j)} \approx \left[\ {}_tp_x^{(\tau)}\right]^{q_x^{(j)}/q_x^{(\tau)}} \approx \left[1 - (t)(q_x^{(\tau)})\right]^{q_x^{(j)}/q_x^{(\tau)}}$.

201) Can also assume that all m single decrement quantities satisfy the UDD assumption, so ${}_tq_x^{'(j)} \approx (t)(q_x^{'(j)})$ and ${}_tp_x^{'(j)} \mu_{x+t}^{(j)} \approx q_x^{'(j)}$ for $0 \le t \le 1$ and $j = 1, ..., m$. Let $0 \le t \le 1$. Recall that ${}_1q_x^{(1)} = q_x^{(1)}$ when $t = 1$.
i) If $m = 2$, then ${}_tq_x^{(1)} \approx q_x^{'(1)} \left(t - \frac{t^2\ q_x^{'(2)}}{2}\right)$.
ii) If $m = 3$, then $q_x^{(1)} \approx q_x^{'(1)} \left[1 - \frac{1}{2}(q_x^{'(2)} + q_x^{'(3)}) + \frac{1}{3}(q_x^{'(2)})(q_x^{'(3)})\right]$.

202) Recall $\delta = \log(1 + i)$. Under the UDD assumption, $\overline{A}_x \approx \frac{i}{\delta} A_x$,
${}^2\overline{A}_x \approx \left(\frac{2i + i^2}{2\delta}\right) ({}^2A_x)$, $\overline{A}_x^{(m)} \approx \frac{i}{i^{(m)}} A_x$, and $\overline{a}_x = \frac{1 - \overline{A}_x}{\delta} \approx \frac{1}{\delta} - \frac{i}{\delta^2} A_x$.

End $\oint$ 9.4 UDD approximation material.

203) **Know**: If g is an increasing function, then $g(t_\alpha)$ is the αth percentile of $g(T)$, while if g is a decreasing function, then $g(t_{1-\alpha})$ is the αth percentile of $g(T)$. Increasing and decreasing are "strictly increasing" and "strictly decreasing" in some texts.

204) The 100 αth percentile premium π is the premium which results in a positive loss at issue with probability α. So want $\alpha = P(e^{-\delta T_x} > \pi \overline{a}_{\overline{T_x}|}) = P\left[e^{-\delta T_x} > \pi\left(\frac{1-e^{-\delta T_x}}{\delta}\right)\right] =$ $P\left(\overline{Z}_x > \frac{\pi}{\pi+\delta}\right) = P\left[T_x < \frac{-1}{\delta}\log\left(\frac{\pi}{\pi+\delta}\right)\right]$. This results in $\pi = \frac{\delta e^{-\delta t_\alpha}}{1-e^{-\delta t_\alpha}} \stackrel{E}{=} \frac{\delta(1-\alpha)^{\delta/\mu}}{1-(1-\alpha)^{\delta/\mu}}$.

If the insurance benefit is K instead of 1, then $\pi = \frac{K\delta e^{-\delta t_\alpha}}{1-e^{-\delta t_\alpha}}$. Here $t_\alpha = \frac{-1}{\delta}\log(\frac{\pi}{\pi+\delta})$, and $P(T_x \le t_\alpha) = \alpha$. Get t_α from point 205) for two distributions.

205) **Know:** If $T \sim U(0,\theta)$, then $t_\alpha = \alpha\theta$. If $T_0 \sim U(0,\omega)$, then $T_x \sim U(0,\omega - x)$ has $\theta = \omega - x$. If $T_x \sim EXP(\mu)$ then $t_\alpha = \frac{-\log(1-\alpha)}{\mu}$.