Math 402 Exam 1 is Wed. Feb. 15. **You are allowed 12 sheets of notes and a calculator.** The exam covers HW1-3, and Q1-3. Numbers refer to types of problems on exam. In this class $\log(t) = \ln(t) = \log_e(t)$ while $\exp(t) = e^t$. **Chapters 2, and 3:**

1) **Memorize** the following distributions:

a) exponential($\mu$)= gamma($\nu = 1, \mu$) where $\mu > 0$ is the force of mortality:

$$f(x) = \mu \exp(-\mu x) \; I(x \geq 0).$$

$E(X) = 1/\mu, \quad VAR(X) = 1/\mu^2$. $F(x) = 1 - \exp(-\mu x), \; x \geq 0$. Here $I(x \geq 0) = 1$ if $x \geq 0$ and $I(x \geq 0) = 0$, otherwise. (The parameterization with $\lambda = 1/\mu$ is common. Then $E(X) = \lambda$ and $V(X) = \lambda^2$.) $S(x) = \exp(-\mu x), \; x \geq 0$.

b) uniform($\theta_1, \theta_2$) and De Moivre($\theta$) = uniform($0, \theta$):

$$f(x) = \frac{1}{\theta_2 - \theta_1} I(\theta_1 \leq x \leq \theta_2).$$

$F(x) = (x - \theta_1)/(\theta_2 - \theta_1)$ for $\theta_1 \leq x \leq \theta_2$.
$E(X) = (\theta_1 + \theta_2)/2. \quad VAR(X) = (\theta_2 - \theta_1)^2/12.$

2) The cdf $F(x) = P(X \leq x)$, the survival function $S(x) = P(X > x)$, the pdf $f(x) = F'(x)$, $\mu(x)$ = force of mortality = hazard rate function, $E(X) = \overset{o}{e}_0$ .

3) $(x)$ denotes a person alive at age $x$.

4) Let $X = T_0$ where $T_x = T(x)$ is the time until failure for a person alive at age $x$. Then $T_0 = x + T_x$ given $T_0 > x$. Also, ${}_tp_0 = S_0(t) = P(T_0 > t)$, ${}_tq_0 = F_0(t) = P(T_0 \leq t)$, and $E(T_0) = \int_0^\infty t f_0(t) dt = \int_0^\infty S_0(t) dt$ if $\lim_{t \to \infty} t S_0(t) = 0$.

5) Let $t > 0$. Let $G_x = G_{T_x}$ where $G$ is $T, S, F, \mu$, or $f$. If there is no subscript $x$, then $G = G_0$.

$$i) \;\; {}_tp_x = S_x(t) = \frac{S_0(x+t)}{S_0(x)} = 1 - \;{}_tq_x = P(T_x > t) = P(T_0 > x + t | T_0 > x)$$

$$= \exp\left(-\int_x^{x+t} \mu_y \, dy\right) = \exp\left(-\int_0^t \mu_{x+w} \, dw\right)$$

Note that $S_0(x+t) = S_0(x) S_x(t)$.

$$ii) \;\; {}_tq_x = F_x(t) = 1 - \;{}_tp_x = 1 - \frac{S_0(x+t)}{S_0(x)} = P(T_x \leq t) = P(T_0 \leq x + t | T_0 > x)$$

$$iii) \;{}_tp_x \; \mu_{x+t} \;\; = \;\; f_x(t) = \frac{f_0(x+t)}{S_0(x)} = \frac{d}{dt} F_x(t) = -\frac{d}{dt} S_x(t)$$

$$iv) \;\; \mu_{x+t} = \;\; \mu_x(t) = \;\; \mu_0(x+t) = \frac{f_0(x+t)}{S_0(x+t)} \; = \frac{f_x(t)}{S_x(t)}$$

6) If $t = 1$ the subscript is often suppressed so $p_x = \;{}_1p_x =$ and $q_x = \;{}_1q_x$.

7) The complete expectation of life at age $x$ or the expected future lifetime at age $x$ is $\overset{o}{e}_x = E(T_x) = \int_0^\infty t f_x(t) dt = \frac{1}{S_0(x)} \int_0^\infty t f_0(x+t) dt = \;\; \int_0^\infty \;{}_tp_x \; dt = \int_0^\infty S_x(t) dt$. Note that $\overset{o}{e}_0 = E(T_0)$.

8) $V(T_x) = E(T_x^2) - [E(T_x))]^2$ where $E(T_x) = \overset{o}{e}_x$ and $E(T_x^2) = \int_0^\infty t^2 f_x(t)dt = 2\int_0^\infty t \;\; {}_tp_x \;\; dt = 2\int_0^\infty t \;\; S_x(t) \;\; dt$. Note that $E(T_0^2)$ can be found using $x = 0$.

9) **Memorize**: If $T_x \sim EXP(\mu)$ where $\mu > 0$, then for $t > 0$, $\mu_{x+t} = \mu$, $f_x(t) = \mu e^{-\mu t}$, $F_x(t) = 1 - e^{-\mu t}$, $S_x(t) = e^{-\mu t}$, $E(T_x) = \overset{o}{e}_x = 1/\mu$ and $V(T_x) = 1/\mu^2$. The **exponential distribution** is the only distribution with a constant force of mortality $\mu_x(t) \equiv \mu$. Often you are told $\mu_{x+t} = \mu$ for some constant $\mu < 1$.

10) **Memorize**: If $T_0 \sim U(0, \omega)$, then $T_x$ has a De Moivre$(\omega - x)$ distribution: $T_x \sim U(0, \omega - x)$ with support $0 < t < \omega - x$. For such $t$, $S_x(t) = {}_tp_x = \dfrac{\omega - x - t}{\omega - x} = 1 - \dfrac{t}{\omega - x}$, $\mu_x(t) = \mu_{x+t} = \dfrac{1}{\omega - x - t}$ and $E(T_x) = \overset{o}{e}_x = \dfrac{\omega - x}{2}$. Often need to recognize the distribution from ${}_tp_x$.

11) Suppose $x \geq 0$ and $0 < t < 1$.

| function to approximate | linear or UDD approx | exponential or constant force approx |
|---|---|---|
| $S_0(x+t)$ | $(1-t)S_0(x) + tS_0(x+1)$ | $[S_0(x)]^{1-t} \;\; [S_0(x+1)]^t$ |
| ${}_tp_x$ | $1 - t(q_x)$ | $(p_x)^t = \exp(-\mu t)$ |
| ${}_tq_x \;\; (= 1 - {}_tp_x)$ | $t(q_x)$ | $1 - (p_x)^t = 1 - (1 - q_x)^t$ |
| $\mu_{x+t}$ | $\dfrac{q_x}{1 - t(q_x)}$ | $-\log(p_x) = \mu$ |
| $f_0(t) = {}_tp_x \;\; \mu_{x+t}$ | $q_x$ | $-(p_x)^t \;\; \log(p_x) \;\; = \mu \exp(-\mu t)$ |
| ${}_tq_{x+v}$ | $\dfrac{(t)q_x}{1 - v(q_x)}$ | $1 - (p_x)^t \approx {}_tq_x$ |
| ${}_tp_{x+v}$ | $1 - \dfrac{(t)q_x}{1 - v(q_x)}$ | $(p_x)^t \approx {}_tp_x$ |

12) The *curtate duration at failure* RV $K_x = \lfloor T_x \rfloor$. Here $\lfloor 7.7 \rfloor = 7$. Suppose the person died in the $k$th time interval $(k-1, k]$ which means $T_0$ is in the time interval $(x+k-1, x+k]$, given $T_0 > x$. Then $K_x = k-1$. $K_x$ is a discrete random variable where $k = 0, 1, 2, \ldots$.. Suppose the interval of failure for $T_x$ is $(k, k+1]$ (so $T_0$ fails in interval $(x+k, x+k+1]$). Then $K_x = k$. The probability (mass) function of $K_x$ is

$${}_k|q_x = p_{K_x}(k) = P(K_x = k) = P(k < T_x \leq k+1) = P(x+k < T_0 \leq x+k+1 | T_0 > x) =$$

$${}_kp_x - {}_{k+1}p_x = F_x(k+1) - F_x(k) = S_x(k) - S_x(k+1).$$

13) The *curtate expectation of life* at age $x$ is

$$e_x = E(K_x) = \sum_{k=0}^{\infty} k \;\; P(K_x = k) = \sum_{k=0}^{\infty} k \;\; {}_k|q_x.$$

14) **Know:** The probability that $(x)$ will die between $x+n$ and $x+n+m$ is ${}_{n|m}q_x = P(x+n < T_0 \leq x+n+m | T_0 > x) = P(n < T_x \leq n+m) = {}_np_x - {}_{n+m}p_x = {}_{n+m}q_x - {}_nq_x = {}_np_x \;\; {}_mq_{x+n}$. For $m = 1$, ${}_{n|1}q_x = {}_n|q_x = P(K_x = n) = {}_np_x \;\; q_{x+n} = P(n < T_x \leq n+1)$.

15) multiplication rule: ${}_{n+m}p_x = {}_np_x \;\; {}_mp_{x+n}$

16) $\overset{o}{e}_{x:\overline{n}|}$ = expected number of years lived in $(x, x+n]$ by a (randomly selected) survivor to age $x$. (The $:\overline{n}|$ in the subscript means take the formula for $\overset{o}{e}_x$ but replace the upper limit $\infty$ in the integrand by $n$.) So $\overset{o}{e}_{x:\overline{n}|} = \int_0^n {}_tp_x \; dt = \int_0^n S_x(t) \; dt = \int_0^n t \; f_x(t) \; dt$.

**Chapter 4:** 17) From interest theory, i) the *compound interest factor* $v = \dfrac{1}{1+i}$ and $0 < v < 1$.

ii) The *effective rate of interest* $i = \dfrac{1-v}{v} > 0$. Often $i = 0.05$.

iii) The *force of interest* $\delta = \log(1+i) > 0$. Note that $1 + i = e^{\delta}$ so $v = e^{-\delta}$.

iv) The *effective rate of discount* $d = \dfrac{i}{1+i} = iv = 1 - v > 0$.

18) The life insurance model has a *benefit function* $b_t$ and a *discount function* $v_t$ where $t =$ the length of time from issue of insurance until death (or until insurance payment). Often $v_t = v^t$ and $b_t = 1$ unit. The *present value function* $z_t = b_t v_t$ is the present value, at time $t$ from policy issue, of the benefit payment. Let $T = T_x =$ insured's future lifetime RV and the *claim random variable* or *present value random variable* $Z = z_{T_x} = b_{T_x} v_{T_x}$. Or $K_x = \lfloor T_x \rfloor =$ the curtate future lifetime RV, and $Z = z_{1+K_x} = b_{1+K_x}\ v_{1+K_x}$.

19) $E(Z)$ is the *actuarial present value* (APV) = *expected present value* (EPV) = *net single premium* (NSP) of the insurance, the expected value of the present value of the payment.

20) **Formulas are given for unit payment.** Let $A = E(Z)$ and ${}^2A = E(Z^2)$. For nonunit payment $c$, multiply the unit payment formula for $A$ by $c$ and the unit formula payment for ${}^2A$ by $c^2$.

21) Suppose $(x)$ buys insurance and dies at $t \in (k-1, k]$ years from purchase so $K_x = k - 1$ where $k \in \{0, 1, 2, ...\}$. Consider the following discrete life insurance models.

i) (Discrete) *whole life insurance* makes unit payment at time $t = k$ with $v_t = v^t, t \geq 0$ and $b_t = 1, t \geq 0$. Then $z_t = b_t v_t = v^t, t \geq 0$. The present value random variable $Z_x = z_{1+K_x} = v^{1+K_x}$. Then the actuarial present value APV = EPV = NSP = $A_x = E(Z_x) = E(v^{1+K_x}) = \sum_{k=0}^{\infty} v^{k+1} P(K_x = k)$, and ${}^2A_x = E[(Z_x)^2] = E[(v^{1+K_x})^2] = \sum_{k=0}^{\infty} v^{2(k+1)} P(K_x = k)$.

ii) (Discrete) *n year term insurance* = (discrete) *n year temporary insurance* makes unit payment at time $t = k$ only if $k \leq n$, otherwise no payment is made. Now $v_t = v^t, t \geq 0$, $b_t = \begin{cases} 1, & t \leq n \\ 0, & t > n \end{cases}$ and $\mathrm{z_t = b_t v_t} = \begin{cases} v^t, & t \leq n \\ 0, & t > n. \end{cases}$ The present value random variable (note $1 + K_x \leq n$ if $K_x < n$) is $Z^1_{x:\overline{n}|} = \begin{cases} v^{1+K_x}, & K_x < n \\ 0, & K_x \geq n. \end{cases}$ Then the actuarial present value APV = EPV = NSP = $A^1_{x:\overline{n}|} = E(Z^1_{x:\overline{n}|}) = \sum_{k=0}^{n-1} v^{k+1} P(K_x = k)$, and ${}^2A^1_{x:\overline{n}|} = E[(Z^1_{x:\overline{n}|})^2] = \sum_{k=0}^{n-1} v^{2(k+1)} P(K_x = k)$. The 1 above the $x$ means unit benefit is payable after $(x)$ dies if death is before time $n$.

iii) (Discrete) *n year deferred insurance* makes unit payment at time $t = k$ only if $k > n$ so $k \geq n + 1$, otherwise no payment is made. Now $v_t = v^t, t \geq 0$,

$$b_t = \begin{cases} 0, & t \leq n \\ 1, & t > n \end{cases} \quad \text{and} \quad \mathrm{z_t = b_t v_t} = \begin{cases} 0, & t \leq n \\ v^t, & t > n. \end{cases}$$

The present value random variable (note $1+K_x > n$ if $K_x \geq n$) is ${}_n|Z_x = \begin{cases} 0, & K_x < n \\ v^{1+K_x}, & K_x \geq n. \end{cases}$
Then the actuarial present value APV = EPV = NSP = ${}_n|A_x = E({}_n|Z_x) = \sum_{k=n}^{\infty} v^{k+1} P(K_x = k)$, and ${}^2\ {}_n|A_x = E[({}_n|Z_x)^2] = \sum_{k=n}^{\infty} v^{2(k+1)} P(K_x = k)$.

iv) (Discrete) *n year endowment life insurance* makes unit payment at time $t = k$ if $t < k < n$ and at time $n$ if $t > n$. Then $b_t = 1, t \geq 0$ and $v_t = \begin{cases} v^t, & t \leq n \\ v^n, & t > n, \end{cases}$ and $\mathrm{z_t} = \mathrm{b_t v_t} = \begin{cases} v^t, & t \leq n \\ v^n, & t > n. \end{cases}$ The present value random variable $Z_{x:\overline{n}|} = \begin{cases} v^{K_x+1}, & K_x < n \\ v^n, & K_x \geq n. \end{cases}$

Note that the n year endowment present value random variable $Z_{x:\overline{n}|} = Z^1_{x:\overline{n}|} + Z_{x:\overline{n}|}^{\ \ 1}$, the sum of the n year term and n year pure endowment present value RVs. Then the actuarial present value APV = EPV = NSP = $A_{x:\overline{n}|} = E[Z_{x:\overline{n}|}]$

$$= A^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1} = \sum_{k=0}^{n-1} v^{k+1} P(K_x = k) + v^n P(K_x \geq n) = \sum_{k=0}^{n-1} v^{k+1} P(K_x = k) + v^n \sum_{k=n}^{\infty} P(K_x = k).$$

Similarly, $[Z_{x:\overline{n}|}]^2 = [Z^1_{x:\overline{n}|}]^2 + [Z_{x:\overline{n}|}^{\ \ 1}]^2$ and ${}^2A_{x:\overline{n}|} = {}^2A^1_{x:\overline{n}|} + {}^2A_{x:\overline{n}|}^{\ \ 1}$

$$= \sum_{k=0}^{n-1} v^{2(k+1)} P(K_x = k) + v^{2n} P(K_x \geq n) = \sum_{k=0}^{n-1} v^{2(k+1)} P(K_x = k) + v^{2n} \sum_{k=n}^{\infty} P(K_x = k).$$

v) (Discrete = continuous) *n year pure endowment insurance* makes unit payment at time $n$ only if $t > n$ , otherwise no payment is made. Now $v_t = \begin{cases} v^t, & t \leq n \\ v^n, & t > n, \end{cases}$ $b_t = \begin{cases} 0, & t \leq n \\ 1, & t > n \end{cases}$ and $\mathrm{z_t} = \mathrm{b_t v_t} = \begin{cases} 0, & t \leq n \\ v^n, & t > n. \end{cases}$ The present value random variable $Z_{x:\overline{n}|}^{\ \ 1} = \begin{cases} 0, & T_x \leq n \\ v^n, & T_x > n. \end{cases}$ Then the actuarial present value APV = EPV = NSP = $A_{x:\overline{n}|}^{\ \ 1} = E(Z_{x:\overline{n}|}^{\ \ 1}) = {}_nE_x = v^n P(T_x > n) = v^n S_x(n) = e^{-\delta n} S_x(n)$ and ${}^2A_{x:\overline{n}|}^{\ \ 1} = E[(Z_{x:\overline{n}|}^{\ \ 1})^2] = v^{2n} P(T_x > n) = v^{2n} S_x(n) = e^{-2\delta n} S_x(n)$. The 1 above the $\overline{n}|$ means unit benefit is payable after $(x)$ dies if death is after time $n$. Also $V(Z_{x:\overline{n}|}^{\ \ 1}) = v^{2n}\ {}_np_x\ {}_nq_x$. Note the book does not use $\overline{Z}$ and $\overline{A}$ for this insurance because payment is made iff $T_x > n$ iff $K_x \geq n$ so the discrete insurance and continuous insurance are technically equivalent.

22) $Z_x = Z^1_{x:\overline{n}|} + {}_n|Z_x$, $A_x = A^1_{x:\overline{n}|} + {}_n|A_x$, $[Z_x]^2 = [Z^1_{x:\overline{n}|}]^2 + [{}_n|Z_x]^2$, ${}^2A_x = {}^2A^1_{x:\overline{n}|} + {}^2\ {}_n|A_x$, $Z_{x:\overline{n}|} = Z^1_{x:\overline{n}|} + Z_{x:\overline{n}|}^{\ \ 1}$, $A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1}$, $[Z_{x:\overline{n}|}]^2 = [Z^1_{x:\overline{n}|}]^2 + [Z_{x:\overline{n}|}^{\ \ 1}]^2$ and ${}^2A_{x:\overline{n}|} = {}^2A^1_{x:\overline{n}|} + {}^2A_{x:\overline{n}|}^{\ \ 1}$

23) Suppose $(x)$ buys insurance and dies at $t > 0$ years from purchase so $T = T_x = t$. Consider the following discrete life insurance models (note that 21 v) $n$ year pure endowment insurance is both continuous and discrete).

i) (Continuous) *whole life insurance* makes unit payment at time $t = k$ with $v_t = v^t, t \geq 0$ and $b_t = 1, t \geq 0$. Then $z_t = b_t v_t = v^t, t \geq 0$. The present value random variable

$\overline{Z}_x = z_T = v^T$. Then the actuarial present value APV = EPV = NSP =

$\overline{A}_x = E(\overline{Z}_x) = E(v^T) = E(e^{-\delta T}) = \int_0^\infty v^t f_T(t)\,dt = \int_0^\infty e^{-\delta t} f_T(t)\,dt = \int_0^\infty v^t\ {}_tp_x\ \mu_{x+t}\,dt$, and

${}^2\overline{A}_x = E[(\overline{Z}_x)^2] = E[(v^T)^2] = E(e^{-2\delta T}) = \int_0^\infty v^{2t} f_T(t)\,dt = \int_0^\infty e^{-2\delta t} f_T(t)\,dt = \int_0^\infty v^{2t}\ {}_tp_x\ \mu_{x+t}\,dt.$

ii) (Continuous) *n year term insurance* makes unit payment at time $t > 0$ only if $t \le n$, otherwise no payment is made. Now $v_t = v^t, t \ge 0$,

$$b_t = \begin{cases} 1, & t \le n \\ 0, & t > n, \end{cases} \qquad z_t = b_t v_t = \begin{cases} v^t, & t \le n \\ 0, & t > n, \end{cases} \text{ and } \overline{Z}^1_{x:\overline{n}|} = \begin{cases} v^{T_x}, & T \le n \\ 0, & T > n. \end{cases}$$

Then the actuarial present value APV = EPV = NSP =

$$\overline{A}^1_{x:\overline{n}|} = E(\overline{Z}^1_{x:\overline{n}|}) = \int_0^n e^{-\delta t} f_T(t)\ dt = \int_0^n v^t f_T(t)\ dt = \int_0^n v^t\ {}_tp_x\ \mu_{x+t}\ dt, \text{ and}$$

$${}^2\overline{A}^1_{x:\overline{n}|} = E[(\overline{Z}^1_{x:\overline{n}|})^2] = \int_0^n e^{-2\delta t} f_T(t)\ dt = \int_0^n v^{2t} f_T(t)\ dt = \int_0^n v^{2t}\ {}_tp_x\ \mu_{x+t}\ dt.$$

The 1 above the $x$ means unit benefit is payable after $(x)$ dies if death is not after time $n$.

iii) (Continuous) *n year deferred insurance* makes unit payment at time $t > 0$ only if $t > n$, otherwise no payment is made. Now $v_t = v^t, t \ge 0$,

$$b_t = \begin{cases} 0, & t \le n \\ 1, & t > n, \end{cases} \qquad z_t = b_t v_t = \begin{cases} 0, & t \le n \\ v^t, & t > n, \end{cases} \quad \text{and } {}_{n|}\overline{Z}_x = \begin{cases} 0, & T \le n \\ v^T, & T > n. \end{cases}$$

Then the actuarial present value APV = EPV = NSP =

$${}_{n|}\overline{A}_x = E({}_{n|}\overline{Z}_x) = \int_n^\infty e^{-\delta t} f_T(t)\ dt = \int_n^\infty v^t f_T(t)\ dt = \int_n^\infty v^t\ {}_tp_x\ \mu_{x+t}\ dt, \text{ and}$$

$${}^2\ {}_{n|}\overline{A}_x = E[({}_{n|}\overline{Z}_x)^2] = \int_n^\infty e^{-2\delta t} f_T(t)\ dt = \int_n^\infty v^{2t} f_T(t)\ dt = \int_n^\infty v^{2t}\ {}_tp_x\ \mu_{x+t}\ dt.$$

iv) (Continuous) *n year endowment life insurance* makes unit payment at time $t > 0$ if $t < n$ and at time $n$ if $t > n$. Then $b_t = 1, t \ge 0$ and

$$v_t = \begin{cases} v^t, & t \le n \\ v^n, & t > n, \end{cases} \qquad z_t = b_t v_t = \begin{cases} v^t, & t \le n \\ v^n, & t > n, \end{cases} \quad \text{and } \overline{Z}_{x:\overline{n}|} = \begin{cases} v^T, & T \le n \\ v^n, & T > n. \end{cases}$$

Then $\overline{Z}_{x:\overline{n}|} = \overline{Z}^1_{x:\overline{n}|} + Z_{\substack{\ \ 1\\ x:\overline{n}|}}$, $\overline{A}_{x:\overline{n}|} = E[\overline{Z}_{x:\overline{n}|}] = \overline{A}^1_{x:\overline{n}|} + A_{\substack{\ \ 1\\ x:\overline{n}|}}$, $[\overline{Z}_{x:\overline{n}|}]^2 = [\overline{Z}^1_{x:\overline{n}|}]^2 + [Z_{\substack{\ \ 1\\ x:\overline{n}|}}]^2$, and ${}^2\overline{A}_{x:\overline{n}|} = {}^2\overline{A}^1_{x:\overline{n}|} + {}^2A_{\substack{\ \ 1\\ x:\overline{n}|}}$.

24) **KNOW:** Let $T \sim EXP(\mu)$. Then $E(T) = \int_0^\infty t\mu e^{-\mu t}dt = \int_0^\infty e^{-\mu t}dt = 1/\mu$. So $\int_0^\infty tDe^{-t(D)}dt = \int_0^\infty e^{-t(D)}dt = 1/D$ for $D > 0$. Use $\overset{E}{=}$ when exponential RV is used.

25) In 24) and 26), often $\int_0^\infty$ is replaced by $\int_a^b$. If $D > 0$, $\int_0^n De^{-tD}dt = 1 - e^{-nD}$, $\int_n^\infty De^{-tD}dt = e^{-nD}$, $\int_0^n e^{-tD}dt = \frac{1}{D}[1 - e^{-nD}]$, and $\int_n^\infty e^{-tD}dt = \frac{1}{D}\ e^{-nD}$.

26) Whole life insurance with the exponential($\mu$) distribution often has $\overline{Z} = b_T \ v^T$ where $b_t = e^{\theta t}$. Now $\int_0^\infty \mu e^{-\mu t} dt = 1$ so $\int_0^\infty e^{-\mu t} dt = 1/\mu$ if $\mu > 0$. Hence $E[\overline{Z}\ ] = \int_0^\infty b_t e^{-\delta t} \mu e^{-\mu t} dt = \int_0^\infty e^{\theta t} e^{-\delta t} \mu e^{-\mu t} dt = \mu \int_0^\infty e^{-t[\mu+\delta-\theta]} dt = \frac{\mu}{\mu+\delta-\theta}$ provided $\mu + \delta - \theta > 0$. Also $E[(\overline{Z})^j] = \int_0^\infty [b_t e^{-\delta t}]^j \mu e^{-\mu t} dt = \int_0^\infty e^{\theta j t} e^{-\delta j t} \mu e^{-\mu t} dt = \mu \int_0^\infty e^{-t[\mu+\delta j-\theta j]} dt = \frac{\mu}{\mu+\delta j-\theta j}$ provided $\mu+\delta j-\theta j > 0$. Note that $\theta = 0$ corresponds to unit payment.

27) Often unit benefits are not used for continuous insurance. Let $\overline{B}_x = \overline{Z} = z_{T_x} = b_{T_x} v_{T_x}$. Then ${}^j\overline{A} = E[(\overline{Z})^j] = \int_0^\infty (b_t v_t)^j f_T(t) dt$. Note that APV $= \overline{A} = E[\overline{Z}] = E[\overline{B}_x] = \int_0^\infty b_t v_t f_T(t) dt$. The bars on $A$ and $Z$ are often omitted. Usually $v_t = v^t = e^{-\delta t}$.

28) **KNOW:** Let $T \sim EXP(\mu)$. $S(t) = e^{-\mu t}$ for $t > 0$. Often use $Z$ instead of $\overline{Z}$.

i) If $b_t = ce^{\theta t}$ and $Z = b_T v_T$, then $E[Z^j] = E[(b_T v_T)^j] = c^j E[(e^{\theta T} v_T)^j]$. So multiply $c = 1$ formulas by $c^j$. Usually want $j = 1, 2$.

a) Special whole life insurance: $b_t = e^{\theta t}$, $v_t = e^{-\delta t}$, and $Z = b_T v_T = e^{\theta T} e^{-\delta T}$. $E(Z^j) \overset{E}{=} \frac{\mu}{\mu+\delta j-\theta j}$ if $\mu + \delta j - \theta j > 0$. See 26).

b) Whole life insurance: special case of a) with $\theta = 0$. See 28i). $\overline{Z}_x = e^{-\delta T}$. $\overline{A}_x = E(\overline{Z}_x) = E(e^{-\delta T}) \overset{E}{=} \frac{\mu}{\mu+\delta}$, and ${}^2\overline{A}_x = E[(\overline{Z}_x)^2] = E(e^{-2\delta T}) \overset{E}{=} \frac{\mu}{\mu+2\delta}$.
$V(\overline{Z}_x) = {}^2\overline{A}_x - (\overline{A}_x)^2$.

c) Whole life annuity. See 32). $\overline{Y}_x = \frac{1-\overline{Z}_x}{\delta}$.

$E[\overline{Y}_x] = \overline{a}_x = \int_0^\infty e^{-\delta t} S_T(t) dt \overset{E}{=} \frac{1}{\mu+\delta}$. $\quad V(\overline{Y}_x) = \frac{V(\overline{Z}_x)}{\delta^2} = \frac{{}^2\overline{A}_x - (\overline{A}_x)^2}{\delta^2}$.

**Chapter 5:** 29) $a_{\overline{n}|} = v + v^2 + v^3 + \cdots + v^n = \sum_{j=1}^n v^j = \frac{1-v^n}{i}$,

$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + v^3 + \cdots + v^{n-1} = \sum_{j=0}^{n-1} v^j = \frac{1-v^n}{d}$ where $d = iv = \frac{i}{1+i}$, and

$\overline{a}_{\overline{n}|} = \int_0^n v^t dt = \frac{1-v^n}{\delta}$.

30) A (discrete annual) immediate whole life annuity pays $(x)$ 1 unit at times $t = 1, 2, \ldots$, as long as $(x)$ survives. For integer $t$, $P(K_x \geq t) = P(T_x > t) = {}_tp_x$. Let $Y_x = a_{\overline{K_x}|}$ be the present value random variable and let $a_x = E(Y_x)$ be the APV = EPV = NSP of the annuity. Then $a_x = E(Y_x) = \sum_{t=1}^\infty \ v^t \ {}_tp_x$. Then $Y_x = \frac{1-v^{K_x}}{i} = \frac{1}{i}[1-(1+i)Z_x]$. Note the immediate annuity has an $i$ in the denominator. Also, $V(Y_x) = \frac{{}^2A_x - (A_x)^2}{d^2}$.

31) A (discrete annual) whole life annuity-due pays $(x)$ 1 unit at times $t = 0, 1, 2, \ldots$, as long as $(x)$ survives. Let $\ddot{Y}_x$ be the present value random variable and let $\ddot{a}_x = E(\ddot{Y}_x)$ be the APV = EPV = NSP of the annuity. Then $\ddot{a}_x = E(\ddot{Y}_x) = a_x + 1 = \sum_{t=0}^\infty \ v^t \ {}_tp_x$. $\ddot{Y}_x = \ddot{a}_{\overline{1+K_x}|} = \frac{1-v^{1+K_x}}{d} = \frac{1-Z_x}{d} = Y_x + 1$. Note the $d$ in the denominator for an

annuity–due. Then $E(\ddot{Y}_x) = a_x + 1$ and $V(\ddot{Y}_x) = V(Y_x)$.

32) A continuous whole life annuity makes a continuous payment at an annual rate of 1 unit per year as long as $(x)$ survives. The present value RV $\overline{Y}_x = \overline{a}_{\overline{T_x}|} = \dfrac{1 - v^{T_x}}{\delta} = \dfrac{1 - \overline{Z}_x}{\delta}$. The APV is $\overline{a}_x = E(\overline{Y}_x) = \int_0^\infty v^t \ {}_tp_x dt = \int_0^\infty e^{-\delta t} \ S_x(t) dt \stackrel{E}{=} \dfrac{1}{\mu + \delta}$.

$V(\overline{Y}_x) = \dfrac{V(\overline{Z}_x)}{\delta^2} = \dfrac{{}^2\overline{A}_x - (\overline{A}_x)^2}{\delta^2}$. Note the $\delta$ in the denominator of the continuous annuity.

33) The (discrete) immediate $n$ year temporary annuity pays $(x)$ 1 unit at times $t = 1, ..., n$ if $K_x \geq n$ and at times $t = 1, ..., k-1$ if $1 \leq K_x = k-1 \leq n-1$. No payment is made if $K_x = 0$. The present value RV $Y_{x:\overline{n}|} = \sum_{t=1}^{n} Z_{\overset{1}{x:\overline{t}|}} = \begin{cases} a_{\overline{K_x}|}, & K_x < n \\ a_{\overline{n}|}, & K_x \geq n. \end{cases}$ The APV $a_{x:\overline{n}|} = E(Y_{x:\overline{n}|}) = \sum_{t=1}^{n} v^t \ {}_tp_x$.

34) The (discrete) $n$ year temporary annuity-due pays $(x)$ 1 unit at times $t = 0, 1, ..., n-1$ if $K_x \geq n$ and at times $t = 0, 1, ..., k-1$ if $K_x = k-1 < n$. The present value RV $\ddot{Y}_{x:\overline{n}|} = \sum_{t=0}^{n-1} Z_{\overset{1}{x:\overline{t}|}} = Y_{x:\overline{n}|} + 1 - Z_{x:\overline{n}|} = \dfrac{1 - Z_{x:\overline{n}|}}{d}$. The APV $\ddot{a}_{x:\overline{n}|} = E(\ddot{Y}_{x:\overline{n}|}) = \sum_{t=0}^{n-1} v^t \ {}_tp_x = \dfrac{1 - A_{x:\overline{n}|}}{d} = a_{x:\overline{n}|} + 1 - \ {}_nE_x$.

35) A (continuous) temporary $n$ year annuity makes a continuous payment at annual rate of 1 unit a year for $n$ years if $T_x > n$ and for $T_x$ years if $T_x < n$. The present value RV is $\overline{Y}_{x:\overline{n}|} = \dfrac{1 - \overline{Z}_{x:\overline{n}|}}{\delta} = \begin{cases} \overline{a}_{\overline{T_x}|}, & T_x \leq n \\ \overline{a}_{\overline{n}|}, & T_x > n. \end{cases}$ Then $\overline{a}_{x:\overline{n}|} = E(\overline{Y}_{x:\overline{n}|}) = \int_0^n e^{-\delta t} S_x(t) dt$. $V(\overline{Y}_{x:\overline{n}|}) = \dfrac{{}^2\overline{A}_{x:\overline{n}|} - (\overline{A}_{x:\overline{n}|})^2}{\delta^2}$.

36) A (discrete) immediate $n$ year deferred whole life annuity makes no payment if $K_x \leq n$. If $K_x = k-1 \geq n+1$, then unit payment is made at times $t = n+1, n+2, ..., k-1$. The present value RV ${}_n|Y_x = Y_x - Y_{x:\overline{n}|} = \sum_{t=n+1}^{\infty} Z_{\overset{1}{x:\overline{t}|}}$. The APV

${}_n|a_x = E({}_n|Y_x) = a_x - a_{x:\overline{n}|} = \sum_{t=n+1}^{\infty} v^t \ {}_tp_x$.

37) A (discrete) $n$ year deferred whole life annuity-due makes no payment if $K_x < n$. If $K_x = k-1 \geq n$, then unit payment is made at times $t = n, n+1, n+2, ..., k-1$. The present value RV ${}_n|\ddot{Y}_x = \ddot{Y}_x - \ddot{Y}_{x:\overline{n}|} = Z_{\overset{1}{x:\overline{n}|}} + \ {}_n|Y_x = \sum_{t=n}^{\infty} Z_{\overset{1}{x:\overline{t}|}}$. The APV

${}_n|\ddot{a}_x = E({}_n|\ddot{Y}_x) = \ddot{a}_x - \ddot{a}_{x:\overline{n}|} = v^n \ {}_np_x + \ {}_n|a_x = \sum_{t=n}^{\infty} v^t \ {}_tp_x$.

38) A (continuous) $n$ year deferred annuity makes no payment if $T_x \leq n$. If $T_x = t > n$ then continuous payment at annual unit rate is made from time $n$ to time $t$. The present value RV is ${}_n|\overline{Y}_x = \overline{Y}_x - \overline{Y}_{x:\overline{n}|} = \begin{cases} 0, & T_x \leq n \\ v^n \ \overline{a}_{\overline{T_x - n}|}, & T_x > n. \end{cases}$ Then ${}_n|\overline{a}_x = E({}_n|\overline{Y}_x) =$

$\overline{a}_x - \overline{a}_{x:\overline{n}|} = \int_n^\infty e^{-\delta t} S_x(t)dt$. $E[({}_n|\overline{Y}_x)^2] = \frac{2}{\delta}\ v^{2n}\ {}_np_x\ [\overline{a}_{x+n} -\ {}^2\overline{a}_{x+n}]$ where $\overline{a}_{x+n} = \int_0^\infty e^{-\delta t} S_{x+n}(t)dt$ and ${}^2\overline{a}_{x+n} = \int_0^\infty e^{-2\delta t} S_{x+n}(t)dt$.

39) Contingent annuities paid $m$thly are paid $m$ times a year with payment $1/m$ where $m \geq 1$. So annual payment is 1 unit per year. A discrete immediate $m$thly whole life annuity pays $1/m$ units at the end of each $m$thly time interval while $(x)$ survives. The APV is $a_x^{(m)}$. A discrete $m$thly whole life annuity-due pays $1/m$ units at the beginning of each $m$thly time interval while $(x)$ survives. The APV is $\ddot{a}_x^{(m)} = a_x^{(m)} + \frac{1}{m}$. The **Woolhouse approximation** is $a_x^{(m)} \approx a_x + \frac{m-1}{2m} \approx a_x + \frac{m-1}{2m} + \frac{m^2-1}{12m^2}(\mu_x + \delta)$. Then $\ddot{a}_x^{(m)} = a_x^{(m)} + \frac{1}{m} \approx \ddot{a}_x - \frac{m-1}{2m}$ where $\ddot{a}_x$ is given by the illustrative life table. Also $\ddot{a}_x \approx a_x + 1$. Also $\overline{a}_x \approx \ddot{a}_x - 0.5 \approx a_x + 0.5 \approx \ddot{a}_x - 0.5 - (\mu_x + \delta)/12$ since a continuous annuity is the limiting case of an $m$thly annuity as $m \to \infty$. The approximation is good for $m \geq 12$.

**End of Math 401 Material**

**Chapter 10**

40) Multiple life functions consider failure or survival of a *status* of multiple lives. Insurance is payable when the status fails. Annuities are payable as long as the status survives. For 2 life functions the $x$ and $y$ are separated by a colon. So think of $(xy)$ as $(x{:}y)$, and $(\overline{xy})$ as $(\overline{x:y})$. Notation $x+n : y+n$ is also used. Let $T_x \perp\!\!\!\perp T_y$ mean that $T_x$ and $T_y$ are independent. Usually assume $T_{x_1} \perp\!\!\!\perp T_{x_2} \perp\!\!\!\perp \ldots \perp\!\!\!\perp T_{x_k}$.

41) A **joint life status** for $(xy)$ fails as soon as $x$ or $y$ dies. Let $T_{xy} = \min(T_x, T_y) =$ time until 1st death. Convert $q$'s to $p$'s, then convert back to $q$'s if needed.

42) If $n = 10$, $x = 40$, and $y = 20$, want $T_{40:20}$, $S_{T_{40:20}}(t) = S_{40:20}(t) = {}_tp_{40:20}$, $F_{40:20}(t) = {}_tq_{40:20}$, $f_{40:20}(t)$, $\mu_{40:20}(t)$, ${}_{10|}q_{40:20}$, $\overset{o}{e}_{40:20}$, $K_{40:20}$, $e_{40:20}$ and $e_{40:20:\overline{10|}}$.

43) **Know:** Consider a joint life status $(xy)$ and $T_{xy}$.

i) survival function: $S_{xy}(t) = {}_tp_{xy} = P(T_{xy} > t)$. If $T_x \perp\!\!\!\perp T_y$, then ${}_tp_{xy} = ({}_tp_x)({}_tp_y)$.

ii) cdf: $F_{xy}(t) = {}_tq_{xy} = P(T_{xy} \le t)$. If $T_x \perp\!\!\!\perp T_y$, then ${}_tq_{xy} = {}_tq_x + {}_tq_y - ({}_tq_x)({}_tq_y)$.

iii) pdf: $f_{xy}(t) = \dfrac{d}{dt} F_{xy}(t) = \dfrac{-d}{dt} S_{xy}(t)$. If $T_x \perp\!\!\!\perp T_y$, then $f_{xy}(t) = {}_tp_{xy}(\mu_{x+t} + \mu_{y+t})$.

iv) force of mortality: $\mu_{xy}(t) = \dfrac{f_{xy}(t)}{S_{xy}(t)}$. If $T_x \perp\!\!\!\perp T_y$, then $\mu_{xy}(t) = \mu_{x+t} + \mu_{y+t} \equiv \mu_{x+t:y+t}$.

v) a) ${}_{n|}q_{xy} = {}_np_{xy} - {}_{n+1}p_{xy} = P(n < T_{xy} \le n+1)$. Let $p_{x+n:y+n} = \dfrac{{}_{n+1}p_{xy}}{{}_np_{xy}}$ and $q_{x+n:y+n} = 1 - p_{x+n:y+n}$. Then ${}_{n|}q_{xy} = {}_np_{xy}(1 - p_{x+n:y+n}) = {}_np_{xy}\; q_{x+n:y+n}$. If $T_x \perp\!\!\!\perp T_y$, then ${}_{n|}q_{xy} = ({}_np_x)({}_np_y) - ({}_{n+1}p_x)({}_{n+1}p_y)$.

b) ${}_{n|m}q_{xy} = {}_np_{xy}\; {}_mq_{x+n:y+n} = {}_np_{xy} - {}_{n+m}p_{xy} = P(n < T_{xy} \le n+m)$. See 14)–15) for more formulas.

c) ${}_{n+m}p_{xy} = {}_np_{xy}\; {}_mp_{x+n:y+n}$.

vi) $\overset{o}{e}_{xy} = E(T_{xy}) = \int_0^\infty t f_{xy}(t)dt = \int_0^\infty {}_tp_{xy} dt$.

vii) $E[(T_{xy})^2] = \int_0^\infty t^2 f_{xy}(t)dt = 2\int_0^\infty t\; {}_tp_{xy}\; dt$.

viii) Let $K_{xy} = \lfloor T_{xy} \rfloor$ be the *curtate duration at failure of the joint status* $(xy)$ (the number of whole years of survival left to the status $(xy)$). Then $P(K_{xy} = k) = {}_{n|}q_{xy}$.

ix) The *curtate expectation of future lifetime for the joint status* is $e_{xy} = \sum_{k=1}^{\infty} {}_kp_{xy} = E[K_{xy}]$ is the average number of whole years of survival left to the joint status $(xy)$.

x) The *temporary curtate lifetime* is $e_{xy:\overline{n|}} = \sum_{k=1}^{n} {}_kp_{xy}$ = average number of whole years of survival within the next $n$ years (for time $t \in (0, n]$) of the joint status $(xy)$.

xi) Still have ${}_tp_{xy} + {}_tq_{xy} = 1$.

44) $p_{xy} = {}_1p_{xy} = S_{T_{xy}}(1)$ and $q_{xy} = {}_1q_{xy} = 1 - {}_1p_{xy} = F_{T_{xy}}(1)$.

45) If $T_x \sim EXP(\mu_x) \perp\!\!\!\perp T_y \sim EXP(\mu_y)$, then $T_{xy} = \min(T_x, T_y) \sim EXP(\mu_x + \mu_y)$.

46) A two life **last survivor status** for $(\overline{xy})$ fails after both $x$ and $y$ die. Let $T_{\overline{xy}} = \max(T_x, T_y) =$ time until 2nd death. Then $T_{xy} + T_{\overline{xy}} = T_x + T_y$. Convert $p$'s to

$q$'s, then convert back to $p$'s if needed.

47) If $n = 10$, $x = 40$ and $y = 20$, want $T_{\overline{40:20}}$, $S_{T_{\overline{40:20}}}(t) = S_{\overline{40:20}}(t) = {}_tp_{\overline{40:20}}$, $F_{\overline{40:20}}(t) = {}_tq_{\overline{40:20}}$, $f_{\overline{40:20}}(t)$, $\mu_{\overline{40:20}}(t)$, ${}_{10}|q_{\overline{40:20}}$, $\overset{o}{e}_{\overline{40:20}}$, $K_{\overline{40:20}}$, $e_{\overline{40:20}}$ and $e_{\overline{40:20}:\overline{10}|}$.

48) **Know:** Consider a last survivor status $(\overline{xy})$ and $T_{\overline{xy}}$.

i) survival function: $S_{T_{\overline{xy}}}(t) = S_{\overline{xy}}(t) = {}_tp_{\overline{xy}} = P(T_{\overline{xy}} > t) = {}_tp_x + {}_tp_y - {}_tp_{xy}$. If $T_x \perp\!\!\!\perp T_y$, then ${}_tp_{\overline{xy}} = 1 - ({}_tq_x)({}_tq_y) = {}_tp_x + {}_tp_y - ({}_tp_x)({}_tp_y)$.

ii) cdf: $F_{\overline{xy}}(t) = {}_tq_{\overline{xy}} = P(T_{\overline{xy}} \leq t) = 1 - S_{\overline{xy}}(t)$. If $T_x \perp\!\!\!\perp T_y$, then ${}_tq_{\overline{xy}} = ({}_tq_x)({}_tq_y) = F_x(t)F_y(t) = F_{T_x}(t)F_{T_y}(t)$.

iii) pdf: $f_{\overline{xy}}(t) = \dfrac{d}{dt}F_{\overline{xy}}(t) = \dfrac{-d}{dt}S_{\overline{xy}}(t) = f_x(t) + f_y(t) - f_{xy}(t) =$
$({}_tp_x)(\mu_{x+t}) + ({}_tp_y)(\mu_{y+t}) - ({}_tp_{xy})(\mu_{x+t} + \mu_{y+t}) = ({}_tp_x)(\mu_{x+t}) + ({}_tp_y)(\mu_{y+t}) - ({}_tp_{xy})(\mu_{x+t:y+t})$.

iv) force of mortality: $\mu_{\overline{xy}}(t) = \dfrac{f_{\overline{xy}}(t)}{S_{\overline{xy}}(t)} = \dfrac{({}_tp_x)(\mu_{x+t}) + ({}_tp_y)(\mu_{y+t}) - ({}_tp_{xy})(\mu_{x+t:y+t})}{{}_tp_x + {}_tp_y - {}_tp_{xy}}$.

If $T_x \perp\!\!\!\perp T_y$, then $\mu_{\overline{xy}}(t) = \dfrac{({}_tq_x)({}_tp_y)(\mu_{y+t}) + ({}_tq_y)({}_tp_x)(\mu_{x+t})}{{}_tp_{\overline{xy}}}$.

v) a) ${}_n|q_{\overline{xy}} = {}_np_{\overline{xy}} - {}_{n+1}p_{\overline{xy}} = P(n < T_{\overline{xy}} \leq n+1) = {}_n|q_x + {}_n|q_y - {}_n|q_{xy} = P(K_{\overline{xy}} = n)$. See 14)–15) for more formulas.

b) ${}_{n|m}q_{\overline{xy}} = {}_np_{\overline{xy}} - {}_{n+m}p_{\overline{xy}} = P(n < T_{\overline{xy}} \leq n+m)$. See 14) for more formulas.

vi) $\overset{o}{e}_{\overline{xy}} = E(T_{\overline{xy}}) = \int_0^\infty t f_{\overline{xy}}(t)dt = \int_0^\infty {}_tp_{\overline{xy}}\, dt = \overset{o}{e}_x + \overset{o}{e}_y - \overset{o}{e}_{xy}$.

vii) $E[(T_{\overline{xy}})^2] = \int_0^\infty t^2 f_{T_{\overline{xy}}}(t)dt = 2\int_0^\infty t\, {}_tp_{\overline{xy}}\, dt$.

viii) Let $K_{\overline{xy}} = \lfloor T_{\overline{xy}} \rfloor$ be the curtate duration at failure of the status $(\overline{xy})$ (the number of whole years of survival left to the last survivor status $(\overline{xy})$).

ix) $e_{\overline{xy}} = \sum_{k=1}^{\infty} {}_kp_{\overline{xy}} = e_x + e_y - e_{xy} = E[K_{\overline{xy}}]$ is the average number of whole years of survival left to the last survivor status $(xy)$.

x) $e_{\overline{xy}:\overline{n}|} = \sum_{k=1}^{n} {}_kp_{\overline{xy}} = e_{x:\overline{n}|} + e_{y:\overline{n}|} - e_{xy:\overline{n}|}$ = average number of whole years of survival within the next $n$ years (for time $t \in (0, n]$) of the last survivor status $(\overline{xy})$.

49) $p_{\overline{xy}} = {}_1p_{\overline{xy}} = S_{\overline{xy}}(1)$ and $q_{\overline{xy}} = {}_1q_{\overline{xy}} = 1 - {}_1p_{\overline{xy}} = F_{\overline{xy}}(1)$.

50) $T_{xy}$ is one of $T_x$ or $T_y$, and $T_{\overline{xy}}$ is the other. Hence $T_{xy} + T_{\overline{xy}} = T_x + T_y$, and $T_{\overline{xy}} = T_x + T_y - T_{xy}$. Similarly, $P(T_{xy} > t) + P(T_{\overline{xy}} > t) = P(T_x > t) + P(T_y > t)$, and $P(T_{\overline{xy}} > t) = P(T_x > t) + P(T_y > t) - P(T_{xy} > t)$. See point 48) i) and vi).

51) $E[\min(X, j)] = \int_0^j x f_X(x)dx + \int_j^\infty j f_X(x)dx = \int_0^j x f_X(x)dx + jS_X(j)$.

52) $E[\max(X, j)] = \int_0^j j f_X(x)dx + \int_j^\infty x f_X(x)dx = jF_X(j) + \int_j^\infty x f_X(x)dx$.

53) P[$(x)$ fails before $(y)$] $= P(T_x < T_y) = {}_\infty q^1_{xy} = \int_0^\infty \int_t^\infty f_{T_x,T_y}(t,s)dsdt = \int_0^\infty \int_t^\infty f_{T_y|T_x}(s|t)ds f_{T_x}(t)dt = \int_0^\infty P(T_y > t|T_x = t)f_{T_x}(t)dt$. If $T_x \perp\!\!\!\perp T_y$, then

$P(T_x < T_y) = {}_{\infty}q^1_{xy} = E[S_{T_y}(T_x)] = \int_0^\infty S_y(t) f_x(t) dt = \int_0^\infty {}_tp_y \ {}_tp_x \ \mu_{x+t} \ dt = \int_0^\infty {}_tp_{xy} \ \mu_{x+t} \ dt.$

54) If $T_x \perp\!\!\!\perp T_y$, then P[$(x)$ fails after $(y)$] $= P(T_x > T_y) = {}_{\infty}q^2_{xy} = 1 - {}_{\infty}q^1_{xy}$.

55) If $T_x \perp\!\!\!\perp T_y$, then P[$(x)$ fails before $(y)$ and within $n$ years] $= {}_nq^1_{xy} = \int_0^n {}_tp_{xy} \ \mu_{x+t} \ dt$.

56) If $T_x \perp\!\!\!\perp T_y$, then P[$(x)$ fails after $(y)$ and within $n$ years] $= {}_nq^2_{xy} = \int_0^n F_y(t) f_x(t) dt = {}_nq_x - {}_nq^1_{xy}$. (The 2 means (x) is the 2nd failure.)

57) If $T_x \perp\!\!\!\perp T_y$, then P[$(y)$ fails before $(x)$ and within $n$ years] $= {}_nq_{x\overset{1}{y}} = \int_0^n {}_tp_{xy} \ \mu_{y+t} \ dt = \int_0^n S_x(t) f_y(t) dt$. (The 1 means (y) is the 1st failure.)

58) If $T_x \perp\!\!\!\perp T_y$, then P[$(y)$ fails after $(x)$ and within $n$ years] $= {}_nq_{x\overset{2}{y}} = \int_0^n F_x(t) f_y(t) dt = {}_nq_y - {}_nq_{x\overset{1}{y}}$. Note that the superscript 2 is for the 2nd failure and the 1 for the 1st.

59) ${}_nq^1_{xy} + {}_nq_{x\overset{1}{y}} = {}_nq_{xy}$

60) ${}_nq^2_{xy} + {}_nq_{x\overset{2}{y}} = {}_nq_{\overline{xy}}$

61) $\overset{o}{e}_{xy:\overline{n}|} = \int_0^n {}_tp_{xy} \ dt$.

62) $E[(T_{xy})^2] = 2\int_0^\infty t \ {}_tp_{xy} \ dt$, and $E[(T_{\overline{xy}})^2] = 2\int_0^\infty t \ {}_tp_{\overline{xy}} \ dt$. See 43 vii) and 48 vii).

63) A generalized DeMoivre $GD(\alpha, \theta)$ distribution has survival function $S_0(t) = \left(\frac{\theta - t}{\theta}\right)^\alpha$ for $0 < t < \theta$ where $\alpha > 0$. Often $\theta = \omega - x$. If $T_x \sim$ DeMoivre$(\omega - x)$, then $\alpha = 1$. If $T_x \sim GD(\alpha, \omega - x)$, then for $0 < t < \omega - x$, $S_x(t) = {}_tp_x = \left(\frac{\omega - x - t}{\omega - x}\right)^\alpha$, $F_x(t) = {}_tq_x = 1 - \left(\frac{\omega - x - t}{\omega - x}\right)^\alpha$, $f_x(t) = {}_tp_x \ \mu_{x+t} = \frac{\alpha(\omega - x - t)^{\alpha - 1}}{(\omega - x)^\alpha}$, $\mu_x(t) = \mu_{x+t} = \frac{\alpha}{\omega - x - t}$, $E(T_x) = \overset{o}{e}_x = \frac{\omega - x}{\alpha + 1}$, and $V(T_x) = \frac{\alpha(\omega - x)^2}{(1+\alpha)^2(2+\alpha)}$.

If $T_{x_i}$ are independent $GD(\alpha_i, \omega - x)$, then $T_{x_1 x_2 \ldots x_k} = \min(T_{x_1}, \ldots T_{x_k}) \sim GD(\sum_{i=1}^k \alpha_i, \omega - x)$. So if $T_x \sim GD(\alpha_x, \omega - x) \perp\!\!\!\perp T_y \sim GD(\alpha_y, \omega - x)$, then $T_{xy} \sim GD(\alpha_x + \alpha_y, \omega - x)$. Note: only need $\omega_x - x = \omega_y - y \equiv \omega - x$.

Hence if $T_x \sim U(0, \omega - x) \perp\!\!\!\perp T_y \sim U(0, \omega - x)$, then $T_{xy} \sim GD(2, \omega - x)$ since DeMoivre$(\omega - x) \sim U(0, \omega - x) \sim GD(1, \omega_x)$ has $\alpha_x = \alpha_y = 1$. $T_0 \sim GD(\alpha, \omega) \Rightarrow T_x \sim GD(\alpha, \omega - x)$.

Insurance and pensions for $(xy)$ and $(\overline{xy})$ are like those for $(x)$, but replace the subscript $x$ by $(xy)$ or $(\overline{xy})$. Points 64) through 70) are illustrative.

64) $T_x \perp\!\!\!\perp T_y$, discrete whole life insurance for $(xy)$ has $Z_{xy} = v^{1+K_{xy}}$, $A_{xy} = E[Z_{xy}] =$

$\sum_{k=0}^{\infty} v^{k+1}(\ {}_{k}|q_{xy})$ and ${}^2A_{xy} = E[(Z_{xy})^2] = \sum_{k=0}^{\infty} v^{2(k+1)}(\ {}_{k}|q_{xy})$.

65) $T_x \perp\!\!\!\perp T_y$, discrete whole life insurance for $(\overline{xy})$ has $A_{\overline{xy}} = E[Z_{\overline{xy}}] = \sum_{k=0}^{\infty} v^{k+1}(\ {}_{k}|q_{\overline{xy}}) = A_x + A_y - A_{xy}$ and ${}^2A_{\overline{xy}} = E[(Z_{\overline{xy}})^2] = \sum_{k=0}^{\infty} v^{2(k+1)}(\ {}_{k}|q_{\overline{xy}}) =\ {}^2A_x +\ {}^2A_y -\ {}^2A_{xy}$.

66) $T_x \perp\!\!\!\perp T_y$, continuous whole life insurance for $(xy)$ has $\overline{Z}_{xy} = v^{T_{xy}}$, $\overline{A}_{xy} = E[\overline{Z}_{xy}] = \int_0^{\infty} e^{-\delta t} f_{xy}(t)dt = \int_0^{\infty} e^{-\delta t}\ {}_{t}p_{xy}\mu_{x+t:y+t}dt$ and ${}^2\overline{A}_{xy} = E[(\overline{Z}_{xy})^2] = \int_0^{\infty} e^{-2\delta t} f_{xy}(t)dt = \int_0^{\infty} e^{-2\delta t}\ {}_{t}p_{xy}\mu_{x+t:y+t}dt$.

67) $T_x \perp\!\!\!\perp T_y$, continuous whole life insurance for $(\overline{xy})$ has $\overline{Z}_{\overline{xy}} = v^{T_{\overline{xy}}}$, $\overline{A}_{\overline{xy}} = E[\overline{Z}_{\overline{xy}}] = \int_0^{\infty} e^{-\delta t} f_{\overline{xy}}(t)dt = \overline{A}_x + \overline{A}_y - \overline{A}_{xy}$ and ${}^2\overline{A}_{\overline{xy}} = E[(\overline{Z}_{\overline{xy}})^2] = \int_0^{\infty} e^{-2\delta t} f_{\overline{xy}}(t)dt =$
${}^2\overline{A}_x +{}^2 \overline{A}_y -{}^2 \overline{A}_{xy}$.

68) $T_x \perp\!\!\!\perp T_y$, discrete annual immediate life annuity for $(xy)$ has $a_{xy} = E[Y_{xy}] = \sum_{k=1}^{\infty} v^k(\ {}_{k}p_{xy})$.

69) $T_x \perp\!\!\!\perp T_y$, discrete annual life annuity-due for $(\overline{xy})$ has $\ddot{a}_{\overline{xy}} = E[\ddot{Y}_{\overline{xy}}] = \sum_{k=0}^{\infty} v^k(\ {}_{k}p_{\overline{xy}})$.

70) $T_x \perp\!\!\!\perp T_y$, a continuous temporary $n$ year annuity for $(xy)$ has $\overline{a}_{xy:\overline{n}|} = E(\overline{Y}_{xy:\overline{n}|}) = \int_0^{n} v^t\ {}_{t}p_{xy}\ dt = \int_0^{n} e^{-\delta t} S_{T_{xy}}(t)dt$.

71) Given a joint distribution for $(T_x, T_y)$, a) $S_{T_x,T_y}(n,n) = P(T_{xy} > n) =\ {}_{n}p_{xy}$ and b) $F_{T_x,T_y}(n,n) = P(T_{\overline{xy}} \leq n) =\ {}_{n}q_{\overline{xy}}$.

72) **Know:** Let $T_{x_1}, ..., T_{x_m}$ be independent EXP$(\mu_i)$ RVs. Let $u = (x_1 \cdots x_m)$ or $u = x_1 \cdots x_m$. Then $T = T_u = T_{x_1 \cdots x_m} = \min(T_{x_1}, ..., T_{x_m}) \sim EXP(\sum_{i=1}^{m} \mu_i)$. Then $\mu_T(t) = \sum_{i=1}^m \mu_i$, $S_T(t) = \exp(-t\sum_{i=1}^m \mu_i)$, $\overset{o}{e}_u = E(T) = 1/(\sum_{i=1}^{m} \mu_i)$ and $V(T) = 1/(\sum_{i=1}^m \mu_i)^2$. a) For whole life insurance, $\overline{A}_u = E[\overline{Z}_u] = \frac{\sum_{i=1}^m \mu_i}{\delta + \sum_{i=1}^m \mu_i}$, and ${}^2\overline{A}_u = E[(\overline{Z}_u)^2] = \frac{\sum_{i=1}^m \mu_i}{2\delta + \sum_{i=1}^m \mu_i}$.

b) For a whole life annuity, $\overline{a}_u = E[\overline{Y}_u] = \frac{1}{\delta + \sum_{i=1}^m \mu_i}$, and $V[\overline{Y}_u] = \frac{{}^2\overline{A}_u - (\overline{A}_u)^2}{\delta^2}$.

Usually $m = 2$, $x_1 = x$, and $x_2 = y$.

73) For an annuity, $\overline{a}_{\overline{xy}} = \overline{a}_x + \overline{a}_y - \overline{a}_{xy}$, and if $T_x \sim EXP(\mu_x)$, then $\overline{a}_x = \frac{1}{\delta + \mu_x}$.