Math 401 Exam 2 is Wed. Oct. 23. **You are allowed 10 sheets of notes and a calculator.** The exam covers HW2-6 and Q2-7. Numbers refer to types of problems on exam. More emphasis is on HW4-6, Q4-7.

29) A life table displays x and $l_x = l_0 S_0(x)$. Often $d_x = l_x - l_{x+1}$ and other quantities are shown. The l_0 is the radix of the table and often $l_0 = 100000$. The table goes from $x = 0$ to $x = z$. The integer ω is the smallest integer such that $S_0(\omega) = 0$. So $S_0(\omega - 1) > 0$. Often $z = \omega$ or $z = \omega - 1$. Often $\omega = 110$. Note that $l_\omega = 0$.

x	l_x	d_x
0	l_0	d_0
1	l_1	d_1
$\vdots$	$\vdots$	$\vdots$
z	l_z	d_z

30) Know that $l_x =$ (expected) number living to age x out of a group of l_0 newborns. Then ${}_n d_x = l_x - l_{x+n}$ is the (expected) number who die between ages x and $x+n$. So $d_x = {}_1 d_x = l_x - l_{x+1}$ is the number who die between ages x and $x+1$. Given a life table with columns x and l_x, be able to fill in the d_x column.

31) **Know:** Given the life table with columns x and l_x, be able to find d_x, ${}_n q_x$, q_x, ${}_n p_x$, p_x and ${}_n d_x$.

${}_n q_x = \dfrac{{}_n d_x}{l_x} = P(T_0 \le x + n | T_0 > x) = P(T_x \le n)$.

$q_x = \dfrac{d_x}{l_x} = P(T_0 \le x + 1 | T_0 > x) = P(T_x \le 1)$.

${}_n p_x = 1 - {}_n q_x = \dfrac{l_{x+n}}{l_x} = P(T_0 > x + n | T_0 > x) = P(T_x > n)$.

$p_x = 1 - q_x = \dfrac{l_{x+1}}{l_x} = P(T_0 > x + 1 | T_0 > x) = P(T_x > 1)$.

Use $l_x = l_0 S_0(x)$ to see that these quantities are the same as in ch. 5.

32) ${}_n p_x = \exp(-\int_x^{x+n} \mu_y dy) = \exp(-\int_0^n \mu_{x+w} dw)$

33) For USA humans, recognize the graph of i) μ_x which roughly decreases until age $x = 10$ then increases, with rapid increase around $x = 50$, ii) $f_0(x)$ or $l_x \mu_x = l_0 f_0(x)$ which has a peak at 0 and near $x = 80$, iii) $S_0(x)$ or $l_x = l_0 S_0(x)$ which is nonincreasing and decreases rapidly near $x = 70$, and iv) $F_0(x)$ or $l_0 F_0(x)$ which is nondecreasing.

34) $\mu_x = \dfrac{-\frac{d}{dx} l_x}{l_x} = \dfrac{-S_0'(x)}{S_0(x)}$

35) $l_x = l_o \exp(-\int_0^x \mu_y dy) = l_0 \; {}_x p_0$

36) $\mu_{x+t} = \mu_0(t) = \dfrac{f_0(x+t)}{S_0(x+t)} = \dfrac{-\frac{d}{dt} l_{x+t}}{l_{x+t}}$

37) **Know:** ${}_{n|m} q_x = {}_n p_x - {}_{n+m} p_x = {}_{n+m} q_x - {}_n q_x = {}_n p_x \; {}_m q_{x+n}$

38) ${}_{n|m} q_x = P(x + n < T_0 \le x + n + m | T_0 > x) = \dfrac{P(x + n < T_0 \le x + n + m)}{P(T_0 > x)} =$
$\dfrac{F_0(x+n+m) - F_0(x+n)}{S_0(x)} = \dfrac{S_0(x+n) - S_0(x+n+m)}{S_0(x)} = \dfrac{l_{x+n} - l_{x+n+m}}{l_x} =$

$\dfrac{S_0(x+n)}{S_0(x)} \dfrac{S_0(x+n) - S_0(x+n+m)}{S_0(x+n)} = P(n < T_x \leq n+m) = S_x(n) - S_x(n+m) = F_x(n+m) - F_x(n)$.

39) ${}_{n|m}q_x = {}_np_x \; {}_mq_{x+n} = \dfrac{{}_md_{x+n}}{l_x}$

40) The probability that (x) will die between $x+n$ and $x+n+m$ $= P(x+n < T_0 < x+n+m | T_0 > x) = {}_{n|m}q_x = {}_np_x - {}_{n+m}p_x = {}_np_x \; {}_mq_{x+n}$.

41) For $m = 1$, ${}_{n|1}q_x = {}_{n|}q_x = \dfrac{d_{x+n}}{l_x} = P(K_x = n) = {}_np_x \; q_{x+n}$ where $K_x = \lfloor T_x \rfloor$.

42) ${}_nq_x = 1 - {}_np_x = \dfrac{{}_nd_x}{l_x} = \dfrac{l_x - l_{x+n}}{l_x} =$ proportion of those alive at x dying in the interval $(x, x+n] = n$ year mortality rate starting at x.

43) multiplication rule: ${}_{n+m}p_x = {}_np_x \; {}_mp_{x+n}$

44) $d_x = l_x q_x$ and $l_{x+1} = l_x - d_x$

45) $f_0(x) = \mu_0(x) S_0(x) = {}_xp_0 \; \mu_x$

46) Let W_x have the same distribution as $T_0 | T_0 > x$. Then W_x corresponds to T_0 truncated below at x. Such a truncated random variable has pdf and survival function proportional to those of T_0. Hence for $z > x$, $f_{W_x}(z) = \dfrac{f_0(z)}{S_0(x)} = f_{T_0|T_0>x}(z)$, $S_{W_x}(z) = \dfrac{S_0(z)}{S_0(x)} = S_{T_0|T_0>x}(z)$, $F_{W_x}(z) = \dfrac{F_0(z) - F_0(x)}{S_0(x)} = F_{T_0|T_0>x}(z)$, and $\mu_{W_x}(z) = \mu_0(z) = \mu_{T_0|T_0>x}(z)$.

47) $f(y|T_0 > x) = f_{T_0|T_0>x}(y) = \dfrac{f_0(y)}{S_0(x)} = \dfrac{l_y \mu_y}{l_x} = {}_{y-x}p_x \; \mu_y$

48) $f_x(t) = f_0(x+t|T_0 > x) = \dfrac{f_0(x+t)}{S_0(x)} = {}_tp_x \; \mu_{x+t}$

49) $\mu_x(t) = \mu_{x+t} = \dfrac{-S_x'(t)}{S_x(t)} = \dfrac{-\frac{d}{dx} \; {}_tp_x}{{}_tp_x}$

Thus $\dfrac{d}{dx} \; {}_tp_x = - \; {}_tp_x \; \mu_{x+t}$.

50) $\overset{o}{e}_0 = E(T_0) = \int_0^\infty x f_0(x) dx = \int_0^\infty x \; {}_xp_0 \; \mu_x \; dx = \int_0^\infty S_0(x) dx = \int_0^\infty {}_xp_0 \; dx = \dfrac{1}{l_0} \int_0^\infty l_x \; dx$

51) $E(T_0^2) = \int_0^\infty x^2 f_0(x) dx = \int_0^\infty x^2 \; {}_xp_0 \; \mu_x \; dx = 2 \int_0^\infty x \; {}_xp_0 \; dx = \dfrac{2}{l_0} \int_0^\infty x \; l_x \; dx$

52) $\overset{o}{e}_x = E(T_x) = \int_0^\infty t f_x(t) dt = \int_0^\infty t \; {}_tp_x \; \mu_{x+t} \; dt = \int_0^\infty S_x(t) dt = \int_0^\infty {}_tp_x \; dx = \dfrac{1}{l_x} \int_0^\infty l_{x+t} \; dt = \dfrac{1}{l_x} \int_x^\infty l_y \; dy$.

Hence $\overset{o}{e}_x = E(T_x)$ is the expected number of years of life remaining for a (randomly selected) person surviving to x = [expected total number of remaining years lived by the l_x survivors to age $x]/l_x$ = complete expectation of life at age x = expected future lifetime at age x. Also see points 14) and 17).

53) Note that $T_0 = T_x$ if $x = 0$. $E(T_x^2)$ is given by point 18). Plugging in $x = 0$ into 52) and 18) gives 50) and 51).

54) The median of X is equal to $E(X)$ if $E(X)$ exists and the pdf of X is symmetric.

55) If $X \sim U(a, b)$ where usually $a = 0$, then $E(X) = (b + a)/2$ and $V(X) = (b - a)^2/12$. The median is equal to $E(X)$ by symmetry.

56) $\overset{o}{e}_{x:\overline{n}|}$ = expected number of years lived in $(x, x + n]$ by a (randomly selected) survivor to age x.

(The $:\overline{n}|$ in the subscript means take the formula for $\overset{o}{e}_x$ but replace the upper limit ∞ in the integrand by n.)

57) $\overset{o}{e}_{x:\overline{n}|} = \int_0^n {}_tp_x \, dt = \frac{1}{l_x}\int_0^n l_{x+t}dt = \frac{1}{l_x}\int_x^{x+n} l_y dy.$

The right hand side is the expected total number of years lived by all l_x survivors in the interval $(x, x + n]$ divided by the number of survivors l_x.

58) The curtate expectation of life at age x is $e_x = E(K_x)$ = expected number of whole years of future lifetime for a (randomly selected) survivor to age x. Then $e_x = \frac{1}{l_x}\sum_{y=x+1}^{\infty} l_y = \frac{1}{l_x}\sum_{k=1}^{\infty} l_{x+k} = \sum_{k=1}^{\infty} {}_kp_x$. $\overset{o}{e}_x \approx e_x + 0.5$ is of more interest than e_x.

59) The temporary curtate expectation of life at age x = expected number of whole years lived over interval $(x, x + n]$ by a (randomly selected) survivor to age x is $e_{x:\overline{n}|} = \frac{1}{l_x}\sum_{k=1}^{n} l_{x+k} = \sum_{k=1}^{n} {}_kp_x$.

60) The quantities for the life table are for integer values $x = 0, 1, 2, ..., z$. Two methods of interpolation are used for integer $x \geq 0$ and $0 < t < 1$. The *uniform distribution of deaths* **UDD** assumption or *linear* assumption is that the d_x deaths occur uniformly in the interval $(x, x + 1]$. The *exponential* or **constant force** of mortality assumption is that the force of mortality is constant in the interval $(x, x + 1]$.

61) For the linear or UDD approximation, if $x \geq 0$ is an integer and $0 < t < 1$, then $l_{x+t} = (1 - t)l_x + t(l_{x+1}) = l_x - t(d_x)$. Also, $E(T_x) = \overset{o}{e}_x \approx e_x + 0.5$.

62) For the exponential or constant force approximation, if $x \geq 0$ is an integer and $0 < t < 1$, then $l_{x+t} = (l_x)^{1-t} \ (l_{x+1})^t = l_x(p_x)^t$ where $p_x = \exp(-\mu)$ so $\mu = -\log(p_x)$.

63) **Know** how to use both the UDD and constant force assumptions to find approximate the following quantities for integer $x \geq 0$ and $0 < t < 1$. For the UDD or linear approximation, note that l_{x+t} uses linear interpolation and that $f_0(t)$ is constant ("uniform") in the interval $(x, x + 1)$. For the exponential or constant force assumption μ_{x+t} is constant and $f_0(t)$ is "exponential" in the interval $(x, x+1)$. Sometimes want approximations when the subscript x is replaced by $x + v$ where $0 \leq v < 1$ and $0 \leq v + t < 1$. The exact, UDD and exponential constant force approximations are usually close. Note that the exponential constant force approximation does not depend on v.

function to approximate	linear or UDD approx	exponential or constant force approx
$S_0(x+t)$	$(1-t)S_0(x) + tS_0(x+1)$	$[S_0(x)]^{1-t} \ \ [S_0(x+1)]^t$
l_{x+t}	$(1-t)l_x + t(l_{x+1})$	$(l_x)^{1-t} \ \ (l_{x+1})^t = \ l_x(p_x)^t$
${}_tp_x \quad \left(= \dfrac{l_{x+t}}{l_x}\right)$	$1 - t(q_x)$	$(p_x)^t = \exp(-\mu t)$
${}_tq_x \quad (= 1 - \ {}_tp_x)$	$t(q_x)$	$1-(p_x)^t = 1-(1-q_x)^t$
$\mu_{x+t} \quad \left(= \dfrac{\frac{-d}{dt} l_{x+t}}{l_{x+t}}\right)$	$\dfrac{q_x}{1-t(q_x)}$	$-\log(p_x) = \mu$
$f_0(t) = \ {}_tp_x \ \mu_{x+t}$	q_x	$-(p_x)^t \ \log(p_x) \quad = \mu \exp(-\mu t)$
${}_tq_{x+v}$	$\dfrac{(t)q_x}{1-v(q_x)}$	$1-(p_x)^t \approx \ {}_tq_x$
${}_tp_{x+v}$	$1 - \dfrac{(t)q_x}{1-v(q_x)}$	$(p_x)^t \approx \ {}_tp_x$

Poisson Processes

64) A stochastic process $\{X(t) : t \in \tau\}$ is a collection of random variables where the set τ is often $[0, \infty)$. Often t is time and the random variable $X(t)$ is the state of the process at time t.

65) A stochastic process $\{N(t) : t \geq 0\}$ is a counting process if $N(t)$ counts the total number of events that occurred in time interval $(0, t]$. If $0 < t_1 < t_2$, then the random variable $N(t_2) - N(t_1)$ counts the number of events that occurred in interval $(t_1, t_2]$.

66) $N(t)$ is said to possess independent increments if the number of events that occur in disjoint time intervals are independent. Hence if $0 < t_1 < t_2 < t_3 < \cdots < t_k$, then the RVs $N(t_1) - N(0), N(t_2) - N(t_1), ..., N(t_k) - N(t_{k-1})$ are independent.

67) $N(t)$ is said to possess stationary increments if the distribution of events that occur in any time interval depends only on the length of the time interval.

68) A counting process $\{N(t) : t \geq 0\}$ is a *Poisson process with rate* λ for $\lambda > 0$ if i) $N(0) = 0$, ii) the process has independent increments, iii) the number of events in any interval of length t has a Poisson (λt) distribution with mean λt.

69) Hence the Poisson process $N(t)$ has stationary increments, the number of events in $(s, s+t]$ = the number of events in $(s, s+t)$, and for all $t, s \geq 0$, the RV $D(t) = N(t+s) - N(s) \sim$ Poisson (λt). In particular, $N(t) \sim$ Poisson (λt). So $P(D(t) = n) = P(N(t+s) - N(s) = n) = P(N(t) = n) = \dfrac{e^{-\lambda t}(\lambda t)^n}{n!}$ for $n = 0, 1, 2, ...$ Also $E[D(t)] = V[D(t)] = E[N(t)] = V[N(t)] = \lambda t$.

70) Let X_1 be the waiting time until the 1st event, X_2 the waiting time from the

1st event until the 2nd event, ..., X_j the waiting time from the $j-1$th event until the jth event and so on. The X_i are called the waiting times or interarrival times. Let $S_n = \sum_{i=1}^n X_i$ the time of occurrence of the nth event = waiting time until the nth event. For a Poisson process with rate λ, the X_i are iid EXP(λ) with $E(X_i) = 1/\lambda$, and
$S_n \sim$ Gamma (n, λ) with $E(S_n) = n/\lambda$ and $V(S_n) = n/\lambda^2$.

71) If the waiting times = interarrival times are iid EXP(λ) or independent with constant force of mortality λ, then $N(t)$ is a Poisson process with rate λ.

72) Suppose $N(t)$ is a Poisson process with rate λ that counts events of k distinct types where $p_i = P($ type i event$)$. If $N_i(t)$ counts type i events, then $N_i(t)$ is a Poisson process with rate $\lambda_i = \lambda p_i$, and the $N_i(t)$ are independent for $i = 1, ..., k$. Then $N(t) = \sum_{i=1}^k N_i(t)$ and $\lambda = \sum_{i=1}^k \lambda_i$ where $\sum_{i=1}^k p_i = 1$.

73) A counting process $\{N(t) : t \geq 0\}$ is a *nonhomogeneous Poisson process* with *intensity function* or *rate function* $\lambda(t)$, also called a *nonstationary Poisson process*, and has the following properties. i) $N(0) = 0$. ii) The process has independent increments.

iii) $N(t)$ is a Poisson $m(t)$ RV where $m(t) = \int_0^t \lambda(r)dr$, and $N(t)$ counts the number of events that occurred in $(0, t]$ (or $(0, t)$).

iv) Let $0 < t_1 < t_2$. The RV $N(t_2) - N(t_1) \sim$ Poisson $(m(t_2) - m(t_1))$ where $m(t_2) - m(t_1) = \int_{t_1}^{t_2} \lambda(r)dr$ and $N(t_2) - N(t_1)$ counts the number of events that occurred in $(t_1, t_2]$ or (t_1, t_2).

74) If $N(t)$ is a Poisson process with rate λ and there are k distinct events where the probability $p_i(s)$ of the ith event at time s depends s, let $N_i(t)$ count type i events. Then $N_i(t)$ is a nonhomogeneous Poisson process with $\lambda_i(t) = \lambda \int_0^t p_i(s)ds$. Here $\sum_{i=1}^k p_i(s) = 1$ and the $N_i(t)$ are independent for $i = 1, ..., k$.

75) A stochastic process $\{X(t) : t \geq 0\}$ is a *compound Poisson process* if $X(t) = \sum_{i=1}^{N(t)} Y_i$ where $\{N(t) : t \geq 0\}$ is a Poisson process with rate λ and $\{Y_n : n \geq 0\}$ is a family of iid random variables independent of $N(t)$. The parameters of the compound process are λ and $F_Y(y)$ where $E(Y_1)$ and $E(Y_1^2)$ are important. Then $E[X(t)] = \lambda t E(Y_1)$ and $V[X(t)] = \lambda t E(Y_1^2)$.

76) The compound Poisson process has independent and stationary increments. Fix $r, t > 0$. Then ${}_tX_r = X(r+t) - X(r)$ has the same distribution as the RV $X(t)$. Hence $E({}_tX_r) = \lambda t E(Y_1)$ and $V({}_tX_r) = \lambda t E(Y_1^2)$.

77) Let $M_Y(t)$ be the moment generating function (mgf) of Y_1. Then the mgf of the RV $X(t)$ is

$$M_{X(t)}(r) = \exp(\lambda t[M_Y(r) - 1]).$$

Mixture Distributions See p. 19.

78) The distribution of a random variable X is a *mixture distribution* if the cdf of Y has the form

$$F_X(x) = \sum_{i=1}^k \alpha_i F_{W_i}(x)$$

where $0 < \alpha_i < 1$, $\sum_{i=1}^k \alpha_i = 1$, $k \geq 2$, and $F_{W_i}(x)$ is the cdf of a continuous or discrete random variable W_i, $i = 1, ..., k$.

Then

$$E[g(X)] = \sum_{i=1}^k \alpha_i E[g(W_i)].$$

If the cdf of X is $F_X(x) = (1-\epsilon)F_Z(x) + \epsilon F_W(x)$ where $0 \leq \epsilon \leq 1$ and F_Z and F_W are cdfs, then $E[g(X)] = (1-\epsilon)E[g(Z)] + \epsilon E[g(W)]$. In particular, $E(X^2) = (1-\epsilon)E[Z^2] + \epsilon E[W^2] = (1-\epsilon)[V(Z) + (E[Z])^2] + \epsilon[V(W) + (E[W])^2]$.

Often Z is nonsmoker, W is smoker, and ϵ is the probability that a randomly chosen person from the population (of X) is a smoker.

ch 7.

79) A life insurance model is a special cases of a contingent payment model where the payment is made contingent (conditional) on the occurrence of some random event.

80) From interest theory, i) the *compound interest factor* $v = \dfrac{1}{1+i}$ and $0 < v < 1$.

ii) The *effective rate of interest* $i = \dfrac{1-v}{v}$ and $i > 0$. Often $i = 0.05$.

iii) The *force of interest* $\delta = \log(1+i)$ and $\delta > 0$. Note that $1 + i = e^{\delta}$ so $v = e^{-\delta}$.

81) First we will consider models where the rate of earnings and inflation is deterministic, eg $i = 0.05$, but the investment period (time from issue of insurance until death) is random.

82) The model has a *benefit function* b_t and a *discount function* v_t where t = the length of time from issue of insurance until death (or until insurance payment). Often $v_t = v^t$ and $b_t = 1$ unit where $1 + i = e^{\delta}$ and $v = e^{-\delta}$.

83) The *present value function* $z_t = b_t v_t$ is the present value, at time t from policy issue, of the benefit payment.

84) $T = T_x$ = insured's future lifetime RV and the *claim random variable* or *present value random variable* $Z = z_{T_x} = b_{T_x} v_{T_x}$. Or $K_x = \lfloor T_x \rfloor$ = the curtate future lifetime RV, and $Z = z_{1+K_x} = b_{1+K_x} v_{1+K_x}$.

85) $E(Z)$ is the *actuarial present value* (APV) = *expected present value* (EPV) = *net single premium* (NSP) of the insurance, the expected value of the present value of the payment.

86) Suppose $b_t \equiv 1$ or $b_t = 1$ for t in some interval and $b_t = 0$, otherwise. Suppose $v_t = v^t$ for $t > 0$. Let $A_x = E(Z) = g(\delta)$. Let ${}^jA_x = E(Z^j)$. The rule of moments is ${}^jA_x = E(Z^j) = g(j\delta)$. The rule of moments only holds if $b_t \in \{0, 1\}$ for all $t \geq 0$. Typically finding $E(Z)$ and $E(Z^2)$ directly is easier than using the rule of moments.

87) **Formulas are given for unit payment.** For nonunit payment c, multiply the unit payment formula for A by c and the unit formula payment for ${}^2A$ by c^2.

88) Suppose (x) buys insurance and dies at $t \in (k-1, k]$ years from purchase so $K_x = k - 1$ where $k \in \{0, 1, 2, ...\}$. Given v, i or δ and a small table of k and $P(K_x = k)$, be able to find the following quantities for the following 4 discrete life insurance models where a unit payment (eg of \$100000, \$500000 or \$1000000) is made.

i) (Discrete) *whole life insurance* makes unit payment at time $t = k$ with $v_t = v^t, t \geq 0$ and $b_t = 1, t \geq 0$. Then $z_t = b_t v_t = v^t, t \geq 0$. The present value random variable $Z_x = z_{1+K_x} = v^{1+K_x}$. Let $v' = v^2$. Then the actuarial present value APV = EPV = NSP

$$= A_x = E(Z_x) = E(v^{1+K_x}) = \sum_{k=0}^{\infty} v^{k+1} P(K_x = k),$$

and $${}^2\mathrm{A_x} = \mathrm{E}[(\mathrm{Z_x})^2] = \mathrm{E}[(\mathrm{v}^{1+\mathrm{K_x}})^2] = \sum_{\mathrm{k}=0}^{\infty} \mathrm{v}^{2(\mathrm{k}+1)} \mathrm{P}(\mathrm{K_x} = \mathrm{k}) = \sum_{\mathrm{k}=0}^{\infty} (\mathrm{v}')^{(\mathrm{k}+1)} \mathrm{P}(\mathrm{K_x} = \mathrm{k}).$$

ii) (Discrete) *n year term insurance* = (discrete) *n year temporary insurance* makes unit payment at time $t = k$ only if $k \leq n$, otherwise no payment is made. Now
$v_t = v^t, t \geq 0$,

$$b_t = \begin{cases} 1, & t \leq n \\ 0, & t > n \end{cases} \quad \text{and} \quad \mathrm{z_t} = \mathrm{b_t v_t} = \begin{cases} v^t, & t \leq n \\ 0, & t > n. \end{cases}$$

The present value random variable (note $1 + K_x \leq n$ if $K_x < n$)

$$Z^1_{x:\overline{n}|} = \begin{cases} v^{1+K_x}, & K_x < n \\ 0, & K_x \geq n. \end{cases}$$

Then the actuarial present value APV = EPV = NSP =

$$A^1_{x:\overline{n}|} = E(Z^1_{x:\overline{n}|}) = \sum_{k=0}^{n-1} v^{k+1} P(K_x = k),$$

and $${}^2\mathrm{A}^1_{\mathrm{x}:\overline{\mathrm{n}}|} = \mathrm{E}[(\mathrm{Z}^1_{\mathrm{x}:\overline{\mathrm{n}}|})^2] = \sum_{\mathrm{k}=0}^{\mathrm{n}-1} \mathrm{v}^{2(\mathrm{k}+1)} \mathrm{P}(\mathrm{K_x} = \mathrm{k}) = \sum_{\mathrm{k}=0}^{\mathrm{n}-1} (\mathrm{v}')^{(\mathrm{k}+1)} \mathrm{P}(\mathrm{K_x} = \mathrm{k}).$$

The 1 above the x means unit benefit is payable after (x) dies if death is before time n.

iii) (Discrete) *n year deferred insurance* makes unit payment at time $t = k$ only if $k > n$ so $k \geq n + 1$, otherwise no payment is made. Now $v_t = v^t, t \geq 0$,

$$b_t = \begin{cases} 0, & t \leq n \\ 1, & t > n \end{cases} \quad \text{and} \quad \mathrm{z_t} = \mathrm{b_t v_t} = \begin{cases} 0, & t \leq n \\ v^t, & t > n. \end{cases}$$

The present value random variable (note $1 + K_x > n$ if $K_x \geq n$)

$$_n|Z_x = \begin{cases} 0, & K_x < n \\ v^{1+K_x}, & K_x \geq n. \end{cases}$$

Then the actuarial present value APV = EPV = NSP =

$$_n|A_x = E(_n|Z_x) = \sum_{k=n}^{\infty} v^{k+1} P(K_x = k),$$

and $${}^2\ {}_\mathrm{n}|\mathrm{A_x} = \mathrm{E}[({}_\mathrm{n}|\mathrm{Z_x})^2] = \sum_{\mathrm{k}=\mathrm{n}}^{\infty} \mathrm{v}^{2(\mathrm{k}+1)} \mathrm{P}(\mathrm{K_x} = \mathrm{k}) = \sum_{\mathrm{k}=\mathrm{n}}^{\infty} (\mathrm{v}')^{(\mathrm{k}+1)} \mathrm{P}(\mathrm{K_x} = \mathrm{k}).$$

iv) (Discrete = continuous) *n year pure endowment insurance* makes unit payment at time n only if $t > n$, otherwise no payment is made. Now

$$v_t = \begin{cases} v^t, & t \le n \\ v^n, & t > n, \end{cases} \quad b_t = \begin{cases} 0, & t \le n \\ 1, & t > n \end{cases} \quad \text{and} \quad \mathrm{z_t = b_t v_t} = \begin{cases} 0, & t \le n \\ v^n, & t > n. \end{cases}$$

The present value random variable

$$Z_{\substack{1\\x:\overline{n}|}} = \begin{cases} 0, & T_x \le n \\ v^n, & T_x > n. \end{cases}$$

Then the actuarial present value APV = EPV = NSP =

$$A_{\substack{1\\x:\overline{n}|}} = E(Z_{\substack{1\\x:\overline{n}|}}) = \ _nE_x \ = v^n P(T_x > n) = v^n \int_n^\infty f_x(t)\, dt = v^n \int_n^\infty \ _tp_x \ \mu_{x+t}\, dt = v^n \ _np_x$$

$(= v^n P(K_x \ge n) = v^n \sum_{k=n}^\infty P(K_x = k) = v^n S_x(n) = e^{-\delta n} S_x(n)$ and

$$^2A_{\substack{1\\x:\overline{n}|}} = E[(Z_{\substack{1\\x:\overline{n}|}})^2] = v^{2n} P(T_x > n) = v^{2n} \int_n^\infty f_x(t)\, dt = v^{2n} \int_n^\infty \ _tp_x \ \mu_{x+t}\, dt = v^{2n} \ _np_x$$

$= v^{2n} P(K_x \ge n) = v^{2n} \sum_{k=n}^\infty P(K_x = k) = v^{2n} S_x(n) = e^{-2\delta n} S_x(n)$. The 1 above the $\overline{n}|$ means unit benefit is payable after (x) dies if death is after time n.

Also $V(Z_{\substack{1\\x:\overline{n}|}}) = v^{2n} \ _np_x \ _nq_x$.

Note the book does not use $\overline{Z}$ and $\overline{A}$ for this insurance because payment is made iff $T_x > n$ iff $K_x \ge n$ so the discrete insurance and continuous insurance are technically equivalent.

89) The relationship between whole life insurance and n year temporary and n year deferred insurance is

$$Z_x = Z^1_{x:\overline{n}|} + \ _n|Z_x,$$

$$A_x = A^1_{x:\overline{n}|} + \ _n|A_x,$$

$$[Z_x]^2 = [Z^1_{x:\overline{n}|}]^2 + \ [_n|Z_x]^2, \quad \text{and}$$

$$^2A_x = \ ^2A^1_{x:\overline{n}|} + \ ^2 \ _n|A_x.$$

90) Suppose (x) buys insurance and dies at $t \in (k-1, k]$ years from purchase so $K_x = k$ where $k \in \{0, 1, 2, ...\}$. Given a small table of k and $P(K_x = k)$, be able to find the following quantities. (Discrete) *n year endowment life insurance* makes unit payment at time $t = k$ if $t < k < n$ and at time n if $t > n$. Then $b_t = 1, t \ge 0$ and

$$v_t = \begin{cases} v^t, & t \le n \\ v^n, & t > n, \end{cases} \quad \text{and} \quad \mathrm{z_t = b_t v_t} = \begin{cases} v^t, & t \le n \\ v^n, & t > n. \end{cases}$$

The present value random variable

$$Z_{x:\overline{n}|} = \begin{cases} v^{K_x+1}, & K_x < n \\ v^n, & K_x \ge n. \end{cases}$$

Note that the n year endowment present value random variable $Z_{x:\overline{n}|} = Z^1_{x:\overline{n}|} + Z_{\substack{\ \ 1\\x:\overline{n}|}}$, the sum of the n year term and n year pure endowment present value RVs.

Then the actuarial present value APV = EPV = NSP = $A_{x:\overline{n}|} = E[Z_{x:\overline{n}|}]$

$$= A^1_{x:\overline{n}|} + A_{\substack{\ \ 1\\x:\overline{n}|}} = \sum_{k=0}^{n-1} v^{k+1} P(K_x = k) + v^n P(K_x \geq n) = \sum_{k=0}^{n-1} v^{k+1} P(K_x = k) + v^n \sum_{k=n}^{\infty} P(K_x = k).$$

Similarly, $[Z_{x:\overline{n}|}]^2 = [Z^1_{x:\overline{n}|}]^2 + [Z_{\substack{\ \ 1\\x:\overline{n}|}}]^2$ and ${}^2A_{x:\overline{n}|} = {}^2A^1_{x:\overline{n}|} + {}^2A_{\substack{\ \ 1\\x:\overline{n}|}}$

$$= \sum_{k=0}^{n-1} v^{2(k+1)} P(K_x = k) + v^{2n} P(K_x \geq n) = \sum_{k=0}^{n-1} v^{2(k+1)} P(K_x = k) + v^{2n} \sum_{k=n}^{\infty} P(K_x = k).$$

91) Suppose (x) buys insurance and dies at $t > 0$ years from purchase so $T = T_x = t$. Given v, i or δ and the distribution of $T = T_x$, be able to find the following quantities for the following 5 continuous life insurance models where a unit payment (eg of \$100000, \$500000 or \$1000000) is made. Recall $v = \dfrac{1}{1+i} = e^{-\delta}$ and $\delta = \log(1+i) = -\log(v)$. Often use $v^t = e^{-\delta t}$ and $v^{2t} = e^{-2\delta t}$.

The rule of moments for $b_t \in \{0, 1\}$ (unit payment insurance) is if $E[\ \overline{Z}\ ] = \overline{A} = g(\delta)$, then $E[(\overline{Z})^j] = {}^j\overline{A} = g(j\delta)$. This rule is usually used for $j = 2$.

i) (Continuous) *whole life insurance* makes unit payment at time $t = k$ with $v_t = v^t, t \geq 0$ and $b_t = 1, t \geq 0$. Then $z_t = b_t v_t = v^t, t \geq 0$. The present value random variable $\overline{Z}_x = z_T = v^T$. Then the actuarial present value APV = EPV = NSP =

$$\overline{A}_x = E(\overline{Z}_x) = E(v^T) = E(e^{-\delta T}) = \int_0^\infty v^t f_T(t)\, dt = \int_0^\infty e^{-\delta t} f_T(t)\, dt = \int_0^\infty v^t\ {}_tp_x\ \mu_{x+t}\, dt, \text{ and}$$

$${}^2\overline{A}_x = E[(\overline{Z}_x)^2] = E[(v^T)^2] = E(e^{-2\delta T}) = \int_0^\infty v^{2t} f_T(t)\, dt = \int_0^\infty e^{-2\delta t} f_T(t)\, dt = \int_0^\infty v^{2t}\ {}_tp_x\ \mu_{x+t}\, dt.$$

ii) (Continuous) *n year term insurance* makes unit payment at time $t > 0$ only if $t \leq n$, otherwise no payment is made. Now $v_t = v^t, t \geq 0$,

$$b_t = \begin{cases} 1, & t \leq n \\ 0, & t > n, \end{cases} \qquad z_t = b_t v_t = \begin{cases} v^t, & t \leq n \\ 0, & t > n, \end{cases} \text{ and } \overline{\mathrm{Z}}^1_{\mathrm{x}:\overline{\mathrm{n}}|} = \begin{cases} v^{T_x}, & T \leq n \\ 0, & T > n. \end{cases}$$

Then the actuarial present value APV = EPV = NSP =

$$\overline{A}^1_{x:\overline{n}|} = E(\overline{Z}^1_{x:\overline{n}|}) = \int_0^n e^{-\delta t} f_T(t)\ dt = \int_0^n v^t f_T(t)\ dt = \int_0^n v^t\ {}_tp_x\ \mu_{x+t}\ dt, \text{ and}$$

$${}^2\overline{A}^1_{x:\overline{n}|} = E[(\overline{Z}^1_{x:\overline{n}|})^2] = \int_0^n e^{-2\delta t} f_T(t)\ dt = \int_0^n v^{2t} f_T(t)\ dt = \int_0^n v^{2t}\ {}_tp_x\ \mu_{x+t}\ dt.$$

The 1 above the x means unit benefit is payable after (x) dies if death is not after time n.

iii) (Continuous) *n year deferred insurance* makes unit payment at time $t > 0$ only if $t > n$, otherwise no payment is made. Now $v_t = v^t, t \geq 0$,

$$b_t = \begin{cases} 0, & t \leq n \\ 1, & t > n \end{cases} \quad \text{and} \quad \mathrm{z_t = b_t v_t} = \begin{cases} 0, & t \leq n \\ v^t, & t > n. \end{cases}$$

The present value random variable

$$_{n|}\overline{Z}_x = \begin{cases} 0, & T \leq n \\ v^T, & T > n. \end{cases}$$

Then the actuarial present value APV = EPV = NSP =

$$_{n|}\overline{A}_x = E(_{n|}\overline{Z}_x) = \int_n^\infty e^{-\delta t} f_T(t)\; dt = \int_n^\infty v^t f_T(t)\; dt = \int_n^\infty v^t \; _tp_x \; \mu_{x+t} \; dt, \text{ and}$$

$$^2 \; _{n|}\overline{A}_x = E[(_{n|}\overline{Z}_x)^2] = \int_n^\infty e^{-2\delta t} f_T(t)\; dt = \int_n^\infty v^{2t} f_T(t)\; dt = \int_n^\infty v^{2t} \; _tp_x \; \mu_{x+t} \; dt.$$

iv) See 88 iv) for the n year pure endowment life insurance which is both continuous and discrete.

v) (Continuous) *n year endowment life insurance* makes unit payment at time $t > 0$ if $t < n$ and at time n if $t > n$. Then $b_t = 1, t \geq 0$ and

$$v_t = \begin{cases} v^t, & t \leq n \\ v^n, & t > n \end{cases} \quad \text{and} \quad \mathrm{z_t = b_t v_t} = \begin{cases} v^t, & t \leq n \\ v^n, & t > n. \end{cases}$$

The present value random variable

$$\overline{Z}_{x:\overline{n}|} = \begin{cases} v^T, & T \leq n \\ v^n, & T > n. \end{cases}$$

Note that the n year endowment present value random variable $\overline{Z}_{x:\overline{n}|} = \overline{Z}^1_{x:\overline{n}|} + Z_{\substack{1\\x:\overline{n}|}}$, the sum of the n year term and n year pure endowment present value RVs.

Then the actuarial present value APV = EPV = NSP =

$$\overline{A}_{x:\overline{n}|} = E[\overline{Z}_{x:\overline{n}|}] = \overline{A}^1_{x:\overline{n}|} + A_{\substack{1\\x:\overline{n}|}} = \int_0^n v^t f_T(t)\; dt + v^n P(T > n) = \int_0^n v^t \; _tp_x \; \mu_{x+t} \; dt + v^n \; _np_x.$$

Similarly, $[\overline{Z}_{x:\overline{n}|}]^2 = [\overline{Z}^1_{x:\overline{n}|}]^2 + [Z_{\substack{1\\x:\overline{n}|}}]^2$ and $^2\overline{A}_{x:\overline{n}|} = \; ^2\overline{A}^1_{x:\overline{n}|} + \; ^2A_{\substack{1\\x:\overline{n}|}}$

$$= \int_0^n v^{2t} f_T(t)\; dt + v^{2n} P(T_x > n) = \int_0^n v^{2t} \; _tp_x \; \mu_{x+t} \; dt \; + \; v^{2n} \; _np_x.$$

92) **Know**: Often $T_0 \sim EXP(\mu)$ so $T = T_x \sim EXP(\mu)$. This distribution occurs if the force of mortality μ, μ_x or μ_{x+t} is constant. Also $S_x(t) = \; _tp_x = e^{-\mu t}$. Hence $f_x(t) = \; _tp_x \mu_{x+t} = \mu e^{-\mu t}$.

93) **Know**: Often $T_0 \sim U(0, \omega)$ so $T_x \sim U(0, \omega - x)$. The uniform distribution has cdf that is linear and increases from 0 to 1 on its support. Its survival function is linear and decreases from 1 to 0 on its support. Hence l_x is linear and decreases from l_0 to 0 on its support. So $S(t) = 1 - t/\omega$ for $0 \leq t \leq \omega$, and $_tp_x = 1 - t/(\omega - x) = \dfrac{\omega - x - t}{\omega - x}$ for $0 \leq t \leq \omega - x$. Also $\mu_{x+t} = \dfrac{1}{\omega - x - t}$ and $f_x(t) = \; _tp_x \mu_{x+t} = \dfrac{1}{\omega - x}$ for $0 \leq t < \omega - x$.

94) On SOA and CAS exams, often the notation A and Z is used even though the correct notation is $\overline{A}$ and $\overline{Z}$.

95) Whole life insurance with the exponential(μ) distribution often has $\overline{Z} = b_T\ v^T$ where $b_t = e^{\theta t}$. Now $\int_0^\infty \mu e^{-\mu t} dt = 1$ so $\int_0^\infty e^{-\mu t} dt = 1/\mu$ if $\mu > 0$. Hence $E[\overline{Z}\ ] = \int_0^\infty b_t e^{-\delta t} \mu e^{-\mu t} dt = \int_0^\infty e^{\theta t} e^{-\delta t} \mu e^{-\mu t} dt = \mu \int_0^\infty e^{-t[\mu+\delta-\theta]} dt = \dfrac{\mu}{\mu+\delta-\theta}$ provided $\mu + \delta - \theta > 0$. Also $E[(\overline{Z})^j] = \int_0^\infty [b_t e^{-\delta t}]^j \mu e^{-\mu t} dt = \int_0^\infty e^{\theta j t} e^{-\delta j t} \mu e^{-\mu t} dt = \mu \int_0^\infty e^{-t[\mu+\delta j-\theta j]} dt = \dfrac{\mu}{\mu+\delta j-\theta j}$ provided $\mu + \delta j - \theta j > 0$. Note that $\theta = 0$ corresponds to unit payment.

96) For whole life insurance let ξ_α be the α percentile of $\overline{Z}$ so $P(\overline{Z} \le \xi_\alpha) = \alpha$ where $0 < \alpha < 1$. Assume unit payment so $\overline{Z} = v^T = e^{-\delta T}$. To find the α percentile ξ_α of $\overline{Z}$, solve $\alpha = P(\overline{Z} \le \xi_\alpha) = P(e^{-\delta T} \le \xi_\alpha) = P[-\delta T \le \log(\xi_\alpha)] = P\left(T \ge \dfrac{\log(\xi_\alpha)}{-\delta}\right) = S_T\left(\dfrac{-\log(\xi_\alpha)}{\delta}\right)$. So solve $\alpha = S_T\left(\dfrac{-\log(\xi_\alpha)}{\delta}\right)$ for ξ_α. Often $T \sim EXP(\mu)$ so $S_T(t) = e^{-\mu t}$. Then solve $\alpha = \exp\left[\dfrac{\mu}{\delta}\log(\xi_\alpha)\right] = \xi_\alpha^{\mu/\delta}$ for $\xi_\alpha \overset{E}{=} \alpha^{\delta/\mu}$.

97) **KNOW:** Let $T \sim EXP(\mu)$. Then $E(T) = \int_0^\infty t\mu e^{-\mu t} dt = \int_0^\infty e^{-\mu t} dt = 1/\mu$. So $\int_0^\infty tDe^{-t(D)} dt = \int_0^\infty e^{-t(D)} dt = 1/D$ for $D > 0$. Use $\overset{E}{=}$ when exponential RV is used.

98) **KNOW:** Let $T \sim EXP(\mu)$. $S(t) = e^{-\mu t}$ for $t > 0$. Often use Z instead of $\overline{Z}$.

i) If $b_t = ce^{\theta t}$ and $Z = b_T v_T$, then $E[Z^j] = E[(b_T v_T)^j] = c^j E[(e^{\theta T} v_T)^j]$. So multiply $c = 1$ formulas by c^j. Usually want $j = 1, 2$.

a) Special whole life insurance: $b_t = e^{\theta t}$, $v_t = e^{-\delta t}$, and $Z = b_T v_T = e^{\theta T} e^{-\delta T}$. $E(Z^j) \overset{E}{=} \dfrac{\mu}{\mu+\delta j-\theta j}$ if $\mu + \delta j - \theta j > 0$. See 95).

b) Whole life insurance: special case of a) with $\theta = 0$. See 100i). $\overline{Z}_x = e^{-\delta T}$. $\overline{A}_x = E(\overline{Z}_x) = E(e^{-\delta T}) \overset{E}{=} \dfrac{\mu}{\mu+\delta}$, and ${}^2\overline{A}_x = E[(\overline{Z}_x)^2] = E(e^{-2\delta T}) \overset{E}{=} \dfrac{\mu}{\mu+2\delta}$. $V(\overline{Z}_x) = {}^2\overline{A}_x - (\overline{A}_x)^2$.

99) In 95), often $\int_0^\infty$ is replaced by $\int_a^b$. If $D > 0$, $\int_0^n De^{-tD} dt = 1 - e^{-nD}$, $\int_n^\infty De^{-tD} dt = e^{-nD}$, $\int_0^n e^{-tD} dt = \dfrac{1}{D}[1 - e^{-nD}]$, and $\int_n^\infty e^{-tD} dt = \dfrac{1}{D}\ e^{-nD}$.