Math 401 Exam 1 is Wed. Sept. 18. **You are allowed 4 sheets of notes and a calculator.** The exam covers HW1-3, part of HW4, and Q1-3. Numbers refer to types of problems on exam.

In this class $\log(t) = \ln(t) = \log_e(t)$ while $\exp(t) = e^t$.

0) Memorize the following distributions:

a) Bernoulli(p) = binomial($n = 1, p$): $p(x) = p^x(1-p)^{1-x}$ for $x = 0, 1$.
$E(X) = p$. $VAR(X) = p(1-p)$.

b) binomial(n, ρ):

$$p(x) = \binom{n}{x} \rho^x (1-\rho)^{n-x} \quad \text{for} \quad \text{x} = 0, 1, \ldots, \text{n} \quad \text{where} \quad 0 < \rho < 1.$$

$E(X) = n\rho$. $VAR(X) = n\rho(1-\rho)$.

c) exponential(β)= gamma($\nu = 1, \beta$):

$$f(x) = \beta \exp(-\beta x)\ I(x \geq 0) \quad \text{where} \quad \beta > 0.$$

$E(X) = 1/\beta$, $VAR(X) = 1/\beta^2$. $F(x) = 1 - \exp(-\beta x)$, $x \geq 0$. Here $I(x \geq 0) = 1$ if $x \geq 0$ and $I(x \geq 0) = 0$, otherwise. (The parameterization with $\lambda = 1/\beta$ is common. Then $E(X) = \lambda$ and $V(X) = \lambda^2$.)

d) gamma(α, β):

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad \text{where} \quad \alpha, \ \beta \quad \text{and} \quad \text{x} \quad \text{are} \quad \text{positive.}$$

$E(X) = \alpha/\beta$. $VAR(X) = \alpha/\beta^2$.

(The parameterization with $\lambda = 1/\beta$ is common. Then $E(X) = \alpha\lambda$ and $V(X) = \alpha\lambda^2$.)

e) $N(\mu, \sigma^2)$:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

where $\sigma > 0$ and μ and x are real.
$E(X) = \mu$. $VAR(X) = \sigma^2$.

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

f) Poisson(λ):

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{for} \quad \text{x} = 0, 1, \ldots, \quad \text{where} \quad \lambda > 0.$$

$E(X) = \lambda = VAR(X)$.

g) uniform(θ_1, θ_2):

$$f(x) = \frac{1}{\theta_2 - \theta_1} I(\theta_1 \leq x \leq \theta_2).$$

$F(x) = (x - \theta_1)/(\theta_2 - \theta_1)$ for $\theta_1 \leq x \leq \theta_2$.
$E(X) = (\theta_1 + \theta_2)/2$. $VAR(X) = (\theta_2 - \theta_1)^2/12$.

Let $X = T_0 > 0$ be a nonnegative random variable.

Then the **cumulative distribution function** (cdf) $F_0(t) = P(T_0 \le t)$. Since $T_0 > 0$, $F_0(0) = 0, F_0(\infty) = 1$, and $F_0(t)$ is nondecreasing.

The probability density function (**pdf**) $f_0(t) = F_0'(t)$.

The **survival (distribution) function** $S_0(t) = P(T_0 > t)$. $S_0(0) = 1, S_0(\infty) = 0$ and $S_0(t)$ is nonincreasing.

The *hazard (rate) function* $\lambda_0(t) =$ **force of mortality** $= \mu_0(t) = \dfrac{f_0(t)}{1 - F_0(t)} = \dfrac{f_0(t)}{S_0(t)}$ for $t > 0$ and $F_0(t) < 1$. Note that $\mu_0(t) \ge 0$ if $F_0(t) < 1$.

The **cumulative hazard function** $\Lambda_0(t) = \int_0^t \mu_0(y)dy$ for $t > 0$. It is true that $\Lambda_0(0) = 0, \Lambda_0(\infty) = \infty$, and $\Lambda_0(t)$ is nondecreasing.

1) Given one of $F_0(t), f_0(t), S_0(t), \mu_0(t)$ or $\Lambda_0(t)$, be able to find the other 4 quantities for $t > 0$. See HW2: 1,3.

A) $F_0(t) = \int_0^t f_0(y)dy = 1 - S_0(t) = 1 - \exp[-\Lambda_0(t)] = 1 - \exp[-\int_0^t \mu_0(y)dy]$.

B) $f_0(t) = F_0'(t) = -S_0'(t) = \mu_0(t)[1 - F_0(t)] = \mu_0(t)S_0(t) = \mu_0(t)\exp[-\Lambda_0(t)] = \Lambda_0'(t)\exp[-\Lambda_0(t)]$.

C) $S_0(t) = 1 - F_0(t) = 1 - \int_0^t f_0(y)dy = \int_t^\infty f_0(y)dy = \exp[-\Lambda_0(t)] = \exp[-\int_0^t \mu_0(y)dy]$.

$$D) \quad \mu_0(t) = \frac{f_0(t)}{1 - F_0(t)} = \frac{f_0(t)}{S_0(t)} = \frac{F_0'(t)}{1 - F_0(t)} = \frac{-S_0'(t)}{S_0(t)} = -\frac{d}{dt}\log[S_0(t)] = \Lambda_0'(t).$$

E) $\Lambda_0(t) = \int_0^t \mu_0(y)dy = -\log[S_0(t)] = -\log[1 - F_0(t)]$.

Tip: if $F(t) = 1 - \exp[G(t)]$ for $t > 0$, then $\Lambda(t) = -G(t)$ and $S(t) = \exp[G(t)]$.

Tip: For $S(t) > 0$, note that $S(t) = \exp[\log(S(t))] = \exp[-\Lambda(t)]$. Finding $\exp[\log(S(t))]$ and setting $\Lambda(t) = -\log[S(t)]$ is easier than integrating $\mu(t)$.

2) **Actuarial notation**: p corresponds to a probability, possibly conditional, that X is greater than some value. Then $q = 1 - p$ corresponds to the probability that X is less than or equal to that value. Let $X = T_0$,

3) ${}_tp_0 = S_0(t) = P(T_0 > t)$

4) ${}_tq_0 = F_0(t) = P(T_0 \le t)$.

5) Let $T_0 > 0$. If $\lim_{t\to\infty} tS_0(t) = 0$, then $E(T_0) = \int_0^\infty tf_0(t)dt = \int_0^\infty S_0(t)dt$ $= \int_0^\infty [1 - F_0(t)]dt$. Hence in a graph of $F_0(t)$, the mean $E(T_0)$ equals the area to the right of the vertical axis $t = 0$ and between the horizontal line at height 1 and $F_0(t)$.

6) **Actuarial notation**: μ stands for force of mortality and e stands for expectation.

7) $\mu_t = \mu(t) =$ force of mortality $=$ hazard rate function

8) $E(T_0) = \overset{o}{e}_0 =$ complete expectation of life at birth.

9) The median y_M of a distribution satisfies $P(T_0 \le y_M) \ge 0.5$ and $P(T_0 \ge y_M) \ge 0.5$. For a continuous RV T_0, $F_0(y_M) = S_0(y_M) = 0.5$. The median need not be unique.

10) **Actuarial notation**: (x) denotes a person alive at age x.

11) T_x is the time until failure for a person alive at age x. $T_0 = (x + T_x)|(T_0 > x)$.

12) Let $t > 0$. Let $f_x(t) = f_{T_x}(t)$, $F_x(t) = F_{T_x}(t)$, $S_x(t) = S_{T_x}(t)$ and $\mu_x(t) = \mu_{T_x}(t)$.
i)

$$\,_tp_x = \frac{S_0(x+t)}{S_0(x)} = 1 - \,_tq_x = P(T_x > t) = P(T_0 > x + t | T_0 > x) = S_x(t)$$

$$= \exp(-\int_x^{x+t} \mu_r \; dr) = \exp(-\int_0^t \mu_{x+s} \; ds)$$

Note that $S_0(x + t) = S_0(x)S_x(t)$.
ii)

$$\,_tq_x = 1 - \,_tp_x = 1 - \frac{S_0(x+t)}{S_0(x)} = P(T_x \leq t) = P(T_0 \leq x + t | T_0 > x) = F_x(t)$$

iii)

$$f_x(t) = \frac{f_0(x+t)}{S_0(x)} = \,_tp_x \; \mu_{x+t} = \frac{d}{dt}F_x(t) = -\frac{d}{dt}S_x(t)$$

iv)

$$\mu_{x+t} = \frac{f_0(x+t)}{S_0(x+t)} = \mu_0(x+t) = \mu_x(t)$$

13) **Actuarial notation**: If $t = 1$ the subscript is suppressed so
$p_x = \,_1p_x = P(T_x > 1) = P(T_0 > x + 1 | T_0 > x) = \dfrac{S_0(x+1)}{S_0(x)}$ and
$q_x = \,_1q_x = P(T_x \leq x + 1) = P(T_0 \leq x + 1 | T_0 > x) = 1 - p_x$.

14) The expected value of T_x given $T_0 > x$ is called the complete expectation of life at age x or the expected future lifetime at age x and is denoted by

$$\overset{o}{e}_x = E(T_x) = E(T_0 | T_0 > x) - x.$$

15) $T_x > 0$ is a nonnegative RV. If the support of T_0 is $(0, \infty)$ then the support of T_x is $(0, \infty)$. If the support of T_0 is $(0, \omega)$ then the support of T_x is $(0, \omega - x)$.

16) $E(T_x)$, $E(T_x^2)$ $V(T_x)$ is better notation than the text notation like $V(T_x | T_0 > x)$.

17) $\overset{o}{e}_x = E(T_x) = \displaystyle\int_0^\infty t f_x(t) dt = \frac{1}{S_0(x)} \int_0^\infty t f_0(x+t) dt =$
$\displaystyle\frac{1}{S_0(x)} \int_0^\infty S_0(x+t) dt = \int_0^\infty \,_tp_x \; dt.$

18) $V(T_x) = E(T_x^2) - [E(T_x))]^2$ where $E(T_x) = \overset{o}{e}_x = \int_0^\infty \,_tp_x \; dt$ and $E(T_x^2) = 2\int_0^\infty \; t \; \,_tp_x \; dt$.
Also, $V(T_x) = E(T_0^2 | T_0 > x) - [E(T_0 | T_0 > x)]^2$.

19) **Memorize**: Know that if $X \sim EXP(\beta)$ where $\beta > 0$, then $\mu(x) = \beta$ for $x > 0$, $f(x) = \beta e^{-\beta x}$ for $x > 0$, $F(x) = 1 - e^{-\beta x}$ for $x > 0$, $S(x) = e^{-\beta x}$ for $x > 0$, $\Lambda(x) = \beta x$ for $x > 0$, $E(X) = 1/\beta$ and $V(X) = 1/\beta^2$. The **exponential distribution** is the only

distribution with a constant force of mortality $\mu(x) \equiv \beta$. Constant force of mortality (hazard) means that a used product is as good as a new product.

The pdf can also be wriiten as $f(x) = \frac{1}{\lambda} e^{-x/\lambda}$ for $x > 0$ and $\lambda > 0$. Then $\lambda = 1/\beta$ and $E(X) = \lambda = 1/\beta$. Usually $E(X) > 1$ and the force of mortality is $\beta < 1$ for human lifetime. Use $E(X) > 1$ to determine if you are given β or λ. For example, $X \sim EXP(0.1)$ means $E(X) = 1/0.1 = 10$ so $\beta = 0.1$ and $\lambda = 10$, while $X \sim EXP(10)$ means $E(X) = 10$ so $\beta = 0.1$ and $\lambda = 10$.

20) **Memorize:** For $t > 0$, if $T_x \sim EXP(\mu)$, then $S_x(t) = {}_tp_x = e^{-\mu t}$, $\mu_x(t) = \mu_{x+t} = \mu$ and $E(T_x) = \overset{o}{e}_x = 1/\mu$. If $T_0 EXP(\mu)$, then $T_x \sim EXP(\mu)$.

21) **Memorize**: If the age at failure RV T_0 has a uniform $(0, \omega)$ distribution, then this distribution is known as **de Moivre's law** or De Moivre's law. Then $T_x \sim U(0, \omega - x)$ with support $0 < t < \omega - x$. For such t, $S_x(t) = {}_tp_x = \dfrac{\omega - x - t}{\omega - x}$,

$\mu_x(t) = \mu_{x+t} = \dfrac{1}{\omega - x - t}$ and $E(T_x) = \overset{o}{e}_x = \dfrac{\omega - x}{2}$.

22) The *curtate duration at failure* RV $K_x = \lfloor T_x \rfloor$. Here $\lfloor 7.7 \rfloor = 7$. Suppose the person died in the kth time interval $(k-1, k]$ which means T_0 is in the time interval $(x + k - 1, x + k]$, given $T_0 > x$. Then $K_x = k - 1$.

23) K_x is a discrete random variable where $k = 0, 1, 2, ...$. The probability (mass) function of K_x is

$${}_{k|}q_x = p_{K_x}(k) = P(K_x = k) = P(k < T_x \le k+1) = P(x + k < T_0 \le x + k + 1 | T_0 > x) =$$

$${}_kp_x - {}_{k+1}p_x = F_x(k+1) - F_x(k) = S_x(k) - S_x(k+1).$$

24) Hence

$${}_{k|}q_x = P(K_x = k) = {}_kp_x - {}_{k+1}p_x.$$

25) Since K_x is only defined for $T_0 > x$, $E(K_x)$ is better notation than $E(K_x | T_0 > x)$.

26) The *curtate expectation of life* at age x is

$$e_x = E(K_x) = \sum_{k=0}^{\infty} k \; {}_{k|}q_x.$$

27) $E(K_x^2) = \sum_{k=0}^{\infty} k^2 \; {}_{k|}q_x$, and $V(K_x) = E(K_x^2) - [E(K_x)]^2$.

28) Suppose the interval of failure for T_x is $(k, k+1]$ (so T_0 fails in interval $(x + k, x + k + 1]$). Then $K_x = k$. Let R_x be the continuous RV that represents the fractional part the time lived during the interval of failure of T_x. If failure occurs at time $R_x = r$, then $T_x = k + r$ where $0 < r \le 1$, the support of R_x. In general, $T_x = K_x + R_x$.