Exam 2 is Th. March 7. **You are allowed 9 sheets of notes and a calculator.**

41) An important fact about simple interest is that for simple interest $A(t) = K[1+it]$, the amount of interest earned each year is constant = Ki. Only the original principal K earns interest from year to year, and interest accumulated in any given year does not earn interest in future years. Usually simple interest is only applied for time periods of less than a year.

42) Suppose the annuity pays X instead of \$1 for n years. Then for an annuity-immediate, $AV = Xs_{\overline{n}|}$, $PV = Xa_{\overline{n}|}$. For the annuity-due, $AV = X\ddot{s}_{\overline{n}|}$, $PV = X\ddot{a}_{\overline{n}|}$.

Know everything on the Exam 1 review. Below is new material.

43) Continuous annuities have a δ in the denominator. Hence for a continuous annuity of \$1 "paid continuously in 1 year," the $AV = \bar{s}_{\overline{n}|} = \dfrac{(1+i)^n - 1}{\delta} = \dfrac{i}{\delta} s_{\overline{n}|}$ and $PV = \bar{a}_{\overline{n}|} = \dfrac{1-v^n}{\delta} = \dfrac{1-e^{-n\delta}}{\delta} = \dfrac{i}{\delta} a_{\overline{n}|}$. Multiply the AV and PV by J for a continuous annuity of \$$J$ "paid continuously in 1 year." Use $J = Km$ to approximate $Kms^{(m)}_{\overline{n}|}$ and $Kma^{(m)}_{\overline{n}|}$.

44) Formulas for annuities assume that the interest period and time period are the same. So a monthly time period needs a monthly interest rate j.

45) $\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|} = 1 + a_{\overline{n-1}|}$.

46) $\ddot{s}_{\overline{n}|} = (1+i)s_{\overline{n}|} = s_{\overline{n+1}|} - 1$.

47) Consider n equally spaced payments of K. If the AV is evaluated immediately after the nth payment or if the PV is evaluated one time period before the first payment, then the annuity is an annuity-immediate. If the AV is evaluated one period after the last payment or if the PV is evaluated at the time of the first payment, then the annuity is an annuity-due. Hence the labels on the time diagram of an annuity can be c, c+1, ..., c+n where c is an arbitrary integer. Look for whether the AV is evaluated on the date of the last deposit (annuity-immediate) or one time period after the last deposit (annuity-due).

48) An m-year deferred n-payment annuity-immediate = an $(m+1)$-year deferred n-payment annuity-due is an annuity with n payments K where the PV is valued $m+1$ time periods before the first payment. The time period "year" can be replaced by other time periods. If $K = 1$, the $PV = {}_{m}|a_{\overline{n}|} = {}_{(m+1)}|\ddot{a}_{\overline{n}|} = v^m a_{\overline{n}|} = v^{m+1}\ddot{a}_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}$. Make a time diagram with labels $0, 1, ..., m, m+1, ..., m+n$ with a 1 over the times $m+1, ..., m+n$. Then the first payment is deferred until time $m+1$, the PV evaluated at time m is $a_{\overline{n}|}$, the PV evaluated at time $m+1$ is $\ddot{a}_{\overline{n}|}$. Hence the PV evaluated at time 0 is $v^m a_{\overline{n}|} = v^{m+1}\ddot{a}_{\overline{n}|}$.

49) ${}_{m}|a_{\overline{n}|}$, ${}_{m}|\ddot{a}_{\overline{n}|}$, etc., is PV notation. Suppose the PV is valued at time 0. Then go to time m and pay what the symbol to the right of the vertical line says (for ${}_{m}|a_{\overline{n}|}$, pay an annuity-immediate for n years with 1s at times $m+1, ..., m+n$)). If the PV is worth J at time m, then the PV is $v^m J$ at time 0.

50) $s_{\overline{n}|} = (1+i)^n a_{\overline{n}|}$ since $AV(n) = PV a(n) = PV(1+i)^n$. $a_{\overline{n}|} = v^n s_{\overline{n}|}$ since $v^n = (1+i)^{-n}$.

51) An infinite period annuity that results as $n \to \infty$ is a perpetuity. A perpetuity-immediate values the PV 1 period before the first payment. If the payments are 1, then $PV = a_{\overline{\infty}|} = a_{\overline{\infty}|i} = 1/i = \lim_{n\to\infty} a_{\overline{n}|}$. A perpetuity-due values the PV at the time of

the first payment with $PV = \ddot{a}_{\overline{\infty}|} = \ddot{a}_{\overline{\infty}|i} = 1/d = \lim_{n\to\infty} \ddot{a}_{\overline{n}|} = (1+i)a_{\overline{\infty}|i} = 1 + a_{\overline{\infty}|i}$ if payments are 1. The AV of a perpetuity does not exist.

52) Another interpretation of a perpetuity is that a deposit X will generate $K = Xi$ interest at the end of the year. So if the interest payment is paid to the holder of the annuity at the end of the year, then X is left in the account and will generate $K = Xi$ indefinitely. So $X = K/i$ and if $X = 1000000$ and $i = 7\%$, the annuity will pay $K = 70000$ and $PV = 1000000 = 70000a_{\overline{\infty}|0.07} = 70000/i$.

53) An $m$thly payable annuity-immediate makes payments of $J = K/m$ at the end of every $1/m$ year for n years. If $K = 1$, the AV at the time of the last payment is $AV = s^{(m)}_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{i^{(m)}} = s_{\overline{n}|i}\ \dfrac{i}{i^{(m)}}$. If $K = 1$, the PV one time period ($(1/m)$th year) before the first payment is $PV = a^{(m)}_{\overline{n}|i} = \dfrac{1 - v_i^n}{i^{(m)}} = a_{\overline{n}|i}\ \dfrac{i}{i^{(m)}}$. Here i is the effective annual interest rate, and there are a total of nm payments. $AV(n) = Jms^{(m)}_{\overline{n}|i}$. $PV = Jma^{(m)}_{\overline{n}|i}$.

54) An $m$thly payable annuity-due is similar to 53) except the AV is evaluated one time period ($(1/m)$th year) after the last payment and the PV is evaluated at the time of the first payment. If $K = 1$, $AV = \ddot{s}^{(m)}_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{d^{(m)}} = \ddot{s}_{\overline{n}|i}\ \dfrac{d}{d^{(m)}} = s_{\overline{n}|i}\ \dfrac{i}{d^{(m)}}$, while $PV = \ddot{a}^{(m)}_{\overline{n}|i} = \dfrac{1 - v_i^n}{d^{(m)}} = \ddot{a}_{\overline{n}|i}\ \dfrac{d}{d^{(m)}} = a_{\overline{n}|i}\ \dfrac{i}{d^{(m)}}$.

55) The AV of n payments of 1 k periods after the nth deposit is $AV(n+k) = s_{\overline{n}|i}(1+i)^k = s_{\overline{n+k}|i} - s_{\overline{k}|i}$ since $AV(n) = s_{\overline{n}|i}$ and $AV(n+k) = AV(n)(1+i)^k$.

56) An $n+k$ annuity immediate has AV immediately after the final payment of $s_{\overline{n+k}|}$. The first n payments are worth $s_{\overline{n}|}$ at time n and accumulate to $s_{\overline{n}|}(1+i)^k$ at time $n+k$ while the final k payments accumulate to $s_{\overline{k}|}$ at time $n+k$. Similarly, the first k payments are worth $s_{\overline{k}|}$ at time k and accumulate to $s_{\overline{k}|}(1+i)^n$ at time $n+k$ while the final n payments accumulate to $s_{\overline{n}|}$ at time $n+k$. Hence $s_{\overline{n+k}|} = s_{\overline{k}|}(1+i)^n + s_{\overline{n}|} = s_{\overline{n}|}(1+i)^k + s_{\overline{k}|}$.

57) Consider an $n+k$ payment annuity-immediate with equally spaced payments of 1 per period with interest rate i_1 per payment period up to the time of the nth payment followed by a rate of i_2 per payment period from the nth period on. The AV immediately after the last payment is $AV(n+k) = s_{\overline{n}|i_1}(1+i_2)^k + s_{\overline{k}|i_2}$ since the first n payments are worth $s_{\overline{n}|i_1}$ at time n and accumulate to $s_{\overline{n}|i_1}(1+i_2)^k$ at time $n+k$ while the final k payments are worth $s_{\overline{k}|i_2}$ at time $n+k$. The PV one period before the first payment is $PV(0) = a_{\overline{n}|i_1} + v_{i_1}^n a_{\overline{k}|i_2}$ since the first n payments have $PV(0) = a_{\overline{n}|i_1}$ while the last k payments have $PV(n) = a_{\overline{k}|i_2}$ which has a PV at time 0 of $v_{i_1}^n a_{\overline{k}|i_2}$.

58) Suppose there are $n+k$ regularly spaced payment periods with payments of J for the first n periods and payments of L for the last k periods. The AV right after the final payment is $AV = J\ s_{\overline{n}|}\ (1+i)^k\ +\ L\ s_{\overline{k}|}$ since the first n payments have $AV(n) = J\ s_{\overline{n}|}$ which accumulates to $J\ s_{\overline{n}|}\ (1+i)^k$ at time $n+k$ and the last k payments have $AV =\ L\ s_{\overline{k}|}$ at time $n+k$. If $L > J$, an equivalent annuity makes $n+k$ payments of J and k payments of $L - J$ at times $n+1, ..., n+k$. Hence $AV = Js_{\overline{n+k}|}\ +\ (L-J)\ s_{\overline{k}|}$.

59) $\dfrac{a_{\overline{Jn}|}}{a_{\overline{n}|}} = 1 + v^n + v^{2n} + \cdots + v^{(J-1)n}$.

60) $\dfrac{a_{\overline{n}|}}{a_{\overline{p}|}} = \dfrac{\ddot{a}_{\overline{n}|}}{\ddot{a}_{\overline{p}|}} = \dfrac{a^{(m)}_{\overline{n}|}}{a^{(m)}_{\overline{p}|}} = \dfrac{\ddot{a}^{(m)}_{\overline{n}|}}{\ddot{a}^{(m)}_{\overline{p}|}} = \dfrac{\overline{a}_{\overline{n}|}}{\overline{a}_{\overline{p}|}}$

61) Formulas for $a_{\overline{n}|}$ and $s_{\overline{n}|}$ tend to hold if bars are added: so tend to hold for $\overline{a}_{\overline{n}|}$ and $\overline{s}_{\overline{n}|}$. See 50), 55) and 56).

62) Consider an annuity-immediate with n payments of J. Let $AV(n) = M = J\left[\dfrac{(1+i)^n - 1}{i}\right] = Js_{\overline{n}|}$. Then M, J, n, and i are variables. Given 3 of the variables, it is possible to solve for the fourth. Then $n = \dfrac{\ln(1 + \frac{Mi}{J})}{\ln(1+i)}$.

Similarly $PV = PV(0) = L = J\left[\dfrac{1 - v^n}{i}\right] = Ja_{\overline{n}|}$, and given 3 of the variables, it is possible to solve for the fourth. Then $n = \dfrac{\ln(1 - \frac{Li}{J})}{\ln(v)}$.

Typically n will not be an integer, for example $n = 14.207$. Let $m = [n]$ be the integer part of n so $m = [14.207] = 14$. Then make $m = [n]$ payments of J and then an additional payment of X to complete the annuity. If X is made at time $[n]$, then the final payment of $X + J$ is known as a balloon payment. Then $AV = M = Js_{\overline{m}|} + X$ and $PV = L = Ja_{\overline{m}|} + Xv^m$. Make a time diagram to see these formulas.

An alternative is to make payment X one time period after the time period of the final payment of J, so at time $m + 1$. Then from a time diagram, $PV = L = Ja_{\overline{m}|} + Xv^{m+1}$. Often use PV since the PV is a loan made at time 0 that is paid off with the annuity.

63) Referring to 62), solving for an unknown interest rate usually needs a financial calculator, or tedious trial and error for a multiple choice problem. On a BA II Plus for an annuity immediate, if the PV = 15400 on 18 equally spaced payments of 1300, enter 18 then press N, enter 15400 then press +/−, then press PV, then enter 1300, then press PMT, then press CPT then press I/Y. (See E1 review above 7 for why the PV is entered as a negative number.) Should get 4.8303 which is in percent, so the interest is 0.048303.

64) An $m$thly perpetuity-immediate uses $a^{(m)}_{\overline{\infty}|} = \dfrac{1}{i^{(m)}} = \lim_{n\to\infty} a^{(m)}_{\overline{n}|}$.

An $m$thly perpetuity-due uses $\ddot{a}^{(m)}_{\overline{\infty}|} = \dfrac{1}{d^{(m)}} = \lim_{n\to\infty} \ddot{a}^{(m)}_{\overline{n}|}$.

65) $s^{(m)}_{\overline{1}|} = i/i^{(m)}$ and $a^{(m)}_{\overline{n}|i} = a_{\overline{n}|i} \; \dfrac{i}{i^{(m)}} = \dfrac{1 - v_i^n}{i^{(m)}}$.

66) Consider an $m$thly annuity-immediate that pays $J = K/m$ every $1/m$th year for n years. The coefficient on $a^{(m)}_{\overline{n}|}$ is the sum of the payments for each interest period (so 12 J if the period is a year with 12 monthly payments). So the PV is $mJa^{(m)}_{\overline{n}|}$. Similarly the PV of an $m$thly annuity-due that pays $J = K/m$ every $1/m$th year for n years is $PV = mJ\ddot{a}^{(m)}_{\overline{n}|}$. So an annuity that pays 100 at the end of each month for 10 years at an effective annual interest rate of 5% has $PV = 1200a^{(12)}_{\overline{10}|0.05}$ with $J = 100$ and $m = 12$.

67) Suppose an annuity pays 1 at times 1, ..., n, but the accumulation function is $a(t)$ where interest varies. Want AV at time $t = n$. Now 1 at time $t = j$ has $AV = a(n)/a(j)$ at time n. See E1 review 19). Hence the annuity AV immediately after the final payment is $AV = a(n)\left[\dfrac{1}{a(1)} + \dfrac{1}{a(2)} + \cdots + \dfrac{1}{a(n)}\right]$.

68) A common problem uses the $AV_1(n)$ of one fund at time n to buy a second fund that makes k future payments and has a $PV_2(n)$ at time n. Usually set $AV_1(n) = PV_2(n)$ and solve for an unknown. See HW4: 3, 4 and HW5: 4.

69) Consider an n–payment annuity with an arithmetic progression as shown below.

```
      P    P+Q  P+2Q     P+(j-1)Q         P+(n-1)Q
--------------------------------------------------
|     |    |    |     ... |     ...        |
0     1    2    3         j                n
```

So P is the first payment and Q is the common difference. Then the PV one time period before the first payment is $PV = A_A = Pa_{\overline{n}|} + Q\left[\dfrac{a_{\overline{n}|} - nv^n}{i}\right]$, and the AV immediately after the last payment of $AV = S_A = (1+i)^n A_A = Ps_{\overline{n}|} + Q\left[\dfrac{s_{\overline{n}|} - n}{i}\right]$.

70) Consider an n–payment annuity with an arithmetic progression as shown below.

```
P    P+Q  P+2Q              P+(j-1)Q         P+(n-1)Q
--------------------------------------------------
|     |    |    |     ... |     ...        |        |
0     1    2    3         j                n-1      n
```

Then the PV evaluated on the date of the first payment is $PV = \ddot{A}_A = (1+i)A_A = P\ddot{a}_{\overline{n}|} + Q\left[\dfrac{a_{\overline{n}|} - nv^n}{d}\right]$, and the AV one period after the last payment is $AV = \ddot{S}_A = (1+i)^n \ddot{A}_A = P\ddot{s}_{\overline{n}|} + Q\left[\dfrac{s_{\overline{n}|} - n}{d}\right]$.

71) An increasing annuity-immediate has $P = Q = 1$. Then 69) can be shown to have PV one payment before the first payment of $PV = (Ia)_{\overline{n}|} = \dfrac{\ddot{a}_{\overline{n}|} - nv^n}{i}$ and an AV at the time of the final payment of $PV = (Is)_{\overline{n}|} = \dfrac{\ddot{s}_{\overline{n}|} - n}{i} = \dfrac{s_{\overline{n+1}|} - (n+1)}{i}$.

72) An increasing annuity-due replaces i by d in the denominator of the RHS. So the PV at the time of the first payment is $PV = (I\ddot{a})_{\overline{n}|} = \dfrac{\ddot{a}_{\overline{n}|} - nv^n}{d}$ and an AV one time period after the final payment of $PV = (I\ddot{s})_{\overline{n}|} = \dfrac{\ddot{s}_{\overline{n}|} - n}{d}$.

73) A decreasing annuity-immediate has $P = n$ and $Q = -1$, so the n payments are $n, n-1, n-2, ..., 3, 2, 1$. The PV one time period before the first payment is $PV = (Da)_{\overline{n}|} = \dfrac{n - a_{\overline{n}|}}{i}$, and the AV immediately after the final payment is $AV = (Ds)_{\overline{n}|} = \dfrac{n(1+i)^n - s_{\overline{n}|}}{i} = (1+i)^n (Da)_{\overline{n}|}$.

74) The due form has AV and PV 1 period after the immediate form, so $AV_{due} = (1+i)AV_{immediate}$ and $PV_{due} = (1+i)PV_{immediate}$. Also $AV_{immediate} = vAV_{due}$ and $PV_{immediate} = vPV_{due}$. For both forms, $AV = (1+i)^n PV$ and $PV = v^n AV$.

75) An increasing perpetuity-immediate with payments 1, 2, 3, ... at times 1, 2, 3, ... has PV at the time before the first payment of $PV = (Ia)_{\overline{\infty}|} = \dfrac{1}{id} = \dfrac{1}{i} + \dfrac{1}{i^2}$.

76) An increasing perpetuity-due with payments 1, 2, 3, ... at times 0, 1, 2, 3, ... has PV at the time of the first payment of $PV = (I\ddot{a})_{\overline{\infty}|} = \dfrac{1}{d^2} = \lim_{n\to\infty} (I\ddot{a})_{\overline{n}|}$.

77) Although the general P and Q forms 69) and 70) can be used to evaluate an annuity in arithmetic progression, such an annuity can be written as a combination of a decreasing or increasing annuity and a level payment annuity.

```
      P    P+Q  P+2Q     P+(j-1)Q        P+(n-1)Q

 -------------------------------------------------
 |    |    |    |     ... |     ...         |
 0    1    2    3         j                 n
```

78) i) Consider PV and suppose $Q > 0$. Then write down $Q(Ia)_{\overline{n}|}$.

ii) The annuity in i) pays $Q, Q+Q, ..., Q+(n-1)Q$. So add level payments of $P-Q$, so the other term in the combination is $(P-Q)a_{\overline{n}|}$.

Hence the PV of the annuity is $Q(Ia)_{\overline{n}|} + (P-Q)a_{\overline{n}|}$.

79) i) Consider PV and suppose $Q < 0$. Then write down $|Q|(Da)_{\overline{n}|}$.

ii) The annuity in i) pays $n|Q|, (n-1)|Q|, ..., |Q|$. So add level payments of $P - n|Q|$, so the other term in the combination is $(P - n|Q|)a_{\overline{n}|}$.

Hence the PV of the annuity is $|Q|(Da)_{\overline{n}|} + (P - n|Q|)a_{\overline{n}|}$.

80) For AV, replace a by s for the immediate form. So use $s_{\overline{n}|}$ and either $(Is)_{\overline{n}|}$ or $(Ds)_{\overline{n}|}$. So the AV in 78) is $Q(Is)_{\overline{n}|} + (P-Q)s_{\overline{n}|}$ while the
AV in 79) is $|Q|(Ds)_{\overline{n}|} + (P - n|Q|)s_{\overline{n}|}$.

Similar formulas work for the PV and AV of the due form but with $\ddot{a}$ replacing a and $\ddot{s}$ replacing s.

81) Suppose an increasing perpetuity-immediate has first payment P and increases by Q thereafter. Then $PV = \dfrac{P}{i} + \dfrac{Q}{i^2}$.

```
      1    2    3         n      n      n ...

 -------------------------------------------------
 |    |    |    |     ... |      |      |
 0    1    2    3         n      n+1    n+2   ...
```

82) Consider a perpetuity-immediate with payments of 1, 2, ..., n at the end of each year and then payments of n for each subsequent year. Then the PV of the first n payments one time period before the first payment is $(Ia)_{\overline{n}|}$ while the PV of the remaining payments evaluated at time n is $na_{\overline{\infty}|}$ which has PV at time 0 of $v^n n a_{\overline{\infty}|}$. Hence the PV of the perpetuity-immediate one time period before the first payment is
$PV = (Ia)_{\overline{n}|} + v^n \dfrac{n}{i} = \dfrac{\ddot{a}_{\overline{n}|}}{i}$.

83) Suppose a lender makes a loan of L at interest rate i and invests the payments K_t at interest rate i'. If F is the AV for the lender at time n years using interest rate i', then $L(1+j)^n = F$ and $j = \left(\dfrac{F}{L}\right)^{1/n} - 1$ is the annual yield rate for the lender.

84) Consider an annuity that makes a cost of living adjustment (COLA) so that there are n payments, the first payment is 1, and all subsequent payments are $(1+r)$ times the previous payment. This annuity is in geometric progression with payments $1, (1+r), (1+r)^2, ..., (1+r)^{n-1}$ with PV before the first payment of $PV = \dfrac{1 - (\frac{1+r}{1+i})^n}{i-r}$ and an AV immediately after the last payment of $AV = PV(1+i)^n = \dfrac{(1+i)^n - (1+r)^n}{i-r}$.

85) There are many variants of the problem of finding the AV of investments and reinvested interest. Use time diagrams of investments and of reinvested interest. Three examples are a) 83). b) Suppose L is invested and generates interest at rate i per time period which is reinvested at rate j. Then L generates interest Li per period. Hence the AV immediately after the last interest payment consists of L and the reinvested interest with $AV = L + Lis_{\overline{n}|j}$.

```
     Li   Li   Li        Li
 -------------------------------        --------------------------------
 |    |    |    |     ... |             |   |   |    |    | ...     |
 0    1    2    3         n             0   1   2    3    4         n
                                payments    1   1    1    1         1
                                interest        i    2i   3i        (n-1)i
```

c) Suppose n deposits of 1 generate interest at rate i per time period which are reinvested at rate j. Then at time 2 interest i is invested, at time 3 interest on the first two payments of 1 is $2i$ et cetera as shown in the above right time diagram. Then the AV immediately after the nth payment is $AV = n + i(Is)_{\overline{n-1}|j}$ where n is the value of the n payments of 1 and the 2nd term is the AV of the reinvested interest.

86) Amortizing a loan reduces the outstanding balance of a loan by making payments that pay interest and reduce the principal. Let $OB_t = B_t$ be the outstanding balance immediately after the tth payment $K_t = R_t$. Let I_t be the interest paid at the end of period t. Let $PR_t = P_t$ be the principal repayment at the end of period t.

```
     K1   K2...Kt   K(t+1) ... Kn           K   K ...K    K  ...  K
 --------------------------------      --------------------------------
 |    |    | ...|    | ...      |      |   |   | ...|    |  ...  |
 0    1    2    t    t+1        n      0   1   2    t    t+1     n
 L              OBt                    L            OBt
```

The retrospective method says $OB_t = L(1+i)^t - K_1(1+i)^{t-1} - K_2(1+i)^{t-2} - \cdots - K_{t-1}(1+i) - K_t =$ AV of loan L at time t $-$ AV of the 1st t payments immediately after tth payment. If $K_t \equiv K$, then $OB_t = L(1+i)^t - Ks_{\overline{t}|}$.

The prospective method says $OB_t = K_{t+1}v + K_{t+2}v^2 + \cdots + K_n v^{n-t} =$ PV(t) of the remaining payments immediately after the tth payment. If $K_t \equiv K$, then $OB_t = Ka_{\overline{n-t}|}$.

87) $L = K_1 v + K_2 v^2 + \cdots + K_n v^n$. If $K_t \equiv K$, then $L = Ka_{\overline{n}|}$.

88) $I_t = i \;\; OB_{t-1}$, $PR_t = K_t - I_t$, $OB_t = OB_{t-1} - PR_t$, $OB_0 = L$. If $K_t \equiv K$, then $K = L/a_{\overline{n}|}$, $I_t = K(1 - v^{n-t+1})$, $PR_t = Kv^{n-t+1} = PR_{t-1}(1+i) = PR_1(1+i)^{t-1}$.

89) The retrospective method can be better if L is given but you need to figure out n and the last payment K_n. The prospective method can be better if L is not known.

duration t	payment K_t	interest paid $I_t = (i)OB_{t-1}$	principal repaid $PR_t = K_t - I_t$	outstanding balance $OB_t = OB_{t-1} - PR_t$
0	$-$	$-$	$-$	$OB_0 = L$
1	K_1	$I_1 = (i)OB_0$	$PR_1 = K_1 - I_1$	$OB_1 = OB_0 - PR_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	K_t	$I_t = (i)OB_{t-1}$	$PR_t = K_t - I_t$	$OB_t = OB_{t-1} - PR_t$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	K_n	$I_n = (i)OB_{n-1}$	$PR_n = K_n - I_n$	$OB_n = OB_{n-1} - PR_n = 0$
total	$\sum_{j=1}^n K_j$	$\sum_{j=1}^n I_j$	$\sum_{j=1}^n PR_j = L$	

90) the above table shows an amortization schedule. Note that the sum of the principal repayments is equal to the loan amount L. So $\sum_{j=1}^n PR_j = L$. Also the outstanding balance at time n is 0, and $L + \sum_{j=1}^n I_j = \sum_{j=1}^n (I_j + PR_j) = \sum_{j=1}^n K_j$. Also $OB_t = OB_{t-1}(1+i) - K_t = OB_{t-1} + I_t - K_t = OB_{t-1} - PR_t$.

91) If payments $K_t \equiv K$, then the sum of payments is nK, and the sum of principal repayments is $\sum_{j=1}^n PR_j = K a_{\overline{n}|} = L$. Also $\sum_{j=1}^n I_j = nK - L$.

92) A fund designed to accumulate a specified amount of money in a specified amount of time by making regular deposits is a sinking fund (SF). There is
a) interest rate on the loan i
b) interest rate on the sinking fund j
c) a periodic interest payment $I_t = Li$
d) a periodic sinking fund deposit (SFD) such that $L = SFD s_{\overline{n}|j}$ so $SFD = L/s_{\overline{n}|j}$. Hence the SF accumulates to L at time $t = n$.
e) Note that the total periodic payment is $J = Li + SFD$, the sum of the quantities found in c) and d).

93) You may need to find the quantities in 92) as well as the payment K for an amortized loan that would pay off the loan L in 92). Then $K = L/a_{\overline{n}|i}$. See HW7 4.

94) If $i = j$ then the sinking fund approach and the amortization approach are equivalent in that the periodic payments J and K in 92) and 93) satisfy $J = K$, but often $i > j$ and the lender makes more money under the sinking fund approach. Technically the borrower owns the sinking fund until time n, but often the sinking fund is held by a financial institute like a bank. Typically the interest the borrower can make on the SF satisfies $j < i$.

95) The result in 94) implies that $\dfrac{1}{a_{\overline{n}|i}} = i + \dfrac{1}{s_{\overline{n}|i}}$.

96) The general approach to SF problems is to answer two questions:
i) what interest is paid to the lender (find Li), and
ii) what is the SFD ($= L/s_{\overline{n}|j}$)?
Then the total periodic payment made by the the borrower to repay the loan is $J = Li + \dfrac{L}{s_{\overline{n}|j}} = L\left[i + \dfrac{1}{s_{\overline{n}|j}}\right]$.

97) SF with nonlevel SFDs: suppose the sinking fund has interest rates i and j, the total annual payment is J the 1st n years and K the next m years. The SF accumulates to L at time $n+m$, so $(J - Li)s_{\overline{n+m}|j} + (K - J - Li)s_{\overline{m}|j} = L$. Note that the interest payment is Li and the SFD is $J - Li$ for years 1, ..., n and $K - Li$ for years $n+1, ..., n+m$.

```
                                                              K-J-Li... K-J-Li
     J-Li     J-Li  K-Li ...  K- Li        J-Li ...           J-Li...   J-Li
--------------------------------     --------------------------------
|   |    ... |      |         |      |    |    ... |      |         |
0   1        n      n+1  ...  n+m    0    1        n      n+1  ...  n+m
```

Ch. 4. 98) A bond makes periodic interest payments Fr for periods 1, ..., n as well as a payment C at time n where C is the redemption value of the bond and r is the coupon rate of interest per time period. F is the face value of the bond, and $i = j$ is the yield rate of interest per time period. Note that n is the number of payments.

```
     Fr   Fr     ... Fr    Fr+C
-----------------------------
|   |    |      ... |     |
0   1    2          n-1   n
```

99) The bond price P is the PV of future cash flows. The coupon rate r is fixed for the life of the bond. Bonds are sold in a bond market and the yield rate j is determined by market forces. Typically interest payments are made twice a year so $n = 2Y$ where Y is the number of years of the bond, but other payment periods are possible. When the payment period is 6 months, often two bond interest rates $r^{(m)}$ and $j^{(m)}$are given as nominal annual interest rates compounded semiannually where $m = 2$. Then $r = r^{(2)}/2$ and $j = j^{(2)}/2$. In general you may need to convert a given interest rate to the interest rate for the time period. Note that $\left(1 + \frac{i^{(m_1)}}{m_1}\right)^{m_1} = \left(1 + \frac{i^{(m_2)}}{m_2}\right)^{m_2} = 1 + i$ where i is the annual interest rate. Hence if $m_1 < m_2$ and m_2/m_1 is a positive integer, then $\left(1 + \frac{i^{(m_1)}}{m_1}\right) = \left(1 + \frac{i^{(m_2)}}{m_2}\right)^{m_2/m_1}$.

100) Let g be the coupon rate applied to C to determine the amount of the coupon = periodic interest payment. Then $Cg = Fr$ and $g = Fr/C$. The basic formula for the bond price is $P = Fra_{\overline{n}|j} + Cv_j^n = Cga_{\overline{n}|j} + Cv_j^n$. The premium discount formula is $P = C + (Fr - Cj)a_{\overline{n}|j} = C + (Cg - Cj)a_{\overline{n}|j}$. Let $K = Cv_j^n$, then Makeham's formula is $P = \frac{g}{j}(C - Cv_j^n) + Cv_j^n = \frac{g}{j}(C - K) + K = K + \frac{g}{j}(C - K)$. See HW7 5.

101) If $F = C$ then $r = g$, and the bond is redeemable at par. If $F = C$ and $r = j$, then P = F. If $g = j$, then $P = C$. If $g > j$ then $P > C$, and the bond is redeemable at a premium with the premium $= P - C = (Cg - Cj)a_{\overline{n}|j}$. If $g < j$ then $P < C$ and the bond is redeemable at a discount with the discount $= C - P = -(Cg - Cj)a_{\overline{n}|j} = (Cj - Cg)a_{\overline{n}|j}$. See the premium discount formula in 100). Note that both the premium and the discount are positive. To determine whether the bond is redeemable at a premium or discount, compute g and see whether $g > j$ or $g < j$. If the discount is $D = C - P$ then the price is $P = C - D$. If the premium is $E = P - C$, then the price is $P = E + C$.

102) If the bond is redeemable at par then $F = C$. If no information is given, then assume $F = C$ and that the interest payment period is 6 months. Sometimes you are told that the bond is redeemable at 950 or redeemable at 95% of the face value (so $C = 0.95F$).