Math 250 Exam 4 review. Thursday April 30. Bring a TI–30 calculator but NO NOTES. Emphasis on sections 5.5, 6.1, 6.2, 6.3, 3.7, 6.6, 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7, 8.8, 9.1, 9.2, 9.3, 9.4, 9.5, 7.4; HW1-24; Q1-23. Know for trig functions that 0.707 $\approx \sqrt{2}/2$ and 0.866 $\approx \sqrt{3}/2$.

From Math 150, for derivatives know power rule, product rule, quotient rule, chain rule and rules from reference p. 5. For integration know power rule, u-substitution and rules 1-20 from reference p. 6.

Know everything from Exam 1, 2 and 3 reviews, including F1)-F24).

The following problems are **very important for exam 4 and the final**. The notation F*** means it was on 3 out of 3 of the last 3 finals.

F25*)** Given parametric equations $x = x(t)$ and $y = y(t)$, find the equation of the tangent line to the curve at point where (or when) $t = t_0$. Let $(x_0, y_0) = (x(t_0), y(t_0))$. Then find $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ as a function of t. Then $m = \left.\dfrac{dy}{dx}\right|_{t=t_0}$ and $y - y_0 = m(x - x_0)$ is the point slope equation of the line. Now $y = mx + b$ is the point intercept equation of the line. Note that $y = mx + (y_0 - mx_0)$. Also $y_0 = mx_0 + b$, so $b = y_0 - mx_0$. Note that m **is a real number**, and is free of t. See F07 11, F08 9, S08 10.

F26*) As a variant to F25) also eliminate t to find a Cartesian (x, y) equation of the curve. Then identify the curve. See F08 9cd.

Tips: i) $(x-h)^2 + (y-k)^2 = r^2$ is a circle centered at (h, k) with radius r, and often $r = 1$.

ii) $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ is the standard ellipse (with $(h, k) = (0, 0)$) shifted to have center (h, k).

iii) $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ is the standard hyperbola (with $(h, k) = (0, 0)$) shifted to have "center or point of symmetry" (h, k). This hyperbola intersects the x axis. Note that $x^2 - y^2 = 1$ if $x = \sec t$ and $y = \tan t$.

iv) $\dfrac{(y-h)^2}{a^2} - \dfrac{(x-k)^2}{b^2} = 1$ is the standard hyperbola (with $(h, k) = (0, 0)$) shifted to have "center or point of symmetry" (h, k). This hyperbola intersects the y axis.

v) If $x = h + \cos t$ and $y = k + \sin t$, then $(x-h)^2 + (y-k)^2 = 1 \quad (= \cos^2 t + \sin^2 t)$ is a circle.

vi) If $x = h + \sin t$ and $y = k + \cos t$, then $(x-h)^2 + (y-k)^2 = 1 \quad (= \sin^2 t + \cos^2 t)$ is a circle.

vii) If $x = h + r \sin t$ and $y = k + r \cos t$, or if $x = h + r \cos t$ and $y = k + r \sin t$, then $(x-h)^2 + (y-k)^2 = r^2 \quad [= r^2(\sin^2 t + \cos^2 t)$ or $r^2(\cos^2 t + \sin^2 t)]$ is a circle.

vii) If $x = x(t)$ and $y = g(x(t))$ then $y = g(x)$. If $x = g(y(t))$ and $y = y(t)$ then $x = g(y)$. For example, if $x = \sin t$ and $y = \sin^2 t$, then $y = x^2$ is a parabola.

viii) If $x = x(t)$ can be solved for $t = f(x)$, then $y = y(t) = g(t) = g(f(x))$. If $y = y(t)$ can be solved for $t = g(y)$, then $x = f(t) = f(g(y))$.

Eg, if $x(t) = x = mt + b$ then $t = (x - b)/m$, and $y = y(t) = g(t) = g((x-b)/m)$. If $x(t) = x = \sqrt{t}$, then $x^2 = t$ and $y = g(t) = g(x^2)$ for $t \ge 0$. If $x(t) = x = t^2$, then $\sqrt{x} = t$ and $y = g(\sqrt{x})$ for $x \ge 0$. If $x = e^{2t}$, then $2t = \ln(x)$ so $t = \ln(x)/2$.

In Section 6.3 (partial fractions), **completing the square** is an important technique. Recall that $\int \frac{dx}{x^2+d^2} = \frac{1}{d}\tan^{-1}\left(\frac{x}{d}\right) + C$. Now $x^2+bx+c = \left(x+\frac{b}{2}\right)^2 - \frac{b^2}{4} + c = x^2+bx+\frac{b^2}{4}-\frac{b^2}{4}+c$ is known as "completing the square."

Note that $ax^2+Bx+D = a(x^2+bx+c)$ where $ab = B$ and $ac = D$. Suppose $d^2 = c - b^2/4 > 0$. So $\int \frac{k\ dx}{ax^2+Bx+D} = \frac{k}{a}\int \frac{dx}{x^2+bx+c} = \frac{k}{a}\int \frac{dx}{(x+\frac{b}{2})^2+d^2} = \frac{k}{a}\int \frac{du}{u^2+d^2} = \frac{k}{ad}\tan^{-1}\left(\frac{x+b/2}{d}\right) + C$ where $u = x+b/2$ and $du = dx$.

F27) Use *completing the square* to find $\int \frac{k\ dx}{ax^2+Bx+D} = \frac{k}{a}\int \frac{dx}{x^2+bx+c}$ where a, b, c and k are constants and $d^2 = c - b^2/4a > 0$.

Cartesian coordinates and equations use (x, y) while **polar coordinates** use (r, θ) where $x = r\cos(\theta)$, $y = r\sin(\theta)$, $r^2 = x^2+y^2$ and $\tan(\theta) = y/x$. So $\cos(\theta) = x/r$ and $\sin(\theta) = y/r$.

A problem is that for $\theta \in [0, 2\pi)$, each value of $\tan(\theta)$ occurs twice: $\tan(\theta-\pi) = \tan(\theta) = \tan(\theta+\pi)$, so θ must be chosen so that (r, θ) lies in the same quadrant as (x, y).

Another problem is that $(r, \theta) = (r, \theta+2n\pi)$ for all integers n, and $(-r, \theta) = (r, \theta+\pi)$. So if $r > 0$ the point (r, θ) is plotted, but if $r < 0$, then plot $(|r|, \theta+\pi)$.

In graphing polar functions such as $r = f(\theta)$, take $r > 0$, so that if $r(\theta_0) = f(\theta_0) < 0$, then plot the point $(|r(\theta_0)|, \theta_0+\pi)$. If $r_0 > 0$, then plot the point $(r(\theta_0), \theta_0)$.

A point in polar coordinates is a ray with angle θ and length $|r|$. An angle measured counterclockwise has positive sign while an angle measured clockwise has negative sign. So $\theta = \pi/4$ and $\theta = -\pi/4$ make sense.

Sketching $r = f(\theta)$ or the more general polar equation $G(r, \theta) = 0$:

i) Sketch one or more concentric circles of radius r_0 (if you has a ruler, you might take one inch = 1 radian if r went from 0 to 2 or one inch = π if r went from 0 to 2π).

ii) Compute $r = f(\theta)$ for $\theta = 0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4$, and 2π. (The horizontal axis corresponds to $\theta = 0$ at the right and $\theta = \pi$ on the left. The vertical axis corresponds to $\theta = \pi/2$ on the top and $\theta = 3\pi/2$ on the bottom. Sketch the two diagonal axes corresponding to $\theta = \pi/4$ at the upper right and to $\theta = 3\pi/4$ at the upper left.)

iii) If $r(\theta_0) = f(\theta_0) < 0$, plot the point $(|r(\theta_0)|, \theta_0+\pi)$.

iv) Sketch $r = f(\theta)$ in Cartesian coordinates so you can read off r from the graph.

Recognize the following polar equations: i) $r = r_0$ is a circle. ii) $\theta = \theta_0$ is a line. Let $d > 0$. Often $d = 1$. iii) $r = d\cos(\theta)$ is a circle. iv) $r = d\sin(\theta)$ is a circle.
v) $r = d\cos(2\theta)$ is a 4 leaved rose. vi) $r = d\sin(2\theta)$ is 4 leaved rose.
vii) $r = d\cos(3\theta)$ is a 3 leaved rose. viii) $r = d\sin(3\theta)$ is 3 leaved rose.
ix) $r = d\sin(k\theta)$ is a $2k$ leaved rose when k is even and a k leaved rose when $k > 1$ is odd. x) $r = d\cos(k\theta)$ is a $2k$ leaved rose when k is even and a k leaved rose when $k > 1$ is odd.
xi) $r = d[1-\cos(\theta)]$, $r = d[1+\cos(\theta)]$, $r = d[1-\sin(\theta)]$ and $r = d[1+\sin(\theta)]$ are cardioids.

F28)** Let $\mathcal{R}$ be the region bounded by the polar curve $r = f(\theta) > 0$ and rays $\theta = a$ and $\theta = b$ where f is continuous and $0 < b - a < 2\pi$. Then the area of region $\mathcal{R}$ is $A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta \quad (= \int_a^b \frac{1}{2}r^2 d\theta)$. Often you need to find a and b. F07 13, S08 13

Tips: Suppose $r = d\sin(k\theta)$ or $r = d\cos(k\theta)$ and you find the area of a circle ($k = 1$), of one loop of a k leaved rose ($k > 1$ odd) or of one loop of a $2k$ leaved rose ($k \geq 2$ even).

Sketch $r = d\sin(k\theta)$ in Cartesian coordinates to see that r increases from 0 to d as θ increases from 0 to $\pi/(2k)$, and r decreases from d to 0 as θ increases from $\pi/(2k)$ to π/k. Hence $a = 0$ and $b = \pi/k$ and the area of one loop (the entire circle for $k = 1$) is $A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta = \int_0^{\pi/k} \frac{1}{2}[d\sin(k\theta)]^2 d\theta = \frac{d^2}{2}\int_0^{\pi/k} \sin^2(k\theta)d\theta = \frac{d^2}{2}\int_0^{\pi/k}[\frac{1}{2} - \frac{1}{2}\cos(2k\theta)]d\theta$ $= \frac{d^2}{4}\theta\Big|_0^{\pi/k} - \frac{d^2}{8k}\sin(2k\theta)\Big|_0^{\pi/k} = \frac{d^2\pi}{4k}$.

Sketch $r = d\cos(k\theta)$ in Cartesian coordinates to see that r increases from 0 to d as θ increases from $-\pi/(2k)$ to 0, and r decreases from d to 0 as θ increases from 0 to $\pi/(2k)$. Hence $a = -\pi/(2k)$ and $b = \pi/(2k)$ and the area of one loop (the entire circle for $k = 1$) is $A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta = \int_{-\pi/(2k)}^{\pi/(2k)} \frac{1}{2}[f(\theta)]^2 d\theta = 2\frac{1}{2}\int_0^{\pi/(2k)}[f(\theta)]^2 d\theta = \int_0^{\pi/(2k)} d^2\cos^2(k\theta)d\theta = d^2\int_0^{\pi/(2k)}[\frac{1}{2} + \frac{1}{2}\cos(2k\theta)]d\theta = \frac{d^2}{2}\theta\Big|_0^{\pi/(2k)} + \frac{d^2}{4k}\sin(2k\theta)\Big|_0^{\pi/(2k)} = \frac{d^2\pi}{4k}$. **Know the technique, not just the final answer.**

F29)** Find the area A between $r = f_1(\theta)$ and $r = f_2(\theta)$ where the curves intersect at $\theta = a$ and $\theta = b$ and $f_1(\theta) \leq f_2(\theta)$. Then $A = \int_a^b \frac{1}{2}([f_2(\theta)]^2 - [f_1(\theta)]^2)d\theta$. **Variant**: Set up the integral by finding a, b, f_1 and f_2, but do not compute the integral. F08 12

Tips: Find a and b by solving $f_1(\theta) = f_2(\theta)$ for θ. Typically one curve is a circle and the other is a cardioid and a sketch is given. From the sketch determine which curve is f_2 and which is f_1. The area will not be negative, so if you get a negative area then swap f_1 and f_2.

Arc length is like the perimeter or circumference.

F30*)** Find the arc length of curves. F07 12, S08 14, F08 11.

i) The arc length of $r = f(\theta)$ for $a < \theta < b$ is $L = \int_a^b \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\ d\theta$ $(= \int_a^b \sqrt{[r^2 + (\frac{dr}{d\theta})^2}\ d\theta\ \ )$.

ii) The arc length of the curve C given by parametric equations $x(t) = f(t)$ and $y(t) = g(t)$ for $a < t < b$ is $L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\ dt \quad (= \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt\ \ )$.

iii) The arc length of a curve $f(x)$ on $[a, b]$ is $L = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx$ where $f'(x)$ is the derivative of f.

iv) The arc length of a curve $g(y)$ on [c,d] is $L = \int_c^d \sqrt{1 + [g'(y)]^2}\ dy$ where $g'(y)$ is the derivative of g.