Math 250 Exam 3 review. Thursday April 9. Bring a TI–30 calculator but NO NOTES. Emphasis on sections 5.5, 6.1, 6.2, 6.3, 3.7, 6.6, 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7, 8.8; HW1-19; Q1-18. Know for trig functions that $0.707 \approx \sqrt{2}/2$ and $0.866 \approx \sqrt{3}/2$.

From Math 150, for derivatives know power rule, product rule, quotient rule, chain rule and rules from reference p. 5. For integration know power rule, u-substitution and rules 1-20 from reference p. 6.

Know everything from Exam 1 and 2 reviews, including F1)-F17).

The following problems are **very important for exam 3 and the final**. The notation F*** means it was on 3 out of 3 of the last 3 finals.

F18*) Use ratio test and $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ for a series $\sum_{n=1}^{\infty} c^n \frac{n^n}{n!}$ or series $\sum_{n=1}^{\infty} c^n \frac{n!}{n^n}$. See F08 4c.

Ex. $\sum_{n=1}^{\infty} \frac{3^n\ n!}{n^n}$ has $\lim_{n\to\infty}\left|\frac{3^{n+1}(n+1)!}{(n+1)^{n+1}}\ \frac{n^n}{3^n\ n!}\right| = \lim_{n\to\infty}\left|\frac{3^n\ (3)\ (n+1)\ [n!]}{(n+1)\ (n+1)^n}\ \frac{n^n}{3^n\ n!}\right| =$ $3\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = 3\lim_{n\to\infty}\frac{1}{\left(\frac{n+1}{n}\right)^n} = 3\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = 3/e > 1$. So the series diverges by the ratio test.

Ex. $\sum_{n=1}^{\infty} \frac{n^n}{3^n\ n!}$ has $\lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{3^{n+1}(n+1)!}\ \frac{3^n\ n!}{n^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\ (n+1)^n}{3^n(3)(n+1)[n!]}\ \frac{3^n\ n!}{n^n}\right| =$ $\frac{1}{3}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \frac{1}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \frac{e}{3} < 1$. So the series converges by the ratio test.

F19)** Use root test and L'Hospital's rule for a series $\sum_{n=2}^{\infty}\left[\frac{\ln(n^2+1)}{c\ln(n)}\right]^n$. See F07 6b, S08 7b.

Ex. $\sum_{n=2}^{\infty}\left[\frac{\ln(n^2+1)}{3\ln(n)}\right]^n$ has $L = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{\ln(n^2+1)}{3\ln(n)} = \lim_{n\to\infty}\frac{\frac{1}{n^2+1}\ 2n}{3\ \frac{1}{n}}$ where LHOP was used on a limit of the form ∞/∞. So $L = \frac{2}{3}\lim_{n\to\infty}\frac{n^2}{n^2+1} = 2/3 < 1$, and the series converges by the root test.

F20) Use root test and $\lim_{n\to\infty} n^{1/n} = 1$ for a series $\sum_{n=1}^{\infty}\frac{c_n^n}{n^p}$ or $\sum_{n=1}^{\infty}\frac{n^p}{c_n^n}$.

Note that $\lim_{n\to\infty} n^{p/n} = \lim_{n\to\infty}[n^{1/n}]^p = 1^p = 1$.

A **power series** $\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_k(x-a)^k + \cdots$. Here x is a variable and the constants c_n are the coefficients of the series.

The **interval of convergence** of a power series $\sum_{n=j}^{\infty} c_n(x-a)^n$ contains all values of x for which the series converges. The interval is from $a-R$ to $a+R$ where R is the **radius of convergence.**

Power Series Theorem: Given a power series $\sum_{n=j}^{\infty} c_n(x-a)^n$, there are 3 possibilities.

i) $R = 0$ and the interval of convergence is $\{a\}$ $(= [a, a])$. So the power series converges iff $x = a$.

ii) $R = \infty$ and the interval of convergence is $(-\infty, \infty)$. So the power series converges for all real x.

iii) $0 < R < \infty$ and the interval of convergence can have one of 4 forms: $(a-R, a+R)$, $[a - R, a + R)$, $(a - R, a + R]$, or $[a - R, a + R]$. Note that the power series converges for x between $a - R$ and $a + R$, but convergence at the endpoints $x = a - R$ and $x = a + R$ needs to be checked.

Given a power series $\sum_{n=j}^{\infty} c_n(x - a)^n$, find the radius of convergence R and the interval of convergence where x is between $a - R$ and $a + R$.

Tips: i) Get a directly from the power series. So a series with x^n has $a = 0$, a series with $(x - c)^n$ has $a = c$ and a series with $(x + d)^n$ has $a = -d$ since $(x + d) = (x - (-d))$.

ii) Use the ratio test or (less likely) root test to find R. Note that $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{c_{n+1}(x - a)^{n+1}}{c_n(x - a)^n}| = |x - a| \lim_{n\to\infty} |\frac{c_{n+1}}{c_n}| = |x - a|L$. So by the ratio test, the series converges if $|x - a|L < 1$ or $|x - a| < 1/L = R$ or if $x - a$ is in between $-R$ and R or if x is between $a - R$ and $a + R$.

iii) Check for convergence at the endpoints $x = a - R$ and at $x = a + R$. You can not check the endpoints using the ratio or root test: another series test must be used.

iv) Using ii) and iii) gives R and the interval on convergence where x is between $a - R$ and $a + R$. Usually $0 < R < \infty$ and the interval has one of the 4 forms from the Power Series Theorem iii).

v) If the power series is an alternating series, then $|(-1)^n| = 1$ so the $(-1)^n$ term drops out when you use the ratio or root test.

F21*)** Given a power series $\sum_{n=j}^{\infty} c_n(bx - d)^n$, find the interval of convergence. See F07 10, S08 12, F08 6.

Tips: i) Could rewrite the series using $(bx - d)^n = [b(x - d/b)]^n = b^n(x - d/b)^n$. So $a = d/b$.

ii) Use the ratio test or root test directly. Note that $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{c_{n+1}(bx - d)^{n+1}}{c_n(bx - d)^n}| = |bx - d| \lim_{n\to\infty} |\frac{c_{n+1}}{c_n}| = |bx - d|L$. So by the ratio test, the series converges if $|bx - d|L < 1$ or $|b||x - d/b| < 1/L$ or if $|x - d/b| < \frac{1}{|b|\ L}$ or if $x - d/b$ is in between $\frac{-1}{|b|\ L}$ and $\frac{1}{|b|\ L}$ or if x is between $\frac{d}{b} - \frac{1}{|b|\ L}$ and $\frac{d}{b} + \frac{1}{|b|\ L}$. Also check convergence at the endpoints $x = \frac{d}{b} - \frac{1}{|b|\ L}$ and $x = \frac{d}{b} + \frac{1}{|b|\ L}$.

Note: If the interval of convergence $I = (a - R, a + R) = (L, U)$, then R is half of the interval width: $R = (U - L)/2$.

Evaluation Theorem (term by term differentiation and term by term integration): Suppose the power series $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_k(x-a)^k + \cdots$ has radius of convergence R. For x in $(a-R, a+R)$,

i) $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.

ii) $\int f(x)dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

where both series i) and ii) have radius of convergence R.

Notes: i) The original series and series i) and ii) all converge for x in $(a-R, a+R)$ but convergence may differ at the endpoints $x = a - R$ and $x = a + R$.

ii) An indefinite integral $\int f(x)dx = F(x) + C$ where C is an arbitrary constant. So the antiderivative $F(x) = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$.

iii) $f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx} \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n \frac{d}{dx}(x-a)^n =$
$\sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + kc_k(x-a)^{k-1} + \cdots$.

iv) $\int f(x)dx = \int [\sum_{n=0}^{\infty} c_n(x-a)^n]dx = \sum_{n=0}^{\infty} c_n[\int (x-a)^n dx] = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} =$
$C + \left[c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \cdots + c_k \frac{(x-a)^{k+1}}{k+1} + \cdots \right]$.

v) For $b, d \in (a-R, a+R)$, the definite integral $\int_b^d f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \Big|_b^d =$
$\sum_{n=0}^{\infty} c_n \frac{(d-a)^{n+1}}{n+1} - \sum_{n=0}^{\infty} c_n \frac{(b-a)^{n+1}}{n+1}$. Often $b = a$ and often $a = 0$. If $b = a$, then $\int_a^d f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \Big|_a^d = \sum_{n=0}^{\infty} c_n \frac{(d-a)^{n+1}}{n+1}$.

A function f(x) has a power series representation $f(x) = \sum_{n=j}^{\infty} c_n(x-a)^n$ for $|x-a| < R$ if the left hand side equals the right hand side for $x \in (a-R, a+R)$. Thus the power series converges and is equal to $f(x)$.

The lower limit $n = j$ is crucial. For example $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ while $\frac{x}{1-x} = \sum_{n=1}^{\infty} x^n$. So $j = 1$ and $j = 0$ give different functions.

Let $f^{(n)}(x)$ be the nth derivative of $f(x)$ and let $f^{(n)}(a)$ be the nth derivative evaluated at a. Let $f^{(0)}(x) = f(x)$ and $f^{(0)}(a) = f(a)$. Recall that $f^{(n)}(x) = \frac{d}{dx} f^{(n-1)}(x)$. Also $f'(x) = f^{(1)}(x)$ and $f''(x) = f^{(2)}(x)$.

So $f''(x) = \frac{d}{dx} f'(x)$, $f^{(3)}(x) = \frac{d}{dx} f''(x)$, $f^{(4)}(x) = \frac{d}{dx} f^{(3)}(x)$, et cetera.

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ for $|x-a| < R$, then $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n =$
$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots$
is the **Taylor series for f at a** for $|x-a| < R$

The special case $a = 0$ has $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $|x| < R$. Then $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n =$
$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots$
is the **Maclaurin series for f** for $|x| < R$

$f(x) = \sum_{n=j}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ for $|x-a| < R$ is also a Taylor series for f at a and $f(x) = \sum_{n=j}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ for $|x| < R$ is also a Maclaurin series for f.

Memorize the following Maclaurin series:

i) $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ for $R = 1$.

ii) $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ for $R = \infty$.

iii) $f(x) = \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ for $R = \infty$.

iv) $f(x) = \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ for $R = \infty$.

v) $f(x) = \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ for $R = 1$.

Let the power series $S = \sum_{n=j}^{\infty} c_n (x-a)^n$. Let the nth partial sum $S_n = \sum_{i=j}^{n} c_i (x-a)^i$.

A) If the Taylor series of f at a is $S = \sum_{n=j}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, then $|S - S_n| = |R_n(x)| =$ $|\frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}|$ where z is between x and a (either $x < z < a$ or $a < z < x$).

B) If the series is an alternating series $S = \sum_{n=j}^{\infty} (-1)^n b_n$ where $0 < b_{n+1} \le b_n$ and $\lim_{n\to\infty} b_n = 0$, then $|S - S_n| \le b_{n+1}$ by the **Alternating Series Estimation Theorem**.

F22*)** Let $f(x) = kx^p g(h(x))$ where usually $g(x) = e^x$ or g(x) $= \frac{1}{1-x}$. a) Evaluate $f(x)$ or $\int f(x)dx$ as an infinite series giving the 1st few terms. b) Evaluate $S = \int_0^b f(x)dx$ correct to within an error 0.001 by using A) or B) above. F07 9, S08 11, F08 8.

Tips: i) Usually $g(x)$ is a Maclaurin series $g(x) = \sum_{n=0}^{\infty} c_n x^n$ and then $f(x) = x^p g(x) = \sum_{n=0}^{\infty} c_n x^{n+p}$.

ii) If $g(x) = \sum_{n=0}^{\infty} c_n x^n$ for $|x| < R$, then $f(x) = g(h(x)) = \sum_{n=0}^{\infty} c_n [h(x)]^n$ for $|h(x)| < R$.

iii) Often $g(x) = \dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$. Then $\dfrac{kx^p}{1-h(x)} = kx^p g(h(x)) = kx^p \sum_{n=0}^{\infty} [h(x)]^n$ for $|h(x)| < 1$. Note that $\dfrac{1}{1+G(x)} = \dfrac{1}{1-[-G(x)]}$ has $h(x) = -G(x)$.

iv) To write $\int f(x)dx$ as a series, use Evaluation Theorem b).

F23*)** Find the Maclaurin series or Taylor series of f at a (of f centered at a) if $f(x) = kx^p g(h(x))$. See F07 8, S08 9, F08 8.

F24*) From 1st principles, find the Taylor series of f at a to find the degree d Taylor polynomial $T_d(x)$ of $f(x)$ about a. Recall that the Taylor series of f at a is $f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$ and the degree d Taylor polynomial $T_d(x) = \sum_{n=0}^{d} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$. See F08 7.

Tip. Make a table with headers $n \quad f^{(n)}(x) \quad f^{(n)}(a)$ where $a = 0$ for a Maclaurin series. Recall that $f^{(0)}(x) = f(x)$. Fill in the table for $n = 0, 1, 2, 3$, and 4. Try to find a pattern and plug into the formula in F23).