Exam 3 on Friday, April 15 covers sections (6.1, 6.2), 6.3, (6.4), 6.5, 7.1, 7.2, 7.3, 7.4, 7.5, 8.1, (8.2), 1.1, 1.2, 2.1, 2.2, 2.3, 2.5, and 2.6. Emphasis is on the material on quizzes 5 - 9. You are allowed a NON-GRAPHICAL calculator but NO NOTES.

These common final problems from chapter 6 may again appear on exam 3:

1) **Common Final Problem:** Use the multiplication principle to find *how many* ways to perform the tasks. Important special cases include finding the number of license plates or serial numbers. See S04 6, F03 6, S03 4,7, Q2 4,5, Q3 7, HW2, HW3, E1 6,7,9, E2 5.

2) **Common Final Problem.** Often the multiplication principle will be combined with combinations, permutations, powers and factorials. F04 6, F02 10, S02 15, F01 6, HW4 80, HW3 40a, HW4 38, E1 6.

3) **Common Final Problem.** A coin is tossed n times. How many outcomes are possible? Solution: 2^n.

4) **Common Final Problem.** A hand of k cards is dealt from a deck of 52 cards. How many k-card hands are there? Solution: $C(52, k)$. See S04 4a.

5) **Common Final Problem.** Story problem has k groups of indistinguishable objects of size n_i, $i = 1, ..., k$. *Color* is often used, eg green, red and white balls, LIGHTS, or FLAGS. Find the total number of objects $n = n_1 + ... + n_k$. Then the number of ways to arrange the objects is $\dfrac{n!}{n_1!n_2! \cdots n_k!}$. S03 9, HW4 25,26.

See review for exam 1 for more details.

These results and common final problems from ch. 7 may again appear on exam 3:

Know: If E is an event, then $0 \le P(E) \le 1$.

Know. Events E and F are **mutually exclusive** if E and F have no outcomes in common: $E \cap F = \emptyset$. Events are mutually exclusive if they are disjoint sets.

Additive rule for mutually exclusive events: If E and F are mutually exclusive events, then $P(E \cup F) = P(E) + P(F)$. If $E_1, E_2, ..., E_k$ are mutually exclusive events, then $P(E_1 \cup E_2 \cup \cdots \cup E_k) = P(E_1) + P(E_2) + \cdots + P(E_k)$.

If $S = \{e_1, e_2, ..., e_n\}$, then $\{e_1\}, \{e_2\}, ..., \{e_n\}$ are mutually exclusive. If the occurrence of event E prevents the occurrence of event F and vice verca, then E and F are mutually exclusive.

Complement rule: $P(\overline{E}) = 1 - P(E)$. Hence $P(E) = 1 - P(\overline{E})$. This rule is useful because often one probability is far easier to find directly than the other.

Let $S = \{e_1, ..., e_n\}$. Then the outcomes $e_1, ..., e_n$ are *equally likely* if $P(e_i) = 1/n$ for each $i = 1, ..., n$, and if event E contains m outcomes, then $P(E) = \dfrac{m}{n} = \dfrac{c(E)}{c(S)}$. For this reason, counting techniques of chapter 6 are still needed for chapter 7. Equally likely outcomes often occur when items are selected *randomly* (eg cards), and when *fair* coins and die are tossed.

Know. Let E and F be events from sample space S with $P(F) > 0$. Then the **conditional probability of E given F** is $P(E|F) = \dfrac{P(E \cap F)}{P(F)}$.

Know. Let E and F be events in S. Suppose **any one** of the following conditions holds I1) $P(E \cap F) = P(E)P(F)$. I2) $P(E|F) = P(E)$, or I3) $P(F|E) = P(F)$, then E and F are **independent** events. If any of the three conditions fails, then E and F are *dependent.*

Product rule: $P(E \cap F) = P(F)P(E|F)$.

A **tree diagram** can be useful for finding $P(A_i \cap B_j)$ when there are several A_i and several B_j.

Multiplication rule for independent events: if E and F are *independent*, then $P(E \cap F) = P(E)P(F)$. If A_1, A_2, ..., A_n are n independent events, then $P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2) \cdots P(A_n)$.

6) **Common Final Problem.** Given a story problem, list the outcomes that make up an event (especially for die problems). S04 8, F03 7 (die table), E2 8, HW5 4, 7, 39, 16 and the technique is useful for many other problems.

7) **Common Final Problem.** If a *fair coin* is tossed n times, then the probability of (exactly) r heads (or of r tails) is $\dfrac{C(n,r)}{2^n}$. S03 5a, F02 5a, E2 10, Q5 3, HW6 3, 4a.

8) **Common Final Problem.** There are two groups with n_1 and n_2 members where $n_1 + n_2 = n$. A committee or subgroup of size r is formed by selecting r_1 members from the 1st group and r_2 members from the 2nd group where $r_1 + r_2 = r$. The probability of obtaining such a group if the r committee members were randomly selected from the n is equal to

$$\frac{C(n_1, r_1)C(n_2, r_2)}{C(n, r)}.$$

Defectives and nondefectives, females and males, yellow balls and non-yellow balls, spades and non-spades, queens and non-queens, etc are common ways of getting two groups. See F04 2ab, S04 4bc, F03 4a, F02 4a, S02 16a F01 4a, Q5 2a, HW6 1, 2, 24, HW8 13ab.

9) Also know how to find P(**at least j males**) and P(**at most j males**) See F04 2c, F03 4bc, S03 8b, F02 4b, S02 16bc, F01 4bc, Q5 1b, HW6 1, HW8 13c.

10) **Common Final Problem.** Determine if events A and B are independent. If $P(A \cap B) = P(A)P(B)$ or if $P(A|B) = P(A)$ (or if $P(B|A) = P(B)$), then A and B are independent. *Only one of these conditions needs to be checked.* If $P(A \cap B) \neq P(A)P(B)$ or if $P(A|B) \neq P(A)$ (or if $P(B|A) \neq P(B)$), then A and B are dependent. Again, only one of these conditions needs to be checked. If $P(E) > 0$, $P(F) > 0$ and if E and F are mutually exclusive, then E and F are dependent (actually an extreme form of dependence: if you know that F occurred then you know that E did not occur. So $P(E|F) = P(E \cap F) = 0$ if E and F are mutually exclusive). F04 5d, S04 9d, F02 6c, HW7 5,6, HW8 12

11) **Common Final Problem. General Additive Rule.** $P(E \cup F) = P(E) + P(F) - P(E \cap F)$. Given any three of the above probabilities, use the general additive rule to find the fourth probability. Variants: a) given $P(E), P(F)$ and that E and F are **independent**, then $P(E \cap F) = P(E)P(F)$ and so $P(E \cup F) = P(E) + P(F) - P(E)P(F)$. b) Given $P(E), P(F)$ and that E and F are **mutually exclusive**, then $E \cap F = \emptyset$, so $P(E \cap F) = 0$ and so $P(E \cup F) = P(E) + P(F)$. F02 8a, S03 2, F02 6ab, S02 5, Q3 1a, 2c, Q6, HW5 3, 5, 6, 31, HW8 12

12) **Common Final Problem.** Two events E and F correspond to a **Venn diagram** with 2 circles and 4 regions. Fill out the Venn diagram and then use the results to find various probabilities. The probabilities in 4 regions are $P(E \cap \overline{F}), P(E \cap F), P(\overline{E} \cap F)$

and $P(\overline{E} \cap \overline{F})$. The four regions are mutually exclusive. S04 9, F03 8, F02 6, F01 7b, Q6, and **especially** Q7 1, 2.

SEE REVIEW FOR EXAM 2 FOR MORE INFORMATION.

New Material from Ch. 8, 1, and 2.

The sample space S is partitioned into n subsets $A_1, A_2, ..., A_n$ if a) $A_i \cap A_j = \emptyset$ for $i \neq j$, b) $A_i \neq \emptyset$ for $i = 1, ..., n$, and c) $A_1 \cup A_2 \cup \cdots \cup A_n = S$. Often $n = 2$, and A and the complement $\overline{A}$ form a partition of S.

Let $A_1, A_2, ..., A_n$ partition S, and let E be an event in S, then
a) $P(E) = P(A_1)P(E|A_1) + P(A_2)P(E|A_2) + \cdots + P(A_n)P(E|A_n)$ and
b) **Bayes' rule:** $P(A_j|E) = \dfrac{P(A_j \cap E)}{P(E)} = \dfrac{P(A_j)P(E|A_j)}{P(E)}$

$$= \frac{P(A_j)P(E|A_j)}{P(A_1)P(E|A_1) + P(A_2)P(E|A_2) + \cdots + P(A_n)P(E|A_n)}.$$

In particular, if $n = 2$, $P(E) = P(A)P(E|A) + P(\overline{A})P(E|\overline{A})$ and
$$P(A|E) = \frac{P(A)P(E|A)}{P(A)P(E|A) + P(\overline{A})P(E|\overline{A})}.$$

In a **Bayes' rule** story problem, 2 or more unconditional probabilities are given (or easy to find with the complement rule). Several conditional probabilities are also given (or easy to find with the complement rule). **Make a tree diagram** with the events corresponding to the unconditional events labelling the left branches and the events corresponding to the conditional probabilities labelling the right branches. Above the left branches place the unconditional probabilities and above the right branches place the conditional probabilities. You will be asked to find an unconditional right branch probability and to use Bayes' rule to find P(left branch | right branch).

Tips: the hard conditional probability, P(left branch | right branch), usually appears at the end of the story problem. This tells you how to label the left branches and the right branches of the tree. (The easy conditional probabilities, P(right branch | left branch), can also tell you how to label the tree.) The probabilities of the left branch sum to one. Each subtree of right branches has probabilities that sum to one. Occasionally you are asked to find both a P(right branch | left branch) (directly from the tree) and P(left branch | right branch) (using Bayes rule). **See Q7 3, 4 for how to do Bayes problems.** Jar problems are especially common.

13) **Common Final problem:** Draw a tree diagram and use the tree to find the unconditional probability of a right branch event and use the tree and Bayes rule to find a conditional probability of a left branch event given a right branch event. See F04 8, S04 10, F03 9,S03 3, F02 11, S02 13, F01 8, Q7 3, 4, Q8 3, HW9 29, HW10 27, 32, 38, HW18 19?, 25?.

The horizontal line through a point (x_1, y_1) is $y = y_1$ and the vertical line is $x = x_1$.

14) **Common Final problem:** Given two points (x_1, y_1) and (x_2, y_2), find the **slope-intercept form of the line** $y = mx + b$ passing through the two points, find the **y-intercept** and find the **x-intercept**. **Solution:** The slope intercept equation $y = mx+b$ through the points (x_1, y_1) and (x_2, y_2) has slope $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. After finding the slope

m, find b using the equation $y_1 = mx_1 + b$. That is, $b = y_1 - mx_1$. Variant S03 13b: also give the y-intercept is (0,b) and the x-intercept (a,0) where a is found by solving $0 = mx + b$ for x: $x = a = -b/m$.
S03 13a, F02 15a, Q8 2, HW11 17, 20, 22.

15) **Common Final problem:** Find the slope m of the line **parallel to the line** $Ax + By = C$. **Solution:** Parallel lines have the same slope, so $m = -A/B$. If you need to find the equation of the line parallel to line $L : y = mx + b$ that passes through the point $P = (x_o, y_o)$, plug in $y_o = mx_0 + b$ and solve for $b = y_o - mx_o$. See S04 12a, F02 15b, S02 11b, Q8 5, HW12 35, 36.

16) **Common Final problem:** Find the equation of the line J **perpendicular to the line** $Ax + By = C$ that passes through the point $P = (x_o, y_o)$. **Solution:** Put L in slope intercept form $y = m_L x + b_L$. Then the slope of line J is $m = -1/m_L$. Now $J : y = mx + b$ and b can be found as in 14). See S04 12b, F03 15, S02 11a, Q9?, HW12 44.

17) **Common Final problem:** Given augmented matrices, find the general solution (x, y, z) to the problem. Typically three augmented matrices are given. One has a unique solution, one has no solution, and one has infinitely many solutions. The following examples are illustrative.

i) $\begin{bmatrix} 1 & 0 & 0 & | & -2 \\ 0 & 1 & 0 & | & -5 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$ $\quad \begin{matrix} x = -2 \\ y = -5 \\ z = 0 \end{matrix}$

Hence the general solution is $(x, y, z) = (-2, -5, 0)$.

ii) $\begin{bmatrix} 1 & 0 & 2 & | & -2 \\ 0 & 1 & 1 & | & 3 \\ 0 & 0 & 0 & | & 7 \end{bmatrix}$ $\quad 0 = 7, \textit{impossible}$

Hence there is no solution.

iii) $\begin{bmatrix} 1 & 0 & 2 & | & 1 \\ 0 & 1 & 4 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$ $\quad \begin{matrix} x + 2z = 1 \\ y + 4z = -2 \\ 0 = 0 \end{matrix}$

Hence the general solution is $(x, y, z) = (1-2z, -2-4z, z)$ where z is any real number.

iv) $\begin{bmatrix} 1 & 3 & 2 & | & 1 \\ 0 & 1 & 4 & | & -2 \\ 0 & 0 & 1 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$ $\quad \begin{matrix} x + 3y + 2z = 1 & or & x = 1 - 3(-14) - 2(3) = 37 \\ y + 4z = -2 & or & y = -2 - 4(3) = -14 \\ z = 3 & & 0 = 0 \end{matrix}$

Hence the general solution is $(x, y, z) = (37, -14, 3)$.
See F04 11, S04 13, F03 3, S03 10, F02 9, S02 4, F01 3, Q8 4, Q9?.

19) **Common Final problem:** Given a system of linear equations, write down the corresponding augmented matrix and perform row operations until the augmented matrix is in **reduced row echelon form.** Then obtain the general solution to the system from this matrix. F04 10, S04 16, F03 2, S03 14, F02 1, S02 1, F01 2, Q9?, HW15 36 37 45.