Chapter 7

Testing Statistical Hypotheses

A hypothesis is a statement about a population parameter $\boldsymbol{\theta}$, and in hypothesis testing there are two competing hypotheses called the null hypothesis $Ho \equiv H_0$ and the alternative hypothesis $H_1 \equiv H_A$. Let Θ_1 and Θ_0 be disjoint sets with $\Theta_i \subset \Theta$ where Θ is the parameter space. Then $Ho : \boldsymbol{\theta} \in \Theta_0$ and $H_1 : \boldsymbol{\theta} \in \Theta_1$.

When a researcher wants strong evidence about a hypothesis, usually this hypothesis is H_1. For example, if Ford claims that their latest car gets 30 mpg on average, then $Ho : \mu = 30$ and $H_1 : \mu > 30$ are reasonable hypotheses where $\theta = \mu$ is the population mean mpg of the car.

Definition 7.1. Assume that the data $\boldsymbol{Y} = (Y_1, ..., Y_n)$ has pdf or pmf $f(\boldsymbol{y}|\boldsymbol{\theta})$ for $\boldsymbol{\theta} \in \Theta$. A **hypothesis test** is a rule for rejecting Ho.

Definition 7.2. A **type I error** is rejecting Ho when Ho is true. A **type II error** is failing to reject Ho when Ho is false. $P_{\boldsymbol{\theta}}$(reject Ho) = P_θ(type I error) if $\boldsymbol{\theta} \in \Theta_0$ while $P_{\boldsymbol{\theta}}$(reject Ho) $= 1 -$ $P_{\boldsymbol{\theta}}$(type II error) if $\boldsymbol{\theta} \in \Theta_1$.

Definition 7.3. The **power function** of a hypothesis test is

$$\beta(\boldsymbol{\theta}) = P_{\boldsymbol{\theta}}(\text{Ho is rejected})$$

for $\boldsymbol{\theta} \in \Theta$.

Often there is a rejection region R and an acceptance region. **Reject Ho** if the observed statistic $T(\boldsymbol{y}) \in R$, otherwise **fail to reject Ho.** Then $\beta(\boldsymbol{\theta}) = P_{\boldsymbol{\theta}}(T(\boldsymbol{Y}) \in R) = P_{\boldsymbol{\theta}}(\text{reject Ho})$.

Definition 7.4. For $0 \le \alpha \le 1$, a test with power function $\beta(\theta)$ is a **size α test** if

$$\sup_{\theta \in \Theta_0} \beta(\theta) = \alpha$$

and a **level α test** if

$$\sup_{\theta \in \Theta_0} \beta(\theta) \le \alpha.$$

Notice that for $\theta \in \Theta_0$, $\beta(\theta) = P_\theta(\text{type I error})$ and for $\theta \in \Theta_1$, $\beta(\theta) = 1 - P_\theta(\text{type II error})$. We would like $\beta(\theta) \approx 0$ for $\theta \in \Theta_0$ and $\beta(\theta) \approx 1$ for $\theta \in \Theta_1$, but this may not be possible even if the sample size n is large. The tradeoff is that decreasing the probability of a type I error increases the probability of a type II error while decreasing the probability of a type II error increases the probability of a type I error. The size or level of the test gives an upper bound α on the probability of the type I error. Typically the level is fixed, eg $\alpha = 0.05$, and then we attempt to find tests that have a small probability of type II error. The following example is a level 0.07 and size 0.0668 test.

Example 7.1. Suppose that $Y \sim N(\mu, 1/9)$ where $\mu \in \{0, 1\}$. Let $Ho : \mu = 0$ and $H_1 : \mu = 1$. Let $T(Y) = Y$ and suppose that we reject Ho if $Y \ge 0.5$. Let $Z \sim N(0, 1)$ and $\sigma = 1/3$. Then

$$\beta(0) = P_0(Y \ge 0.5) = P_0(\frac{Y - 0}{1/3} \ge \frac{0.5}{1/3}) = P(Z \ge 1.5) \approx 0.0668.$$

$$\beta(1) = P_1(Y \ge 0.5) = P_1(\frac{Y - 1}{1/3} \ge \frac{0.5 - 1}{1/3}) = P(Z \ge -1.5) \approx 0.9332.$$

7.1 Exponential Families, the Neyman Pearson Lemma, and UMP Tests

Definition 7.5. Consider all level α tests of $Ho : \theta \in \Theta_o$ vs $H_1 : \theta \in \Theta_1$. A **uniformly most powerful (UMP) level α test** is a level α test with power function $\beta_{UMP}(\theta)$ such that $\beta_{UMP}(\theta) \ge \beta(\theta)$ for every $\theta \in \Theta_1$ where β is the power function for any level α test of Ho vs H_1.

The following three theorems can be used to find UMP tests.

Theorem 7.1, The Neyman Pearson Lemma (NPL). Consider testing $H_o : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$ where the pdf or pmf corresponding to θ_i is $f(\boldsymbol{y}|\theta_i)$ for $i = 0, 1$. Suppose the test rejects H_o if $f(\boldsymbol{y}|\theta_1) > kf(\boldsymbol{y}|\theta_0)$, and rejects H_o with probability γ if $f(\boldsymbol{y}|\theta_1) = kf(\boldsymbol{y}|\theta_0)$ for some $k \geq 0$. If

$$\alpha = \beta(\theta_0) = P_{\theta_0}[f(\boldsymbol{Y}|\theta_1) > kf(\boldsymbol{Y}|\theta_0)] + \gamma P_{\theta_0}[f(\boldsymbol{Y}|\theta_1) = kf(\boldsymbol{Y}|\theta_0)],$$

then this test is a UMP level α test.

Proof. The proof is for pdfs. Replace the integrals by sums for pmfs. Following Ferguson (1967, p. 202), a test can be written as a test function $\psi(\boldsymbol{y}) \in [0, 1]$ where $\psi(\boldsymbol{y})$ is the probability that the test rejects H_o when $\boldsymbol{Y} = \boldsymbol{y}$. The Neyman Pearson (NP) test function is

$$\phi(\boldsymbol{y}) = \begin{cases} 1, & f(\boldsymbol{y}|\theta_1) > kf(\boldsymbol{y}|\theta_0) \\ \gamma, & f(\boldsymbol{y}|\theta_1) = kf(\boldsymbol{y}|\theta_0) \\ 0, & f(\boldsymbol{y}|\theta_1) < kf(\boldsymbol{y}|\theta_0) \end{cases}$$

and $\alpha = E_{\theta_0}[\phi(\boldsymbol{Y})]$. Consider any level α test $\psi(\boldsymbol{y})$. Since $\psi(\boldsymbol{y})$ is a level α test,

$$E_{\theta_0}[\psi(\boldsymbol{Y})] \leq E_{\theta_0}[\phi(\boldsymbol{Y})] = \alpha. \tag{7.1}$$

Then the NP test is UMP if the power

$$\beta_\psi(\theta_1) = E_{\theta_1}[\psi(\boldsymbol{Y})] \leq \beta_\phi(\theta_1) = E_{\theta_1}[\phi(\boldsymbol{Y})].$$

Let $f_i(\boldsymbol{y}) = f(\boldsymbol{y}|\theta_i)$ for $i = 0, 1$. Notice that $\phi(\boldsymbol{y}) = 1 \geq \psi(\boldsymbol{y})$ if $f_1(\boldsymbol{y}) > kf_0(\boldsymbol{y})$ and $\phi(\boldsymbol{y}) = 0 \leq \psi(\boldsymbol{y})$ if $f_1(\boldsymbol{y}) < kf_0(\boldsymbol{y})$. Hence

$$\int [\phi(\boldsymbol{y}) - \psi(\boldsymbol{y})][f_1(\boldsymbol{y}) - kf_0(\boldsymbol{y})]d\boldsymbol{y} \geq 0 \tag{7.2}$$

since the integrand is nonnegative. Hence the power

$$\beta_\phi(\theta_1) - \beta_\psi(\theta_1) = E_{\theta_1}[\phi(\boldsymbol{Y})] - E_{\theta_1}[\psi(\boldsymbol{Y})] \geq k(E_{\theta_0}[\phi(\boldsymbol{Y})] - E_{\theta_0}[\psi(\boldsymbol{Y})]) \geq 0$$

where the first inequality follows from (7.2) and the second inequality from Equation (7.1). QED

Theorem 7.2, One Sided UMP Tests via the Neyman Pearson Lemma. Suppose that the hypotheses are of the form $H_o : \theta \leq \theta_o$ vs $H_1 : \theta > \theta_o$ or $H_o : \theta \geq \theta_o$ vs $H_1 : \theta < \theta_o$, or that the inequality in H_o is replaced by equality. Also assume that

$$\sup_{\theta \in \Theta_0} \beta(\theta) = \beta(\theta_o).$$

Pick $\theta_1 \in \Theta_1$ and use the Neyman Pearson lemma to find the UMP test for $H_o^* : \theta = \theta_o$ vs $H_A^* : \theta = \theta_1$. Then the UMP test rejects H_o^* if $f(\boldsymbol{y}|\theta_1) > kf(\boldsymbol{y}|\theta_o)$, and rejects H_o^* with probability γ if $f(\boldsymbol{y}|\theta_1) = kf(\boldsymbol{y}|\theta_o)$ for some $k \geq 0$ where $\alpha = \beta(\theta_o)$. This test is also the UMP level α test for $H_o : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$ if k *does not depend on the value of* $\theta_1 \in \Theta_1$.

Theorem 7.3, One Sided UMP Tests for Exponential Families. Let $Y_1, ..., Y_n$ be a sample with a joint pdf or pmf from a one parameter exponential family where $w(\theta)$ is increasing and $T(\boldsymbol{y})$ is the complete sufficient statistic. Alternatively, let $Y_1, ..., Y_n$ be iid with pdf or pmf

$$f(y|\theta) = h(y)c(\theta)\exp[w(\theta)t(y)]$$

from a one parameter exponential family where θ is real and $w(\theta)$ is increasing. Here $T(\boldsymbol{y}) = \sum_{i=1}^n t(y_i)$. I) Let $\theta_1 > \theta_o$. Consider the test that rejects H_o if $T(\boldsymbol{y}) > k$ and rejects H_o with probability γ if $T(\boldsymbol{y}) = k$ where $\alpha = P_{\theta_o}(T(\boldsymbol{Y}) > k) + \gamma P_{\theta_o}(T(\boldsymbol{Y}) = k)$. This test is the UMP test for
a) $H_o : \theta = \theta_o$ vs $H_A : \theta = \theta_1$,
b) $H_o : \theta = \theta_o$ vs $H_A : \theta > \theta_o$, and
c) $H_o : \theta \leq \theta_o$ vs $H_A : \theta > \theta_o$.
II) Let $\theta_1 < \theta_o$. Consider the test that rejects H_o if $T(\boldsymbol{x}) < k$ and rejects H_o with probability γ if $T(\boldsymbol{y}) = k$ where $\alpha = P_{\theta_o}(T(\boldsymbol{Y}) < k) + \gamma P_{\theta_o}(T(\boldsymbol{Y}) = k)$. This test is the UMP test for
d) $H_o : \theta = \theta_o$ vs $H_A : \theta = \theta_1$
e) $H_o : \theta = \theta_o$ vs $H_A : \theta < \theta_o$, and
f) $H_o : \theta \geq \theta_o$ vs $H_A : \theta < \theta_o$.

Proof. I) Let $\theta_1 > \theta_o$. a) Then

$$\frac{f(\boldsymbol{y}|\theta_1)}{f(\boldsymbol{y}|\theta_o)} = \left[\frac{c(\theta_1)}{c(\theta_o)}\right]^n \frac{\exp[w(\theta_1)\sum_{i=1}^n t(y_i)]}{\exp[w(\theta_0)\sum_{i=1}^n t(y_i)]} > c$$

iff

$$[w(\theta_1) - w(\theta_0)] \sum_{i=1}^n t(y_i) > d$$

iff $\sum_{i=1}^n t(y_i) > k$ since $w(\theta)$ is increasing. Hence the result holds by the NP lemma. b) The test in a) did not depend on $\theta_1 > \theta_o$, so the test is UMP by Theorem 7.2. c) In a), $\theta_o < \theta_1$ were arbitrary, so $\sup_{\theta \in \Theta_0} \beta(\theta) = \beta(\theta_o)$ where $\Theta_o = \{\theta \in \Theta | \theta \leq \theta_o\}$. So the test is UMP by Theorem 7.2. The proof of II) is similar. QED

Remark 7.1. As a mnemonic, note that the *inequality used in the rejection region is the same as the inequality in the alternative hypothesis.* Usually $\gamma = 0$ if f is a pdf. Suppose that the parameterization is

$$f(y|\theta) = h(y)c(\theta)\exp[\tilde{w}(\theta)\tilde{t}(y)]$$

where $\tilde{w}(\theta)$ is decreasing. Then set $w(\theta) = -\tilde{w}(\theta)$ and $t(y) = -\tilde{t}(y)$. In this text, $w(\theta)$ is an increasing function if $w(\theta_o) < w(\theta_1)$ for $\theta_o < \theta_1$ and nondecreasing if $w(\theta_o) \leq w(\theta_1)$. Some texts use "strictly increasing" for "increasing" and use "increasing" for "nondecreasing."

If the data are iid from a one parameter exponential family, then Theorem 7.3 is simpler to use than the Neyman Pearson lemma since the test statistic T will have a distribution from an exponential family. This result makes finding the cutoff value k easier. To find a UMP test via the Neyman Pearson lemma, you need to check that the cutoff value k does not depend on $\theta_1 \in \Theta_1$ and usually need to transform the NP test statistic to put the test in *useful form.* With exponential families, the transformed test statistic is often T.

Example 7.2. Suppose that $X_1, ..., X_{10}$ are iid Poisson with unknown mean λ. Derive the most powerful level $\alpha = 0.10$ test for $H_0 : \lambda = 0.30$ versus $H_1 : \lambda = 0.40$.

Solution: Since

$$f(x|\lambda) = \frac{1}{x!} e^{-\lambda} \exp[\log(\lambda)x]$$

and $\log(\lambda)$ is an increasing function of λ, by Theorem 7.3 the UMP test rejects Ho if $\sum x_i > k$ and rejects Ho with probability γ if $\sum x_i = k$ where $\alpha = 0.1 = P_{Ho}(\sum X_i > k) + \gamma P_{Ho}(\sum X_i = k)$. Notice that

$$\gamma = \frac{\alpha - P_{Ho}(\sum X_i > k)}{P_{Ho}(\sum X_i = k)}. \tag{7.3}$$

Alternatively use the Neyman Pearson lemma. Let

$$r = f(\boldsymbol{x}|0.4)/f(\boldsymbol{x}|0.3) = \frac{e^{-n\lambda_1} \lambda_1^{\sum x_i}}{\prod x_i!} \frac{\prod x_i!}{e^{-n\lambda_0} \lambda_0^{\sum x_i}} = e^{-n(\lambda_1 - \lambda_0)} \left(\frac{\lambda_1}{\lambda_0}\right)^{\sum x_i}.$$

Since $\lambda_1 = 0.4 > 0.3 = \lambda_0$, $r > c$ is equivalent to $\sum x_i > k$ and the NP UMP test has the same form as the UMP test found using the much simpler Theorem 7.3.

k	0	1	2	3	4	5
P(T = k)	0.0498	0.1494	0.2240	0.2240	0.1680	0.1008
F(k)	0.0498	0.1992	0.4232	0.6472	0.8152	0.9160

If Ho is true, then $T = \sum_{i=1}^{10} X_i \sim$ Pois(3) since $3 = 10\lambda_0 = 10(0.3)$. The above table gives the probability that $T = k$ and $F(k) = P(T \le k)$. First find the smallest integer k such that $P_{\lambda=0.30}(\sum X_i > k) = P(T > k) < \alpha = 0.1$. Since $P(T > k) = 1 - F(k)$, find the smallest value of k such that $F(k) > 0.9$. This happens with $k = 5$. Next use (7.3) to find γ.

$$\gamma = \frac{0.1 - (1 - 0.9160)}{0.1008} = \frac{0.1 - 0.084}{0.1008} = \frac{0.016}{0.1008} \approx 0.1587.$$

Hence the $\alpha = 0.1$ UMP test rejects Ho if $T \equiv \sum_{i=1}^{10} X_i > 5$ and rejects Ho with probability 0.1587 if $\sum_{i=1}^{10} X_i = 5$. Equivalently, the test function $\phi(T)$ gives the probability of rejecting Ho for a given value of T where

$$\phi(T) = \begin{cases} 1, & T > 5 \\ 0.1587, & T = 5 \\ 0, & T < 5. \end{cases}$$

Example 7.3. Let $X_1, ..., X_n$ be independent identically distributed random variables from a distribution with pdf

$$f(x) = \frac{2}{\lambda\sqrt{2\pi}} \frac{1}{x} \exp\left[\frac{-(\log(x))^2}{2\lambda^2}\right]$$

where $\lambda > 0$ where and $0 \le x \le 1$.

a) What is the UMP (uniformly most powerful) level α test for $H_o : \lambda = 1$ vs. $H_1 : \lambda = 2$?

b) If possible, find the UMP level α test for $H_o : \lambda = 1$ vs. $H_1 : \lambda > 1$.

Solution. a) By the NP lemma reject Ho if

$$\frac{f(\boldsymbol{x}|\lambda = 2)}{f(\boldsymbol{x}|\lambda = 1)} > k'.$$

The LHS =
$$\frac{\frac{1}{2^n}\exp[\frac{-1}{8}\sum[\log(x_i)]^2]}{\exp[\frac{-1}{2}\sum[\log(x_i)]^2]}.$$
So reject Ho if
$$\frac{1}{2^n}\exp[\sum[\log(x_i)]^2(\frac{1}{2}-\frac{1}{8})] > k'$$
or if $\sum[\log(X_i)]^2 > k$ where $P_{Ho}(\sum[\log(X_i)]^2 > k) = \alpha$.

b) In the above argument, with any $\lambda_1 > 1$, get
$$\sum[\log(x_i)]^2(\frac{1}{2}-\frac{1}{2\lambda_1^2})$$
and
$$\frac{1}{2}-\frac{1}{2\lambda_1^2} > 0$$
for any $\lambda_1^2 > 1$. Hence the UMP test is the same as in a).

Theorem 7.3 gives the same UMP test as a) for both a) and b) since the pdf is a 1P-REF and $w(\lambda^2) = -1/(2\lambda^2)$ is an increasing function of λ^2. Also, it can be shown that $\sum[\log(X_i)]^2 \sim \lambda^2\chi^2_n$, so $k = \chi^2_{n,1-\alpha}$ where $P(W > \chi^2_{n,1-\alpha}) = \alpha$ if $W \sim \chi^2_n$.

Example 7.4. Let $X_1, ..., X_n$ be independent identically distributed (iid) random variables with probability density function
$$f(x) = \frac{2}{\lambda\sqrt{2\pi}}e^x\exp\left(\frac{-(e^x-1)^2}{2\lambda^2}\right)$$
where $x > 0$ and $\lambda > 0$.

a) What is the UMP (uniformly most powerful) level α test for $H_o : \lambda = 1$ vs. $H_1 : \lambda = 2$?

b) If possible, find the UMP level α test for $H_o : \lambda = 1$ vs. $H_1 : \lambda > 1$.

a) By the NP lemma reject Ho if
$$\frac{f(\boldsymbol{x}|\lambda=2)}{f(\boldsymbol{x}|\lambda=1)} > k'.$$
The LHS =
$$\frac{\frac{1}{2^n}\exp[\frac{-1}{8}\sum(e^{x_i}-1)^2]}{\exp[\frac{-1}{2}\sum(e^{x_i}-1)^2]}.$$

So reject Ho if

$$\frac{1}{2^n}\exp[\sum(e^{x_i}-1)^2(\frac{1}{2}-\frac{1}{8})] > k'$$

or if $\sum(e^{x_i}-1)^2 > k$ where $P_1(\sum(e^{X_i}-1)^2 > k) = \alpha$.

b) In the above argument, with any $\lambda_1 > 1$, get

$$\sum(e^{x_i}-1)^2(\frac{1}{2}-\frac{1}{2\lambda_1^2})$$

and

$$\frac{1}{2}-\frac{1}{2\lambda_1^2} > 0$$

for any $\lambda_1^2 > 1$. Hence the UMP test is the same as in a).

Alternatively, use the fact that this is an exponential family where $w(\lambda^2) = -1/(2\lambda^2)$ is an increasing function of λ^2 with $T(X_i) = (e^{X_i}-1)^2$. Hence the same test in a) is UMP for both a) and b) by Theorem 7.3.

Example 7.5. Let $X_1, ..., X_n$ be independent identically distributed random variables from a half normal HN(μ, σ^2) distribution with pdf

$$f(x) = \frac{2}{\sigma\ \sqrt{2\pi}}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

where $\sigma > 0$ and $x > \mu$ and μ is real. **Assume that μ is known.**

a) What is the UMP (uniformly most powerful) level α test for $H_o: \sigma^2 = 1$ vs. $H_1: \sigma^2 = 4$?

b) If possible, find the UMP level α test for $H_o: \sigma^2 = 1$ vs. $H_1: \sigma^2 > 1$.

Solution: a) By the NP lemma reject Ho if

$$\frac{f(\boldsymbol{x}|\sigma^2 = 4)}{f(\boldsymbol{x}|\sigma^2 = 1)} > k'.$$

The LHS =

$$\frac{\frac{1}{2^n}\exp[(\frac{-\sum(x_i-\mu)^2}{2(4)})]}{\exp[(\frac{-\sum(x_i-\mu)^2}{2})]}.$$

So reject Ho if

$$\frac{1}{2^n}\exp[\sum(x_i-\mu)^2(\frac{-1}{8}+\frac{1}{2})] > k'$$

or if $\sum(x_i - \mu)^2 > k$ where $P_{Ho}(\sum(X_i - \mu)^2 > k) = \alpha$.

Under Ho, $\sum(X_i - \mu)^2 \sim \chi^2_n$ so $k = \chi^2_n(1-\alpha)$ where $P(\chi^2_n > \chi^2_n(1-\alpha)) = \alpha$.

b) In the above argument,

$$\frac{-1}{2(4)} + 0.5 = \frac{-1}{8} + 0.5 > 0$$

but

$$\frac{-1}{2\sigma_1^2} + 0.5 > 0$$

for any $\sigma_1^2 > 1$. Hence the UMP test is the same as in a).

Alternatively, use the fact that this is an exponential family where $w(\sigma^2) = -1/(2\sigma^2)$ is an increasing function of σ^2 with $T(X_i) = (X_i - \mu)^2$. Hence the test in a) is UMP for a) and b) by Theorem 7.3.

7.2 Likelihood Ratio Tests

Definition 7.6. Let $Y_1, ..., Y_n$ be the data with pdf or pmf $f(\boldsymbol{y}|\boldsymbol{\theta})$ where $\boldsymbol{\theta}$ is a vector of unknown parameters with parameter space Θ. Let $\hat{\boldsymbol{\theta}}$ be the MLE of $\boldsymbol{\theta}$ and let $\hat{\boldsymbol{\theta}}_o$ be the MLE of $\boldsymbol{\theta}$ if the parameter space is Θ_0 (where $\Theta_0 \subset \Theta$). A likelihood test (LRT) statistic for testing $H_o : \boldsymbol{\theta} \in \Theta_0$ versus $H_1 : \boldsymbol{\theta} \in \Theta_0^c$ is

$$\lambda(\boldsymbol{y}) = \frac{L(\hat{\boldsymbol{\theta}}_o|\boldsymbol{y})}{L(\hat{\boldsymbol{\theta}}|\boldsymbol{y})} = \frac{\sup_{\Theta_0} L(\boldsymbol{\theta}|\boldsymbol{y})}{\sup_{\Theta} L(\boldsymbol{\theta}|\boldsymbol{y})}. \tag{7.4}$$

The **likelihood ratio test** (LRT) has a rejection region of the form

$$R = \{\boldsymbol{y}|\lambda(\boldsymbol{y}) \le c\}$$

where $0 \le c \le 1$, and $\alpha = \sup_{\boldsymbol{\theta} \in \Theta_0} P_{\boldsymbol{\theta}}(\lambda(\boldsymbol{Y}) \le c)$. Suppose $\boldsymbol{\theta}_o \in \Theta_0$ and $\sup_{\boldsymbol{\theta} \in \Theta_0} P_{\boldsymbol{\theta}}(\lambda(\boldsymbol{Y}) \le c) = P_{\boldsymbol{\theta}_o}(\lambda(\boldsymbol{Y}) \le c)$. Then $\alpha = P_{\boldsymbol{\theta}_o}(\lambda(\boldsymbol{Y}) \le c)$.

Rule of Thumb 7.1: Asymptotic Distribution of the LRT. Let $Y_1, ..., Y_n$ be iid. Then under strong regularity conditions, $-2 \log \lambda(\boldsymbol{x}) \approx \chi^2_j$ for large n where $j = r - q$, r is the number of free parameters specified by $\boldsymbol{\theta} \in \Theta_1$, and q is the number of free parameters specified by $\boldsymbol{\theta} \in \Theta_o$. Hence the approximate LRT rejects H_o if $-2 \log \lambda(\boldsymbol{y}) > c$ where $P(\chi^2_j > c) = \alpha$. Thus $c = \chi^2_{j,1-\alpha}$ where $P(\chi^2_j > \chi^2_{j,1-\alpha}) = \alpha$.

Often $\boldsymbol{\theta} = \theta$ is a scalar parameter, $\Theta_0 = (a, \theta_o]$ and $\Theta_1 = \Theta_0^c = (\theta_o, b)$ or $\Theta_0 = [\theta_o, b)$ and $\Theta_1 = (a, \theta_o)$.

Remark 7.2. Suppose the problem wants the rejection region in useful form. Find the two MLEs and write $L(\theta|\boldsymbol{y})$ in terms of a sufficient statistic. Then you should either I) simplify the LRT test statistic $\lambda(\boldsymbol{y})$ and try to find an equivalent test that uses test statistic $T(\boldsymbol{y})$ where the distribution of $T(\boldsymbol{Y})$ is known (ie put the LRT in useful form). Often the LRT rejects H_o if $T > k$ (or $T < k$). Getting the test into useful form can be very difficult. Monotone transformations such as log or power transformations can be useful. II) If you can not find a statistic T with a simple distribution, state that the Rule of Thumb 7.1 suggests that the LRT test rejects H_o if $-2\log\lambda(\boldsymbol{y}) > \chi^2_{j,1-\alpha}$ where $\alpha = P(-2\log\lambda(\boldsymbol{Y}) > \chi^2_{j,1-\alpha})$. Using II) is dangerous because for many data sets the asymptotic result will not be valid.

Example 7.6. Let $X_1, ..., X_n$ be independent identically distributed random variables from a $N(\mu, \sigma^2)$ distribution where the variance σ^2 is known. We want to test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$.

a) Derive the likelihood ratio test.

b) Let λ be the likelihood ratio. Show that $-2\log\lambda$ is a function of $(\overline{X} - \mu_0)$.

c) Assuming that H_0 is true, find $P(-2\log\lambda > 3.84)$.

Solution: a) The likelihood function

$$L(\mu) = (2\pi\sigma^2)^{-n/2}\exp[\frac{-1}{2\sigma^2}\sum(x_i-\mu)^2]$$

and the MLE for μ is $\hat{\mu} = \overline{x}$. Thus the numerator of the likelihood ratio test statistic is $L(\mu_0)$ and the denominator is $L(\overline{x})$. So the test is reject H_0 if $\lambda = L(\mu_0)/L(\overline{x}) \leq c$ where $\alpha = P_{H_0}(\lambda \leq c)$.

b) As a statistic, $\log\lambda = \log L(\mu_0) - \log L(\overline{X}) =$ $-\frac{1}{2\sigma^2}[\sum(X_i - \mu_0)^2 - \sum(X_i - \overline{X})^2] = \frac{-n}{2\sigma^2}[\overline{X} - \mu_0]^2$ since $\sum(X_i - \mu_0)^2 = \sum(X_i - \overline{X} + \overline{X} - \mu_0)^2 = \sum(X_i - \overline{X})^2 + n(\overline{X} - \mu_o)^2$. So $-2\log\lambda = \frac{n}{\sigma^2}[\overline{X} - \mu_0]^2$.

c) $-2\log\lambda \sim \chi^2_1$ and from a chi–square table, $P(-2\log\lambda > 3.84) = 0.05$.

Example 7.7. Let $Y_1, ..., Y_n$ be iid $N(\mu, \sigma^2)$ random variables where μ and σ^2 are unknown. Set up the likelihood ratio test for $Ho : \mu = \mu_o$ versus

$H_A : \mu \neq \mu_o$.

Solution: Under Ho, $\mu = \mu_o$ is known and the MLE

$$(\hat{\mu}_o, \hat{\sigma}_o^2) = (\mu_o, \frac{1}{n}\sum_{i=1}^n (Y_i - \mu_o)^2).$$

Recall that

$$(\hat{\mu}, \hat{\sigma}^2) = (\overline{Y}, \frac{1}{n}\sum_{i=1}^n (Y_i - \overline{Y})^2).$$

Now

$$L(\mu, \sigma^2) = \prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} \exp[\frac{1}{2\sigma^2}(y_i - \mu)^2].$$

Thus

$$\lambda(\boldsymbol{y}) = \frac{L(\hat{\mu}_o, \hat{\sigma}_o^2|\boldsymbol{y})}{L(\hat{\mu}, \hat{\sigma}^2|\boldsymbol{y})} = \frac{\frac{1}{(\hat{\sigma}_o^2)^{n/2}} \exp[\frac{1}{2\hat{\sigma}_o^2}\sum_{i=1}^n (y_i - \mu_o)^2]}{\frac{1}{(\hat{\sigma}^2)^{n/2}} \exp[\frac{1}{2\hat{\sigma}^2}\sum_{i=1}^n (y_i - \overline{y})^2]} =$$

$$\left(\frac{\hat{\sigma}^2}{\hat{\sigma}_o^2}\right)^{n/2} \frac{\exp(n/2)}{\exp(n/2)} = \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_o^2}\right)^{n/2}.$$

The LRT rejects Ho iff $\lambda(\boldsymbol{y}) \leq c$ where $\sup_{\sigma^2} P_{\mu_o,\sigma^2}(\lambda(\boldsymbol{Y}) \leq c) = \alpha$.

On an exam the above work may be sufficient, but to implement the LRT, more work is needed. Notice that the LRT rejects Ho iff $\hat{\sigma}^2/\hat{\sigma}_o^2 \leq c'$ iff $\hat{\sigma}_o^2/\hat{\sigma}^2 \geq k'$. Using

$$\sum_{i=1}^n (y_i - \mu_o)^2 = \sum_{i=1}^n (y_i - \overline{y})^2 + n(\overline{y} - \mu_o)^2,$$

the LRT rejects Ho iff

$$\left[1 + \frac{n(\overline{y} - \mu_o)^2}{\sum_{i=1}^n (y_i - \overline{y})^2}\right] \geq k\text{''}$$

iff

$$\frac{\sqrt{n}\ |\overline{y} - \mu_o|}{\left[\frac{\sum_{i=1}^n (y_i - \overline{y})^2}{n-1}\right]^{1/2}} = \sqrt{n}\frac{|\overline{y} - \mu_o|}{s} \geq k$$

where s is the observed sample standard deviation. Hence the LRT is equivalent to the usual t test with test statistic

$$T_o = \frac{\overline{Y} - \mu_o}{S/\sqrt{n}}$$

that rejects Ho iff $|T_o| \geq k$ with $k = t_{n-1,1-\alpha/2}$ where $P(T \leq t_{n-1,1-\alpha/2}) = 1 - \alpha/2$ when $T \sim t_{n-1}$.

Example 7.8. Suppose that $X_1, ..., X_n$ are iid $N(0, \sigma^2)$ where $\sigma > 0$ is the unknown parameter. With preassigned $\alpha \in (0, 1)$, derive a level α likelihood ratio test for the null hypothesis $H_0 : \sigma^2 = \sigma_0^2$ against an alternative hypothesis $H_A : \sigma^2 \neq \sigma_0^2$.

Solution: The likelihood function is given by

$$L(\sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp(-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2)$$

for all $\sigma^2 > 0$, and $\hat{\sigma}^2(\boldsymbol{x}) = \sum_{i=1}^n x_i^2/n$ is the MLE for σ^2. Under Ho, $\hat{\sigma}_o^2 = \sigma_o^2$ since σ_o^2 is the only value in the parameter space $\Theta_o = \{\sigma_o^2\}$. Thus

$$\lambda(\boldsymbol{x}) = \frac{L(\hat{\sigma}_o^2|\boldsymbol{x})}{L(\hat{\sigma}^2|\boldsymbol{x})} = \frac{\sup_{\Theta_0} L(\sigma^2|\boldsymbol{x})}{\sup_{\sigma^2} L(\sigma^2|\boldsymbol{x})} = \frac{(2\pi\sigma_0^2)^{\frac{-n}{2}} \exp(-\frac{1}{2\sigma_0^2}\sum_{i=1}^n x_i^2)}{(2\pi\hat{\sigma}^2)^{\frac{-n}{2}} \exp(\frac{-n}{2})}.$$

So

$$\lambda(\boldsymbol{x}) = \left(\frac{\hat{\sigma}^2}{\sigma_o^2}\right)^{n/2} \exp(\frac{-n\hat{\sigma}^2}{2\sigma_o^2})e^{n/2} = \left[\frac{\hat{\sigma}^2}{\sigma_o^2} \exp\left(1 - \frac{\hat{\sigma}^2}{\sigma_o^2}\right)\right]^{n/2}.$$

The LRT rejects H_0 if $\lambda(\boldsymbol{x}) \leq c$ where $P_{\sigma_o^2}(\lambda(\boldsymbol{X}) \leq c) = \alpha$.

The function $g(u) = ue^{1-u}I(u > 0)$ monotonically increases for $0 < u < d$, monotonically decreases for $d < u < \infty$, and attains its maximum at $u = d$, for some $d > 0$. So $\lambda(\boldsymbol{x})$ will be small in the two tail areas.

Under H_0, $T = \sum_{i=1} X_i^2/\sigma_o^2 \sim \chi_n^2$. Hence the LR test will reject Ho if $T < a$ or $T > b$ where $0 < a < b$. The a and b correspond to horizontal line drawn on the χ_n^2 pdf such that the tail area is α. Hence a and b need to be found numerically. An approximation that should be good for large n rejects Ho if $T < \chi^2_{n,\frac{\alpha}{2}}$ or $T > \chi^2_{n,1-\frac{\alpha}{2}}$ where $P(\chi_n^2 < \chi_{n,\alpha}^2) = \alpha$.

7.3 Summary

For hypothesis testing there is a null hypothesis Ho and an alternative hypothesis $H_1 \equiv H_A$. A **hypothesis test** is a rule for rejecting Ho. Either reject Ho or fail to reject Ho. A **simple hypothesis** consists of exactly one distribution for the sample. A **composite hypothesis** consists of more than one distribution for the sample.

The **power** $\beta(\boldsymbol{\theta}) = P_{\boldsymbol{\theta}}(\text{reject } H_o)$ is the probability of rejecting H_o when $\boldsymbol{\theta}$ is the true value of the parameter. Often the power function can not be calculated, but you should be prepared to calculate the power for a sample of size one for a test of the form $H_o: f(x) = f_0(x)$ versus $H_1: f(x) = f_1(x)$ or if the test is of the form $\sum t(X_i) > k$ or $\sum t(X_i) < k$ when $\sum t(X_i)$ has an easily handled distribution under H_1, eg binomial, normal, Poisson, or χ^2_p. To compute the power, you need to find k and γ for the given value of α.

Consider all level α tests of $H_o: \theta \in \Theta_o$ vs $H_1: \theta_1 \in \Theta_1$. A **uniformly most powerful** (UMP) level α test is a level α test with power function $\beta_{UMP}(\theta)$ such that $\beta_{UMP}(\theta) \geq \beta(\theta)$ for every $\theta \in \Theta_1$ where β is the power function for any level α test of H_o vs H_1.

One Sided UMP Tests for Exponential Families. Let $Y_1, ..., Y_n$ be iid with pdf or pmf

$$f(y|\theta) = h(y)c(\theta)\exp[w(\theta)t(y)]$$

from a one parameter exponential family where θ is real and $w(\theta)$ is increasing. Let $T(\boldsymbol{y}) = \sum_{i=1}^n t(y_i)$. Then the UMP test for $H_o: \theta \leq \theta_o$ vs $H_A: \theta > \theta_o$ rejects H_o if $T(\boldsymbol{y}) > k$ and rejects H_o with probability γ if $T(\boldsymbol{y}) = k$ where $\alpha = P_{\theta_o}(T(\boldsymbol{Y}) > k) + \gamma P_{\theta_o}(T(\boldsymbol{Y}) = k)$. The UMP test for $H_o: \theta \geq \theta_o$ vs $H_A: \theta < \theta_o$ rejects H_o if $T(\boldsymbol{x}) < k$ and rejects H_o with probability γ if $T(\boldsymbol{y}) = k$ where $\alpha = P_{\theta_o}(T(\boldsymbol{Y}) < k) + \gamma P_{\theta_o}(T(\boldsymbol{Y}) = k)$.

Fact: if f is a pdf, then $\gamma = 0$. For a pmf and $H_A: \theta > \theta_o$,

$$\gamma = \frac{\alpha - P_{\theta_o}[T(\boldsymbol{Y}) > k]}{P_{\theta_o}[T(\boldsymbol{Y}) = k]}.$$

For a pmf and $H_A: \theta < \theta_o$,

$$\gamma = \frac{\alpha - P_{\theta_o}[T(\boldsymbol{Y}) < k]}{P_{\theta_o}[T(\boldsymbol{Y}) = k]}.$$

As a mnemonic, note that the *inequality used in the rejection region is the same as the inequality in the alternative hypothesis.* Suppose that the parameterization is

$$f(y|\theta) = h(y)c(\theta)\exp[\tilde{w}(\theta)\tilde{t}(y)]$$

where $\tilde{w}(\theta)$ is decreasing. Then set $w(\theta) = -\tilde{w}(\theta)$ and $t(y) = -\tilde{t}(y)$.

Recall that $w(\theta)$ is increasing on Θ if $w'(\theta) > 0$ for $\theta \in \Theta$, and $w(\theta)$ is decreasing on Θ if $w'(\theta) < 0$ for $\theta \in \Theta$. Also $w(\theta)$ is nondecreasing on Θ if $w'(\theta) \geq 0$ for $\theta \in \Theta$, and $w(\theta)$ is nonincreasing on Θ if $w'(\theta) \leq 0$ for $\theta \in \Theta$.

The Neyman Pearson Lemma: Consider testing $H_o : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$ where the pdf or pmf corresponding to θ_i is $f(\boldsymbol{y}|\theta_i)$ for $i = 0, 1$. Suppose the test rejects H_o if $f(\boldsymbol{y}|\theta_1) > kf(\boldsymbol{y}|\theta_0)$, and rejects H_o with probability γ if $f(\boldsymbol{y}|\theta_1) = kf(\boldsymbol{y}|\theta_0)$ for some $k \geq 0$. If

$$\alpha = \beta(\theta_0) = P_{\theta_0}[f(\boldsymbol{Y}|\theta_1) > kf(\boldsymbol{Y}|\theta_0)] + \gamma P_{\theta_0}[f(\boldsymbol{Y}|\theta_1) = kf(\boldsymbol{Y}|\theta_0)],$$

then this test is a UMP level α test.

One Sided UMP Tests via the Neyman Pearson Lemma: Suppose that the hypotheses are of the form $H_o : \theta \leq \theta_o$ vs $H_1 : \theta > \theta_o$ or $H_o : \theta \geq \theta_o$ vs $H_1 : \theta < \theta_o$, or that the inequality in H_o is replaced by equality. Also assume that

$$\sup_{\theta \in \Theta_0} \beta(\theta) = \beta(\theta_o).$$

Pick $\theta_1 \in \Theta_1$ and use the Neyman Pearson lemma to find the UMP test for $H_o^* : \theta = \theta_o$ vs $H_A^* : \theta = \theta_1$. Then the UMP test rejects H_o^* if $f(\boldsymbol{y}|\theta_1) > kf(\boldsymbol{y}|\theta_o)$, and rejects H_o^* with probability γ if $f(\boldsymbol{y}|\theta_1) = kf(\boldsymbol{y}|\theta_o)$ for some $k \geq 0$ where $\alpha = \beta(\theta_o)$. This test is also the UMP level α test for $H_o : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$ if k *does not depend on the value of* $\theta_1 \in \Theta_1$.

Fact: if f is a pdf, then $\gamma = 0$ and $\alpha = P_{\theta_o}[f(\boldsymbol{Y}|\theta_1) > kf(\boldsymbol{Y}|\theta_o)]$. So γ is important when f is a pmf. For a pmf,

$$\gamma = \frac{\alpha - P_{\theta_o}[f(\boldsymbol{Y}|\theta_1) > kf(\boldsymbol{Y}|\theta_o)]}{P_{\theta_o}[f(\boldsymbol{Y}|\theta_1) = kf(\boldsymbol{Y}|\theta_o)]}.$$

Often it is too hard to give the UMP test in useful form. Then simply specify when the test rejects H_o and specify α in terms of k (eg $\alpha = P_{H_o}(T > k) + \gamma P_{H_o}(T = k)$).

The problem will be harder if you are asked to put the test in useful form. To find an UMP test with the NP lemma, often the ratio $\frac{f(\boldsymbol{y}|\theta_1)}{f(\boldsymbol{y}|\theta_0)}$ is computed. The test will certainly reject H_o is the ratio is large, but usually the distribution of the ratio is not easy to use. Hence try to get an equivalent test by simplifying and transforming the ratio. Ideally, the ratio can be transformed into a statistic T whose distribution is tabled.

If the test rejects H_o if $T > k$ (or if $T > k$ and with probability γ if $T = k$, or if $T < k$, or if $T < k$ and with probability γ if $T = k$) the test is in **useful form** if for a given α, you find k and γ. If you are asked to find the power (perhaps with a table), put the test in useful form.

Let $Y_1, ..., Y_n$ be the data with pdf or pmf $f(\boldsymbol{y}|\boldsymbol{\theta})$ where $\boldsymbol{\theta}$ is a vector of unknown parameters with parameter space Θ. Let $\hat{\boldsymbol{\theta}}$ be the MLE of $\boldsymbol{\theta}$ and let $\hat{\boldsymbol{\theta}}_o$ be the MLE of $\boldsymbol{\theta}$ if the parameter space is Θ_0 (where $\Theta_0 \subset \Theta$). A LRT statistic for testing $H_o : \boldsymbol{\theta} \in \Theta_0$ versus $H_1 : \boldsymbol{\theta} \in \Theta_0^c$ is

$$\lambda(\boldsymbol{y}) = \frac{L(\hat{\boldsymbol{\theta}}_o|\boldsymbol{y})}{L(\hat{\boldsymbol{\theta}}|\boldsymbol{y})}.$$

The **LRT** has a rejection region of the form

$$R = \{\boldsymbol{y}|\lambda(\boldsymbol{y}) \le c\}$$

where $0 \le c \le 1$ and $\alpha = \sup_{\boldsymbol{\theta} \in \Theta_0} P_{\boldsymbol{\theta}}(\lambda(\boldsymbol{Y}) \le c)$.

Fact: Often $\Theta_o = (a, \theta_o]$ and $\Theta_1 = (\theta_o, b)$ or $\Theta_o = [\theta_o, b)$ and $\Theta_1 = (a, \theta_o)$.

If you are not asked to find the power or to put the LRT into useful form, it is often enough to find the two MLEs and write $L(\theta|\boldsymbol{y})$ in terms of a sufficient statistic. Simplify the statistic $\lambda(\boldsymbol{y})$ and state that the LRT test rejects H_o if $\lambda(\boldsymbol{y}) \le c$ where $\alpha = \sup_{\boldsymbol{\theta} \in \Theta_0} P_{\boldsymbol{\theta}}(\lambda(\boldsymbol{Y}) \le c)$. If the sup is achieved at $\boldsymbol{\theta}_o \in \Theta_0$, then $\alpha = P_{\boldsymbol{\theta}_o}(\lambda(\boldsymbol{Y}) \le c)$.

Put the LRT into useful form if asked to find the power. Try to simplify λ or transform λ so that the test rejects H_o if some statistic $T > k$ (or

$T < k$). Getting the test into useful form can be very difficult. Monotone transformations such as log or power transformations can be useful. If you can not find a statistic T with a simple distribution, use the large sample approximation to the LRT that rejects H_o if $-2\log\lambda(\boldsymbol{x}) > \chi^2_{j,1-\alpha}$ where $P(\chi^2_j > \chi^2_{j,1-\alpha}) = \alpha$. Here $j = r - q$ where r is the number of free parameters specified by $\theta \in \Theta$, and q is the number of free parameters specified by $\theta \in \Theta_o$.

7.4 Complements

Definition 7.7. Let $Y_1, ..., Y_n$ have pdf or pmf $f(\boldsymbol{y}|\theta)$ for $\theta \in \Theta$. Let $T(\boldsymbol{Y})$ be a statistic. Then $f(\boldsymbol{y}|\theta)$ has a **monotone likelihood ratio** (MLR) in statistic T if for any two values $\theta_o, \theta_1 \in \Theta$ with $\theta_o < \theta_1$, the ratio $f(\boldsymbol{y}|\theta_1)/f(\boldsymbol{y}|\theta_o)$ depends on the vector $\boldsymbol{y}$ only through $T(\boldsymbol{y})$, and this ratio is an increasing function of $T(\boldsymbol{y})$ over the possible values of $T(\boldsymbol{y})$.

Remark 7.3. Theorem 7.3 is a corollary of the following theorem, because under the conditions of Theorem 7.3, $f(\boldsymbol{y}|\theta)$ has MLR in $T(\boldsymbol{y}) = \sum_{i=1}^n t(y_i)$.

Theorem 7.4, MLR UMP Tests. Let $Y_1, ..., Y_n$ be a sample with a joint pdf or pmf $f(y|\theta)$ that has MLR in statistic $T(\boldsymbol{y})$. Then the UMP test for $H_o : \theta \le \theta_o$ vs $H_1 : \theta > \theta_o$ rejects H_o if $T(\boldsymbol{y}) > k$ and rejects H_o with probability γ if $T(\boldsymbol{y}) = k$ where $\alpha = P_{\theta_o}(T(\boldsymbol{Y}) > k) + \gamma P_{\theta_o}(T(\boldsymbol{Y}) = k)$. The UMP test for $H_o : \theta \ge \theta_o$ vs $H_1 : \theta < \theta_o$ rejects H_o if $T(\boldsymbol{x}) < k$ and rejects H_o with probability γ if $T(\boldsymbol{y}) = k$ where $\alpha = P_{\theta_o}(T(\boldsymbol{Y}) < k) + \gamma P_{\theta_o}(T(\boldsymbol{Y}) = k)$.

Lehmann and Romano (2005) is an authoritative PhD level text on testing statistical hypotheses. Many of the most used statistical tests of hypotheses are likelihood ratio tests, and several examples are given in DeGroot and Schervish (2001). Scott (2007) discusses the asymptotic distribution of the LRT test.

Birkes (1990) and Solomen (1975) compare the LRT and UMP tests. Rohatgi (1984, p. 725) claims that if the Neyman Pearson and likelihood ratio tests exist for a given size α, then the two tests are equivalent, but this claim seems to contradict Solomen (1975). Exponential families have the MLR property, and Pfanzagl (1968) gives a partial converse.

7.5 Problems

PROBLEMS WITH AN ASTERISK * ARE ESPECIALLY USEFUL.

Refer to Chapter 10 for the pdf or pmf of the distributions in the problems below.

7.1. Let $X_1, ..., X_n$ be iid $N(\mu, \sigma^2)$, $\sigma^2 > 0$. Let $\Theta_o = \{(\mu_o, \sigma^2) : \mu_o \text{ fixed}, \sigma^2 > 0\}$ and let $\Theta = \{(\mu, \sigma^2) : \mu \in \Re, \sigma^2 > 0\}$. Consider testing $H_o : \theta = (\mu, \sigma^2) \in \Theta_o$ vs H_1: not H_o. The MLE $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2) = (\overline{X}, \frac{1}{n}\sum_{i=1}^n (X_i - \overline{X})^2)$ while the restricted MLE is $\hat{\theta}_o = (\hat{\mu}_o, \hat{\sigma}_o^2) = (\mu_o, \frac{1}{n}\sum_{i=1}^n (X_i - \mu_o)^2)$.

a) Show that the likelihood ratio statistic

$$\lambda(\boldsymbol{x}) = (\hat{\sigma}^2/\hat{\sigma}_o^2)^{n/2} = [1 + \frac{n(\overline{x} - \mu_o)^2}{\sum_{i=1}^n (x_i - \overline{x})^2}]^{-n/2}.$$

b) Show that H_o is rejected iff $|\sqrt{n}(\overline{X} - \mu_o)/S| \geq k$ and find k if $n = 11$ and $\alpha = 0.05$. (Hint: show that H_o is rejected iff $n(\overline{X} - \mu_o)^2 / \sum_{i=1}^n (X_i - \overline{X})^2 \geq c$, then multiply both sides by a constant such that the left hand side has a $(t_{n-1})^2$ distribution. Use a t-table to find k.)

7.2. Let $X_1, ..., X_n$ be a random sample from the distribution with pdf

$$f(x|\theta) = \frac{x^{\theta - 1} e^{-x}}{\Gamma(\theta)}, \ x > 0, \ \theta > 0.$$

For a) and b) do not put the rejection region into useful form.

a) Use the Neyman Pearson Lemma to find the UMP size α test for testing $H_0 : \theta = 1$ vs $H_1 : \theta = \theta_1$ where θ_1 is a fixed number greater than 1.

b) Find the uniformly most powerful level α test of

$$H_0\text{: } \theta = 1 \text{ versus } H_1\text{: } \theta > 1.$$

Justify your steps. Hint: Use the statistic in part a).

7.3. Let $H_o : X_1, ..., X_n$ are iid $U(0, 10)$ and $H_1 : X_1, ..., X_n$ are iid $U(4, 7)$. Suppose you had a sample of size $n = 1000$. How would you decide which hypothesis is true?

Problems from old quizzes and exams.

7.4. Let $X_1, ..., X_{10}$ be iid Bernoulli(p). The most powerful level $\alpha = 0.0547$ test of $H_o : p = 1/2$ vs $H_1 : p = 1/4$ rejects H_o if $\sum_{i=1}^{10} x_i \le 2$. H_o is not rejected if $\sum_{i=1}^{10} x_i > 2$. Find the power of this test if $p = 1/4$.

7.5. Let $X_1, ..., X_n$ be iid exponential(β). Hence the pdf is

$$f(x|\beta) = \frac{1}{\beta} \exp(-x/\beta)$$

where $0 \le x$ and $0 < \beta$.

a) Find the MLE of β.

b) Find the level α likelihood ratio test for the hypotheses $H_o : \beta = \beta_o$ vs $H_1 : \beta \ne \beta_o$.

7.6. (Aug. 2002 QUAL): Let $X_1, ..., X_n$ be independent, identically distributed random variables from a distribution with a beta(θ, θ) pdf

$$f(x|\theta) = \frac{\Gamma(2\theta)}{\Gamma(\theta)\Gamma(\theta)} [x(1-x)]^{\theta-1}$$

where $0 < x < 1$ and $\theta > 0$.

a) Find the UMP (uniformly most powerful) level α test for $H_o : \theta = 1$ vs. $H_1 : \theta = 2$.

b) If possible, find the UMP level α test for $H_o : \theta = 1$ vs. $H_1 : \theta > 1$.

7.7. Let $X_1, ..., X_n$ be iid N($\mu_1, 1$) random variables and let $Y_1, ..., Y_n$ be iid N($\mu_2, 1$) random variables that are independent of the X's.

a) Find the α level likelihood ratio test for $H_o : \mu_1 = \mu_2$ vs $H_1 : \mu_1 \ne \mu_2$. You may assume that $(\overline{X}, \overline{Y})$ is the MLE of (μ_1, μ_2) and that under the restriction $\mu_1 = \mu_2 = \mu$, say, then the restricted MLE

$$\hat{\mu} = \frac{\sum_{i=1}^n X_i + \sum_{i=1}^n Y_i}{2n}.$$

b) If λ is the LRT test statistic of the above test, use the approximation

$$-2 \log \lambda \approx \chi^2_d$$

for the appropriate degrees of freedom d to find the rejection region of the test **in useful form** if $\alpha = 0.05$.

7.8. Let $X_1, ..., X_n$ be independent identically distributed random variables from a distribution with pdf

$$f(x) = \frac{2}{\sigma\sqrt{2\pi}} \frac{1}{x} \exp\left(\frac{-[\log(x)]^2}{2\sigma^2}\right)$$

where $\sigma > 0$ and $x \geq 1$.

If possible, find the UMP level α test for $H_o : \sigma = 1$ vs. $H_1 : \sigma > 1$.

7.9. Let $X_1, ..., X_n$ be independent identically distributed random variables from a distribution with pdf

$$f(x) = \frac{2}{\sigma\ \sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

where $\sigma > 0$ and $x > \mu$ and μ is real. **Assume that μ is known.**

a) What is the UMP (uniformly most powerful) level α test for $H_o : \sigma^2 = 1$ vs. $H_1 : \sigma^2 = 4$?

b) If possible, find the UMP level α test for $H_o : \sigma^2 = 1$ vs. $H_1 : \sigma^2 > 1$.

7.10. (Jan. 2001 SIU and 1990 Univ. MN QUAL): Let $X_1, ..., X_n$ be a random sample from the distribution with pdf

$$f(x, \theta) = \frac{x^{\theta-1}e^{-x}}{\Gamma(\theta)},\ \ x > 0,\ \ \theta > 0.$$

Find the uniformly most powerful level α test of

H: $\theta = 1$ versus K: $\theta > 1$.

7.11. (Jan 2001 QUAL): Let $X_1, ..., X_n$ be independent identically distributed random variables from a $N(\mu, \sigma^2)$ distribution where the variance σ^2 is known. We want to test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$.

a) Derive the likelihood ratio test.

b) Let λ be the likelihood ratio. Show that $-2\log\lambda$ is a function of $(\overline{X} - \mu_0)$.

c) Assuming that H_0 is true, find $P(-2\log\lambda > 3.84)$.

7.12. (Aug. 2001 QUAL): Let $X_1, ..., X_n$ be iid from a distribution with pdf

$$f(x) = \frac{2x}{\lambda} \exp(-x^2/\lambda)$$

where λ and x are both positive. Find the level α UMP test for $H_o : \lambda = 1$ vs $H_1 : \lambda > 1$.

7.13. (Jan. 2003 QUAL): Let $X_1, ..., X_n$ be iid from a distribution with pdf

$$f(x|\theta) = \frac{(\log \theta)\theta^x}{\theta - 1}$$

where $0 < x < 1$ and $\theta > 1$. Find the UMP (uniformly most powerful) level α test of $H_o : \theta = 2$ vs. $H_1 : \theta = 4$.

7.14. (Aug. 2003 QUAL): Let $X_1, ..., X_n$ be independent identically distributed random variables from a distribution with pdf

$$f(x) = \frac{x^2 \exp\left(\frac{-x^2}{2\sigma^2}\right)}{\sigma^3 \sqrt{2}\ \Gamma(3/2)}$$

where $\sigma > 0$ and $x \geq 0$.

a) What is the UMP (uniformly most powerful) level α test for $H_o : \sigma = 1$ vs. $H_1 : \sigma = 2$?

b) If possible, find the UMP level α test for $H_o : \sigma = 1$ vs. $H_1 : \sigma > 1$.

7.15. (Jan. 2004 QUAL): Let $X_1, ..., X_n$ be independent identically distributed random variables from a distribution with pdf

$$f(x) = \frac{2}{\sigma\sqrt{2\pi}} \frac{1}{x} \exp\left(\frac{-[\log(x)]^2}{2\sigma^2}\right)$$

where $\sigma > 0$ and $x \geq 1$.

a) What is the UMP (uniformly most powerful) level α test for $H_o : \sigma = 1$ vs. $H_1 : \sigma = 2$?

b) If possible, find the UMP level α test for $H_o : \sigma = 1$ vs. $H_1 : \sigma > 1$.

7.16. (Aug. 2004 QUAL): Suppose X is an observable random variable with its pdf given by $f(x)$, $x \in R$. Consider two functions defined as follows:

$$f_0(x) = \begin{cases} \frac{3}{64}x^2 & 0 \le x \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_1(x) = \begin{cases} \frac{3}{16}\sqrt{x} & 0 \le x \le 4 \\ 0 & \text{elsewhere.} \end{cases}$$

Determine the most powerful level α test for $H_0 : f(x) = f_0(x)$ versus $H_a : f(x) = f_1(x)$ in the simplest implementable form. Also, find the power of the test when $\alpha = 0.01$

7.17. (Sept. 2005 QUAL): Let X be one observation from the probability density function

$$f(x) = \theta x^{\theta-1}, \;\; 0 < x < 1, \;\; \theta > 0.$$

a) Find the most powerful level α test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

b) For testing $H_0 : \theta \le 1$ versus $H_1 : \theta > 1$, find the size and the power function of the test which rejects H_0 if $X > \dfrac{5}{8}$.

c) Is there a UMP test of $H_0 : \theta \le 1$ versus $H_1 : \theta > 1$? If so, find it. If not, prove so.