# A theorem on circle configurations

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**Abstract:** A formula for the radii and positions of four circles in the plane for an arbitrary linearly independent circle configuration is found. Among special cases is the recent extended Descartes Theorem on the Descartes configuration and an analytic solution to the Apollonian problem. The general theorem for $n$-spheres is also considered.

**Keywords:** Apollonian problem, Descartes theorem, Soddy's circles, Minkowski space, $n$-spheres.

## 1. Introduction

There is a family of problems of various degrees of generality concerning circles and, in general, $(n-1)$-dimensional spheres in $\mathbf{R}^n$. Among the most famous is the problem of Apollonius of Perga: find a circle simultaneously tangent to three given circles (Fig. 1.1a); and the problem of Descartes: given three pair-wise tangent circles, find the a fourth tangent to each of them. A solution to the latter, four mutually tangent circles, is known a Descartes configuration (Fig. 1.1b). The associated Descartes formula relates sizes of the circles in a peculiarly elegant way:

$$(a+b+c+d)^2 = 2\,(a^2+b^2+c^2+d^2)\,, \tag{1.1}$$

where $a=1/r_1$, $b=1/r_2$, etc. are the curvatures of the corresponding circles (*reciprocals* of radii). For the history of the this formula, rediscovered a number of times, and its higher-dimensional generalizations, see [Des, Ped3, Coxt69, Sodd, Gos] .

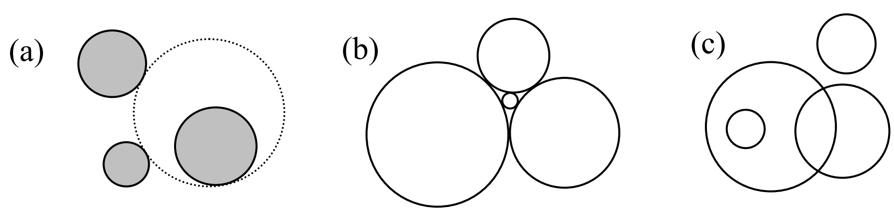



**Figure 1.1**: Four circles ; (a) as a solution to an Apollonian problem; (b) in Descartes configuration (c) in a general configuration

It's easy to observe that the quadratic formula 1.1 may be written in a matrix form:

$$\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = 0, \tag{1.2}$$

i.e., $\mathbf{b}^{\mathrm{T}} D \mathbf{b} = 0$ , where $\mathbf{b}$ is the vector of four curvatures and where $D$ has been termed "Descartes quadratic form".

The Darboux formula has been recently extended to a matrix form so that includes position of the centers of the circles [LMW]. The proof goes via hyperbolic geometry.

In this paper we go beyond this "tangential" canon and present a theorem on an (almost) arbitrary configuration of four circles. It gives a formula relating their radii (curvatures) and positions, given some mutual relations, like tangency, orthogonality, distances, etc. A generalization to $n$-dimensional spheres is also given.

Our proof relies on the fact that circle ($n$-spheres) may be mapped to the vectors of a Minkowski space, a discovery made – to our knowledge – by D. Pedoe. We use the induced inner product in the dual space.

The theorem presented here makes a simple but powerful tool to generate "Descartes-like theorems" for different types of configurations, the Descartes one being just an example, as well as simply to solve particular configurations.

## 2. Geometry of Circles — the Pedoe map

Circles in the plane may be viewed as vectors of a Minkowski space with a pseudo-Euclidean inner product. This beautiful fact ties the ancient geometry of circles with modern geometry of space-time. Let us recall the development of this idea.

**A. Remark on the history of Pedoe product:** In 1826, Jakob Steiner defined the *power* of a point $P$ with respect to a circle $C$:

$$P{*}C \;=\; d^2 - r^2\,, \tag{2.1}$$

where $r$ = radius, $d$ = distance from P to the center of the circle. Consult Fig 2.2a for the motivation: the product of segment lengths $PA{\cdot}PB$ does not depend on the choice of the line through P. Choosing the line tangent to the circle gives expression (2.1). In 1866, G. Darboux [Dar] generalized this to the *power* of a pair of circles, which we will call a ***Darboux product**:*

$$C_1{*}C_2 \;=\; d^2 - r_1^{\,2} - r_2^{\,2}\,. \tag{2.2}$$

If the circles intersect, the product equals $C_1{*}C_2 \;=\; r_1r_2\cos\varphi$ , where $\varphi$ is the angle made by the circles (see Figure 2.2a). Let us note that in the case of more distant circles, the product $C_1{*}C_2$ equals the square of the segment constructed in Figure 2.2c.

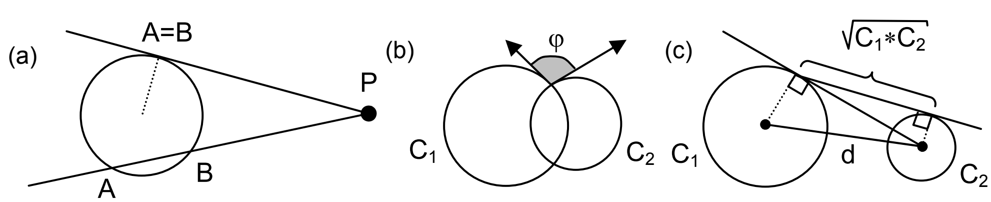



**Figure 2.2:** (a) Power of a point. Geometric interpretation of Darboux product of (a) intersecting circles; (c) distant circles.

The equation of the circle of radius $r$ and centered at $(x_o, y_o)$ has the form:

$$x^2 + y^2 - 2xx_o - 2yy_o \;+\; c \;=\; 0\,, \tag{2.3}$$

where $c = x_o^{\,2} + y_o^{\,2} - r^2$. H. Cox wrote the Darboux product of two circles with centers $(x_1, y_1)$ and $(x_2, y_2)$ and radii $r_1$ and $r_2$, respectively in terms of the coefficients of the corresponding equations:

$$C_1 * C_2 \;=\; c_1 + c_2 - 2x_1x_2 - 2y_1y_2 \tag{2.4}$$

([Cox]). The crucial observation was made in 1970 by D. Pedoe who realized that (2.4) may be interpreted as an inner product [Ped2]. Although (2.4) contains linear terms, $c_1$ and $c_2$, he points out that the equation of a circle is invariant with respect to the multiplication by a scalar, and its general form is

$$a(x^2 + y^2) \;-\; 2px \;-\; 2\,qy \;+\; c \;=\; 0\,. \tag{2.5}$$

One can thus introduce a scalar product $\langle C_1, C_2\rangle$ as

$$2\,\langle C_1, C_2\rangle \;=\; a_1c_2 + c_1a_2 - 2p_1p_2 - 2q_1q_2\,. \tag{2.6}$$

Note that when $a_1$ and $a_2$ are chosen to be $a_1$=$a_2$=1, the scalar product coincides with (half) the Darboux product (2.4). (The factor of 2 is introduced for a later convenience).

**B. Pedoe map.** The previous subsection was to clarify that the discovery of the pseudo-Euclidean geometry of circles should be credited to D. Pedoe, even if he did not follow the path of the Minkowski geometry. Let us formalize this discover and introduce some terminology.

**Definition 2.1:** By a **standard isotropic** 4-dimensional Minkowski space we understand a linear real space $\boldsymbol{M} \cong \mathbf{R}^4$ with an inner product $\langle\;,\;\rangle$ given by the *metric matrix*

$$g \;=\; \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{2.7}$$

For vectors $\mathbf{v} = [v_1\,, v_2, v_3,\; v_4\,]$ and $\mathbf{w} \;=\; [w_1, w_2, w_3, w_4]$, we have

$$\langle \mathbf{v},\mathbf{w}\rangle \;=\; \tfrac{1}{2}\, v_1w_2 + \tfrac{1}{2}\, v_2w_1 - v_3w_3 - v_4w_4 \qquad [\textit{inner product}]$$

$$|\mathbf{v}|^2 \;=\; v_1v_2 \;- v_3^{\,2} - v_4^{\,2}. \qquad [\textit{norm squared}]$$

The basis in which the above inner product is expressed will be called in the **standard isotropic basis**.

**Definition 2.2: The Pedoe projective map** is a map from the set $\Omega$ of circles in the plane to the rays in the standard isotropic Minkowski space **M** (projective space *PM*)

$$\pi\!:\;\; \Omega \rightarrow \boldsymbol{PM}$$

so that the circle $C_r(x_0, y_0)$ with center at $(x_0, y_0)\in\mathbf{R}^2$ and radius $r\in$R is mapped to a ray $\pi$(C) in $\boldsymbol{M}$ spanned by a vector:

$$\pi(C) \;\; = \text{span}\;\{\; [\; 1, x_0^{\,2} + y_0^{\,2} - r^2,\;\; x_0, y_0 \;]^{\mathrm{T}} \;\}\;.$$

and a line $L$ (circle of null curvature) given by equation $ax+by = c$ is mapped to

$$\pi(L) \;\; = \text{span}\;\{\; [\; 0, c/2, a, b \;]^{\mathrm{T}} \;\}\;.$$

Note that Pedoe map may be extended to points (**improper circles**) so that point $P = (x, y)$ has image $\pi(P) = \text{span}\ \{ [\ 1, x^2+y^2, x, y]^T\}$, which is a ray in the light cone, as $|\pi(\boldsymbol{v})|^2 = 0$, $\forall \boldsymbol{v}\in\pi(P)$ . It is easy to see that the rays that represent proper circles are spce-like, i.e., lie in $\boldsymbol{M}$ outside the light cone. For them, we may restrict the Pedoe map to the unit vectors.

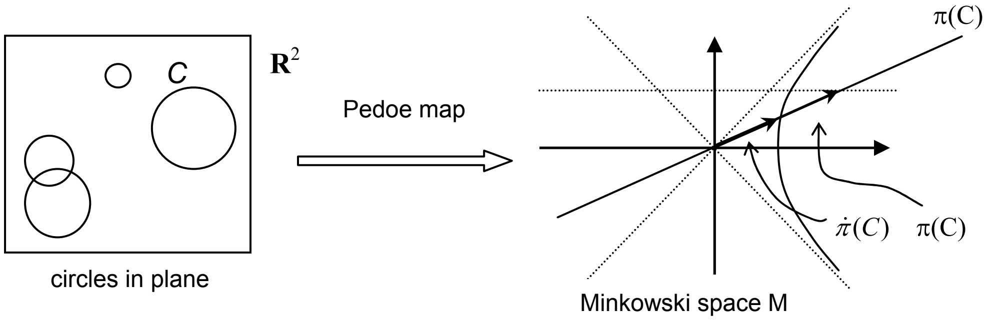



**Figure 2.2:** The Pedoe map carries circles to rays in the Minkowski space

**Definition 2: Pedoe special map** is a specification of $\pi$: $\Omega\to$PM, i.e., is map

$$\dot\pi : \ \Omega \ \to \ \boldsymbol{M}$$

that carries circles to unit space-like vectors of $\boldsymbol{M}$ so that span $\dot\pi(C) = \pi(C)$, $|\dot\pi(C)|^2 = -1$. To remove ambiguity, we demand that the first component of $\dot\pi(C)$ be nonnegative. Vector $\dot\pi(C)$ will be called the **Pedoe vector** of circle $C$. In particular, a circle of radius $r\neq 0$ centered at $(x,y)$ is represented by Pedoe vector

$$\dot\pi(C) \ = \ \begin{bmatrix} b \\ \bar b \\ \dot x \\ \dot y \end{bmatrix} \ \equiv \ \begin{bmatrix} 1/r \\ (x^2+y^2-r^2)/r \\ x/r \\ y/r \end{bmatrix}. \qquad (2.8)$$

Its entries will be called:

$$\begin{aligned} \text{circle curvature:} &\quad b \ = \ 1/\mathrm{r} \\ \text{circle co-curvature:} &\quad \bar b \ = \ (x_0^2+y_0^2-r^2)/r \\ \text{reduced position:} &\quad \dot x \ = \ x/r, \quad \dot y = y/r \end{aligned} \qquad (2.9)$$

(Note that the second term of Pedoe vector is determined by the other three and the requirement of normalization).

**Remark on the geometric meaning:** The *curvature* $b$ has the standard meaning. The *co-curvature* $\bar b$ happens to equal the curvature of the circle that is the image of $C$ via inversion in the unit circle centered at the origin. As for the reduced position $(\dot x, \dot y)$, one can think of it as the position of an "effective" circle one would regard as the circle being viewed from the origin, in the belief that it has radius equal to 1.

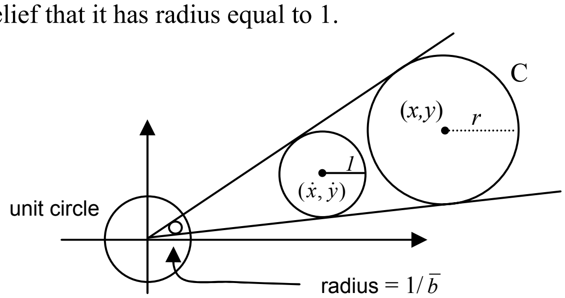



**Figure 2.3:** The geometric meaning of reduced position ( $\dot x$ , $\dot y$ ) and of co-radius $\bar b$

**Remark:** Thus, the following vectors represent the same circle:

$$\mathrm{C} \;\to\; \begin{bmatrix} a \\ c \\ p \\ q \end{bmatrix} \;\sim\; \begin{bmatrix} 1 \\ x_0^2+y_0^2-r^2 \\ x_0 \\ y_0 \end{bmatrix} \;\sim\; \dot{\pi}(C) = \begin{bmatrix} b \\ \bar{b} \\ \dot{x}_0 \\ \dot{y}_0 \end{bmatrix} = \begin{bmatrix} 1/r \\ (x_0^2+y_0^2-r^2)/r \\ x_0/r \\ y_0/r \end{bmatrix},$$

where the first vector is the set of coefficients of Eq. (2.6), the second is scaled by setting $a=1$, and the third, scaled by the radius of the circle, is the Pedoe vector. The first parameterization of vectors has the advantage, that choosing $a = 0$ admits lines into the formalism as special circles. Points $(x,y)$ go to $[1, x^2+y^2, x, y]$ go to the light-like rays, hence they do not permit Pedoe vectors.

**Theorem 2.3 [Pedoe]:** The special Pedoe map is an injection into the unit hyperboloid in ***M*** and the corresponding pseudo-Euclidean inner product is related to the Darboux product of circles:

$$\langle \pi(C_1, \pi(C_2)\rangle = \tfrac{1}{2}\, C_1 * C_2 \,/\, r_1 r_2 \tag{2.10}$$

The above expression will be called the **Pedoe inner product** of circles.

**Proposition 2.4:** Let $\mathbf{C}_i = \dot{\pi}(C_i)$ , i=1,2, denote unit vectors representing a pair of circles. Then:

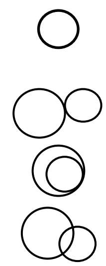
a) $|\mathbf{C}_i|^2 = -1$ (space-like, unit vector) for any circle

b) $\langle \mathbf{C}_1, \mathbf{C}_2 \rangle = \cos\varphi$, if $C_1 \cap C_2 \neq \varnothing$

c) $\langle \mathbf{C}_1, \mathbf{C}_2 \rangle = +1$ if $C_1$ and $C_2$ are tangent externally

d) $\langle \mathbf{C}_1, \mathbf{C}_2 \rangle = -1$ if $C_1$ and $C_2$ are tangent internally

e) $\langle \mathbf{C}_1, \mathbf{C}_2 \rangle = 0$ if $C_1$ and $C_2$ are mutually orthogonal



**Proof:** Simple verification. □

**Remark on basis: isotropic versus orthonormal.** Until now we have dealt with the isotropic basis and non-diagonal metric tensor $g$. But one may want to choose an orthonormal basis in which the metric tensor is diagonalized,

$$\mathrm{g} = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad\sim\quad \mathrm{g}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

so that the the Pedoe inner product signature sign($g$) = (+,–,–,–), is more conspicuous. The scalar product is here simply $\langle \mathbf{v},\mathbf{w}\rangle = v_1w_1 - v_2w_2 - v_3w_3 - v_4w_4$. The Pedoe vector of a circle $C_r(x,y)$ is now:

$$\dot{\pi}(C) = \begin{bmatrix} \frac{1+\rho^2-r^2}{2r} \\ \frac{1-\rho^2+r^2}{2r} \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \frac{b^2+\dot{\rho}^2-1}{2b} \\ \frac{b^2-\dot{\rho}^2+1}{2b} \\ \dot{x} \\ \dot{y} \end{bmatrix}$$

where $\rho^2 = x_0^2 + y_0^2$.

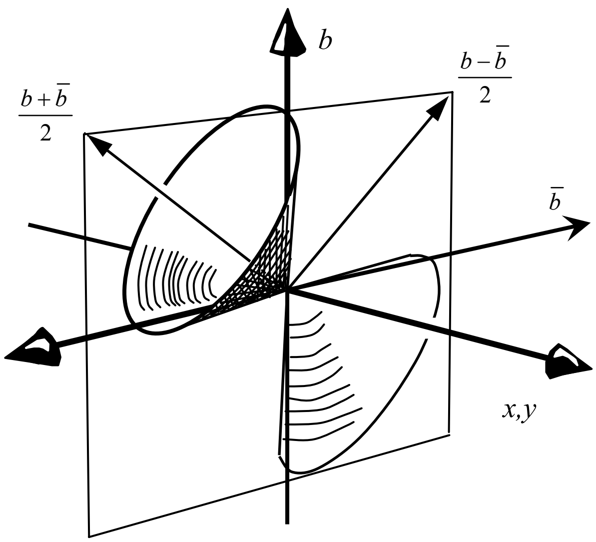



**Figure 2.4:** Isotropic basis versus orthonormal basis in the Minkowski space for circles

Although the orthonormal basis side is more "suggestive" for those accustomed with relativistic physics (see e.g. [Sod]), the path to this form leads from the Darboux product in not as intuitive and clear as to the isotropic basis and coordinates. Figure 2.4 shows the bases and corresponding variables symbolically, squeezing 4 dimensions to a 3-dimensional picture. One of the axes carries variables $x$, and $y$ (the position of the circle's center). (The figure represents exactly the special case of n=1, where spheres are pairs of points on a line).

$$\underbrace{\begin{bmatrix} 1 \\ \boldsymbol{x}_0^2 - r^2 \\ \boldsymbol{x}_0 \end{bmatrix} \sim \begin{bmatrix} 1/r \\ (\boldsymbol{x}_0^2 - r^2)/r \\ \boldsymbol{x}_0 / r \end{bmatrix} = \underset{\text{norm} = -1}{\begin{bmatrix} b \\ \bar{b} \\ \dot{\boldsymbol{x}}_0 \end{bmatrix}}}_{\text{here } g = \left[\begin{array}{cc|c} 0 & ½ & 0 \\ ½ & 0 & 0 \\ \hline 0 & 0 & \mathbf{-1} \end{array}\right]} \quad \overset{\text{change of}}{\underset{\text{basis}}{\Longleftrightarrow}} \quad \underbrace{\underset{\text{norm} = -1}{\begin{bmatrix} (b+\bar{b})/2 \\ (b-\bar{b})/2 \\ \dot{\boldsymbol{x}}_0 \end{bmatrix}} = \begin{bmatrix} \dfrac{1+\boldsymbol{x}_0^2 - r^2}{2r} \\ \dfrac{1-\boldsymbol{x}_0^2 + r^2}{2r} \\ \dfrac{\boldsymbol{x}_0}{r} \end{bmatrix} \sim \begin{bmatrix} 1+\boldsymbol{x}_0^2 - r^2 \\ 1-\boldsymbol{x}_0^2 - r^2 \\ 2\boldsymbol{x}_0 \end{bmatrix}}_{\text{here } g' = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & \mathbf{-1} \end{array}\right]}$$

**Figure 2.5:** Change of basis

Circles correspond to the rays (vectors) outside the cone. Points in $\mathbf{R}^n$, which may be viewed as circles of null radius, correspond to the light-like rays. Similarly, lines, which may be viewed as circles of zero curvature, are rays in the horizontal hyper-plane $b = 0$.

# 3. The theorem on Circle configuration

Consider four circles $C_1, \ldots, C_4$ represented by Pedoe unit vectors (see (2.10)). Define for this set of circles a *configuration matrix f* as the Grammian of the vectors $C_i$, that is:

$$f_{ij} = \langle \mathbf{C}_i, \mathbf{C}_j \rangle \,, \tag{3.1}$$

where $\langle\cdot,\cdot\rangle$ is the Pedoe inner product. On the other hand, we may build a "*data matrix*" as the collective representation of the circles (columns):

$$A = [\mathbf{C}_1 \mid \mathbf{C}_2 \mid \mathbf{C}_3 \mid \mathbf{C}_4 ] \,. \tag{3.2}$$

Now we are ready to state and prove in a shockingly simple way our theorem.

**Theorem 3.1 (Circle Configuration Theorem)**: If circles $C_1, \ldots, C_4$ are linearly independent, then

$$\boxed{AFA^{\mathrm{T}} = G} \tag{3.3}$$

where matrix $G$ is the inverse of the matrix of Minkowski metric, $G = g^{-1}$ , and F is the inverse of the configuration matrix, $F = f^{-1}$.

**Proof:** Definition (3.1) of the configuration matrix $f$ may be written in a cumulative matrix form as

$$f = A^{\mathrm{T}} g A. \tag{3.4}$$

Since $A$ is invertible due to the linear independence of the four circles, we may take the inverse of both sides

$$f^{-1} = A^{-1} g^{-1} (A^{\mathrm{T}})^{-1} \,.$$

Now, by multiplying the sides on the left and on the right by $A$ and $A^{\mathrm{T}}$, respectively, we get (3.3) as desired. □

Let us try to understand of the benefits of this equation. First, introduce four vectors that represent the data on the circles in a way *dual* to what we used until now, namely consider

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} \quad \dot{\boldsymbol{y}} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} \quad \bar{\boldsymbol{b}} = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \\ \bar{b}_4 \end{bmatrix} .$$

Vector $\boldsymbol{b}$ will be called the *curvature vector* of the circle configuration, $\bar{\boldsymbol{b}}$ the *co-curvature vector*, and $\dot{\boldsymbol{x}}$, and $\dot{\boldsymbol{y}}$ are the *reduced position vectors*. The data matrix constructed by supposing these columns is $\mathrm{A}^{\mathrm{T}}$

$$\mathrm{A}^{\mathrm{T}} = [\boldsymbol{b} \mid \bar{\boldsymbol{b}} \mid \dot{\boldsymbol{x}} \mid \dot{\boldsymbol{y}} ] = \begin{bmatrix} b_1 & \bar{b}_1 & \dot{x}_1 & \dot{y}_1 \\ b_2 & \bar{b}_2 & \dot{x}_2 & \dot{y}_2 \\ b_3 & \bar{b}_3 & \dot{x}_3 & \dot{y}_3 \\ b_4 & \bar{b}_4 & \dot{x}_4 & \dot{y}_4 \end{bmatrix} .$$

We shall also need the explicit form of the inverse of the matrix of Minkowski metric:

$$g = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad \Rightarrow \quad G \equiv g^{-1} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{3.5}$$

Now, we see that Theorem 3.1 may equivalently be stated:

**Corollary 3.2:** Let $\mathbf{v}_i$ , $i$ =1,…4, be one of the four vectors: $\boldsymbol{b}$, $\bar{\boldsymbol{b}}$ , $\dot{\boldsymbol{x}}$ , or $\dot{\boldsymbol{y}}$ . Then

$$\mathbf{v}_i^T F\, \mathbf{v}_j \;=\; G_{ij}$$

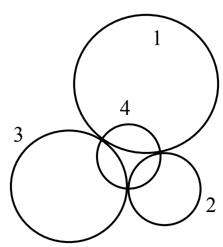
In particular, for the vector of curvatures and reduced positions we have these handy quadratic formulas (given also in the indexed form) :

$$\begin{aligned} \dot{\boldsymbol{x}}^T F\, \dot{\boldsymbol{x}} &= -1 \quad \text{or} \quad \dot{x}_i F_{ij}\, \dot{x}_j = -1, \\ \dot{\boldsymbol{y}}^T F\, \dot{\boldsymbol{y}} &= -1 \quad \text{or} \quad \dot{y}_i F_{ij}\, \dot{y}_j = -1, \\ \mathbf{b}^T F\, \mathbf{b} &= 0 \quad \text{or} \quad b_i F_{ij}\, b_j = 0, \end{aligned} \tag{3.6}$$

were the indices label the four circles, $i,j$ = 1,…4, and where summation over repeated indices is understood. The last equation is a Descartes-like formula —generalized to arbitrary independent circle configurations.

Let us see the theorem in action.

**Example 3.3 [the circle of inversive symmetry]:** Suppose there are given three pairwise externally tangent circles of curvatures $b_1=a$, $b_2=b$, and $b_3=c$. A fourth circle of curvature $b_4=d$ is orthogonal to each of them.



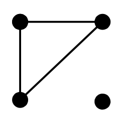



**Figure 3.1:** Circle configuration for Example 3.3. The symbol to the right of the configuration will be explained later.

We set the configuration matrix $F$ and calculate its inverse:

$$f = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad F \equiv f^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Circle configuration theorem gives here

$$\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 & \bar{b}_4 \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_3 & \dot{x}_4 \\ \dot{y}_1 & \dot{y}_2 & \dot{y}_3 & \dot{y}_4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} b_1 & \bar{b}_1 & \dot{x}_1 & \dot{y}_1 \\ b_2 & \bar{b}_2 & \dot{x}_2 & \dot{y}_2 \\ b_3 & \bar{b}_3 & \dot{x}_3 & \dot{y}_3 \\ b_4 & \bar{b}_4 & \dot{x}_4 & \dot{y}_4 \end{bmatrix} = 2\begin{bmatrix} 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

(We multiplied both sides by the factor of 2). The formula for curvatures (an analogue of the Descartes formula) may be read off from the left-top entries:

$$d^2 - ab - ac - ca = 0,$$

which resolves into a well-known formula:

$$d = \pm\sqrt{ab + bc + ca} \ ,$$

or $r_4^{\,2} = r_1r_2r_3 /(r_1+r_2+r_3)$. The theorem predicts also the position of the center of the orthogonal circle in terms of the data of the other circles:

$$\dot{x}_4^2 = \dot{x}_1\dot{x}_2 + \dot{x}_2\dot{x}_3 + \dot{x}_3\dot{x}_1 - 1 \qquad \text{or} \qquad x_4^{\,2} = \frac{r_1x_2x_3 + r_2x_3x_1 + r_3x_1x_2 - r_1r_2r_3}{r_1 + r_2 + r_3},$$

and similarly for $y_4$. Thus the equations fully determine the fourth circle.

An immediate implication of the theorem this a geometric fact:

**Corollary 3.4:** There is no configuration of four mutually perpendicular circles.

**Proof:** The configuration matrix for such an arrangement of circles is $F = I = \text{diag}(1,\ldots,1)$. The "master equation" (3.3) becomes thus

$$AA^{\mathrm{T}} = G\,. \tag{3.6}$$

This is impossible, for the left side is positive definite, while the right side is not. Restating it: Equation (3.3) is a congruence relation, but $F=I$ is not congruent to $G$ since they have different Sylvester's moments, (+, +, +, +) versus (–, +, +, +). □

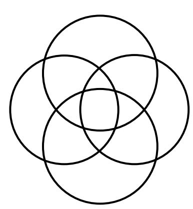


**Figure 3.2:** A failed attempt to draw four mutually orthogonal circles.

This of course is a special case of a more general statement:

**Corollary 3.5:** If the configuration matrix of a hypothetical arrangement of four circles would have eigenvalues different than one strictly positive and three strictly negative, then the configuration cannot be realized.

**Remark on the interpretation of the formulas:** The formula (3.4) has a clear geometric meaning: The four circles $C_1,\ldots,C_4$ — if linearly independent — define a new basis in the Minkowski space. Then $A^{\mathrm{T}}gA = f$ represents the congruence of matrices $g \sim f$, *i.e.,* the transformation of Minkowski metric matrix under a change of basis, where

$g$ = matrix of the Pedoe quadratic form in the standard (Pedoe) basis,
$f$ = matrix of the same Pedoe quadratic form, expressed in the basis made by the four circles.

Now, in order to organize this equation in terms of the curvature vector **b** and (reduced) positions, which correspond to the rows of $A$, we need to move to the *dual* Minkowski space. The "master formula" of Theorem 3.1 does exactly this: it represents congruency of the induced metric matrices $F$ and $G$ in the *dual space*. The columns of matrix $A$ originate as vectors, while the rows, including $\mathbf{b} = [b_1, b_2, b_3, b_4]$, correspond to covectors, representing the dual basis to that of the basis given by the circles.

# 4.  More examples and new configuration families

The Descartes configuration is but one of many families of configurations.  To exemplify the point, let us have a look at two such families (they extend the first two examples of the last section).

**Example 1: Descartes configuration revisited.**

Among the special cases of Theorem 3.1 is the original Descartes formula and its extension discovered by J. Lagarias *et al*. [LMW].  Let us see some details.  Four circles are said to be in a *Descartes configuration*, if the pairs of the circles are mutually tangent in distinct points (see Figure 4.1 for possible arrangements).

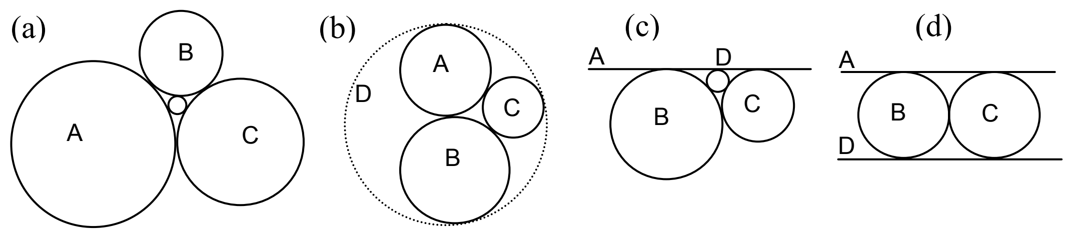



**Figure 4.1**:  Four circles of Descartes configuration.

First, consider the "all-external-tangencies" configuration (Fig. 4.1a).  Construct the configuration matrix *f* and calculate its inverse *F*:

$$f=\begin{bmatrix}-1&1&1&1\\1&-1&1&1\\1&1&-1&1\\1&1&1&-1\end{bmatrix}\qquad F\equiv f^{-1}=\frac{1}{4}\begin{bmatrix}-1&1&1&1\\1&-1&1&1\\1&1&-1&1\\1&1&1&-1\end{bmatrix}.$$

(Notice that $f^2 = 4I$, hence $f^{-1} = 1/4\, f$ readily follows).  Thus our master equation $\mathrm{AFA}^{\mathrm{T}} = \mathrm{G}$ is —after a convenient multiplying both sides by 4 — simply:

$$\begin{bmatrix}b_1&b_2&b_3&b_4\\\bar b_1&\bar b_2&\bar b_3&\bar b_4\\\dot x_1&\dot x_2&\dot x_3&\dot x_4\\\dot y_1&\dot y_2&\dot y_3&\dot y_4\end{bmatrix}\underbrace{\begin{bmatrix}-1&1&1&1\\1&-1&1&1\\1&1&-1&1\\1&1&1&-1\end{bmatrix}}_{4f^{-1}}\begin{bmatrix}b_1&\bar b_1&\dot x_1&\dot y_1\\b_2&\bar b_2&\dot x_2&\dot y_2\\b_3&\bar b_3&\dot x_3&\dot y_3\\b_4&\bar b_4&\dot x_4&\dot y_4\end{bmatrix}=\underbrace{\begin{bmatrix}0&8&0&0\\8&0&0&0\\0&0&-4&0\\0&0&0&-4\end{bmatrix}}_{4g^{-1}}\tag{4.1}$$

Matrix $D = 4\mathrm{f}^{-1}$ (consisting of negative ones on the diagonal and ones everywhere else) is often called the Descartes matrix.  The particular vector equation $\mathbf{b}^{\mathrm{T}}\mathrm{F}\,\mathbf{b} = \mathrm{G}$, part of (4.1), is equivalent to the original Descartes law:

$$a^2 + b^2 + c^2 + d^2 - 2ab - 2ac - 2ad - 2bc - 2bd - 2cd = 0$$

or
$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2 \tag{4.2}$$

In constructing matrix *f*, we have assumed that all circles are tangent externally.  Let us now assume that one, say the fourth, circle contains the other three, like in Figure 4.1b.  Then matrix *f* and its inverse *F* are:

$$f = \begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \quad \Rightarrow \quad F = \frac{1}{4}\begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix} = RDR\,,$$

where $D$ is the Descartes matrix, and $R = \mathrm{diag}(1, 1, 1, -1)$. The Descartes formula for curvatures becomes

$$\mathbf{b}^{\mathrm{T}}(RDR)\mathbf{b} = 0,$$

or

$$(R\mathbf{b})^{\mathrm{T}} D\,(R\mathbf{b}) = 0.$$

This explains why the convention that the circle that contains the other circles is assumed to have a negative sign (hence the notion of the "bend" as a "signed curvature"). This allows one to have a single formula for all cases in Figure 4.1.

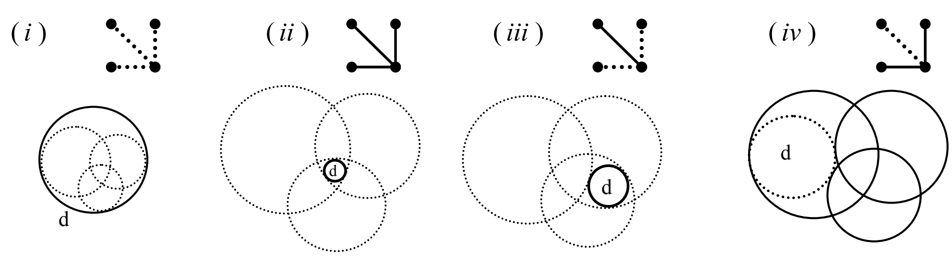
**Corollary 4.2: (Extended Descartes Theorem, [MLW])** Four circles in Descartes configuration with possibly both types of tangencies, internal and external, satisfy the generalized Descartes theorem (4.1) if one assumes that $b$ represents bends.

(Equation (4.1) has been obtained in [LMW] by different means.)

It might be a point of surprise that this convention must be replaced by other agreements, if a single formula is sought for the families of circle configurations other than the Descartes configuration. The notion of bend is **not** of a universal nature (see the following examples).

### Example 2. Beyond Descartes configuration

Let $a$, $b$, and $c$ be three pair-wise orthogonal circles and let $d$ be tangent to each of them. There are four (and algebraically three) distinct realizations, presented here in columns:




$$f = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \qquad f = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \qquad f = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix} \qquad f = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

$$F = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \quad F = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad F = \frac{1}{2}\begin{bmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad F = \frac{1}{2}\begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$$

The quadratic relations may be read from these matrices. For the curvatures, one may arrive at these concise forms:

(i) $2(a^2 + b^2 + c^2) = (-a - b - c + d)^2$ (ii) $2(a^2 + b^2 + c^2) = (a - b - c + d)^2$

(iii) $2(a^2 + b^2 + c^2) = (a + b + c + d)^2$ (iv) $2(a^2 + b^2 + c^2) = (-a + b + c + d)^2$

[Legend to the symbols: dots represent the four circles, thick lines — external tangency, dotted lines — internal tangency, and no line — orthogonality].

As these formulas resemble each other, to account for the varying sign this unification may be proposed:

**Theorem 4.1:** Let three circles of curvature $a$, $b$, and $c$, be mutually orthogonal, and let the fourth circle of curvature $d$ be tangent to each of them. Then

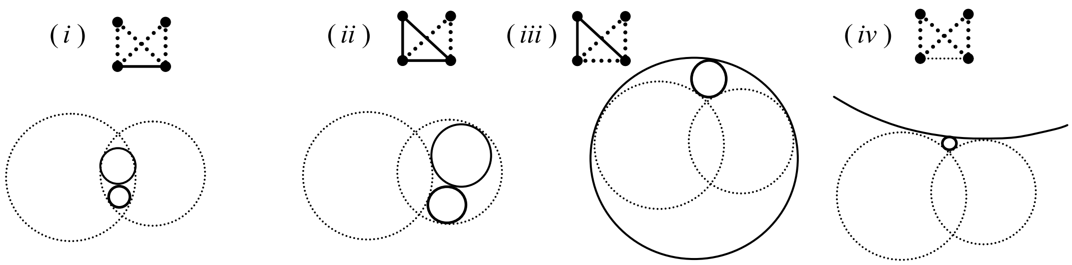
$$2(a^2 + b^2 + c^2) \;=\; (a + b + c + d)^2$$

under the convention that if any of the mutually tangent circles, $a$, $b$ or $c$, contains or is contained in $d$, then its curvature is negative.

Note that there will always be some negative values among $a$, $b$, and $c$. If the fourth circle curvature is sought, one gets one of the formulas $d = \pm a \pm b \pm c \pm \sqrt{2(a^2+b^2+c^2)}$ .

**Example 3 Beyond Descartes configuration again**

Let $a$ and $b$ be a couple of orthogonal circles. Add two mutually tangent circles $c$ and $d$ that are also tangent to each $a$ and $b$. Then we have these four cases:




$$f = \begin{bmatrix} -1 & 0 & -1 & -1 \\ 0 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \end{bmatrix} \qquad f = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \qquad f = \begin{bmatrix} -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \qquad f = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

$$F = \frac{1}{8}\begin{bmatrix} -4 & 4 & -2 & -2 \\ 4 & -4 & -2 & -2 \\ -2 & -2 & -1 & 3 \\ -2 & -2 & 3 & -1 \end{bmatrix} \qquad F = \frac{1}{8}\begin{bmatrix} -4 & -4 & 2 & 2 \\ -4 & -4 & -2 & -2 \\ 2 & -2 & -1 & 3 \\ 2 & -2 & 3 & -1 \end{bmatrix} \qquad F = \frac{1}{8}\begin{bmatrix} -4 & 4 & 2 & -2 \\ 4 & -4 & 2 & -2 \\ 2 & 2 & -1 & -3 \\ -2 & -2 & -3 & -1 \end{bmatrix} \qquad F = \frac{1}{8}\begin{bmatrix} -4 & 4 & 2 & 2 \\ 4 & -4 & 2 & 2 \\ 2 & 2 & -1 & 3 \\ 2 & 2 & 3 & -1 \end{bmatrix}$$

(i) $2[(2a)^2 + (2b)^2 + (c-d)^2\,] \;=\; (\,-2a - 2b + c + d)^2$

(iii) $2[(2a)^2 + (2b)^2 + (c+d)^2\,] \;=\; (2a + 2b + c - d)^2$

(ii) $2[(2a)^2 + (2b)^2 + (c-d)^2\,] \;=\; (2a - 2b + c + d)^2$

(iv) $2[(2a)^2 + (2b)^2 + (c-d)^2\,] \;=\; (2a + 2b + c + d)^2$

We encounter the same problem with signs as in Example 2. (Note that seeking the signs under the squares may be changed as one seeks the unifying formula for all four cases). Here is a solution:

**Theorem 4.2:** Let two circles $C_1$ and $C_2$ of curvature $a$ and $b$ be mutually orthogonal, and let the second pair of circles $C_3$ and $C_4$ of curvature $c$ and $d$ be mutually tangent. Then

$$2[(2a)^2 + (2b)^2 + (c-d)^2\,] \;=\; (2a + 2b + c + d)^2$$

under the convention that

(i) if any of the mutually tangent circles, $C_3$ and $C_4$, contains the pair of perpendicular circles $C_1$ and $C_2$, then its curvature is negative;

(ii) if any of the perpendicular circles, $C_1$ and $C_2$, contains the pair of the tangent circles $C_3$ and $C_4$, then its curvature is negative;

We arrive at this important conclusion:

**Remark 4.3:** The case of the Descartes configuration prompted the introduction of the notion of *bend* as a signed curvature — this allowed one to write a set of formulas in a single equation, see Sec 4. But the particular definition of "bend" — so convenient in Descartes theorem and for Apollonian gaskets — is not universal! It is native to the Descartes Theorem, and a different set of rules for the signs of "bends" may emerge in other configurations. (See also Remark 5.4).

## Example 4: A solution to Apollonius' problem

The problem of Apollonius is to find a circle that is simultaneously tangent to three geometric objects. If we choose these objects to be three circles, we may expect eight possible solutions, which differ by the type of each of the tangency, external versus internal (see Fig. 4.2). A geometric (constructive) solution is known (see [Coxt68]). Let us see how Theorem 3.3 provides analytic solutions.

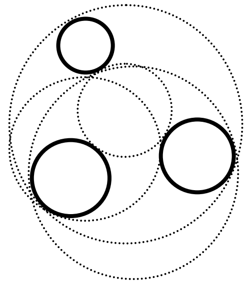


**Figure 4.2:** Four (from among eight) solution to the Apollonian problem.

For the Apollonian problem, the configuration matrix is of the following type:

$$f = \begin{bmatrix} -1 & * & * & \pm 1 \\ * & -1 & * & \pm 1 \\ * & * & -1 & \pm 1 \\ \pm 1 & \pm 1 & \pm 1 & -1 \end{bmatrix}$$

where the last column and the last row are combinations of +1/–1, depending on which of the eight solutions we seek ($2^3=8$). More precisely, define coefficients that represent the configuration of the three given circles:

$$\varphi_{ij} = \frac{d_{ij}^2 - r_i^2 - r_j^2}{2 r_i r_j} = \frac{1}{2}\left( d_{ij}^2 b_i b_j - \frac{b_i^2 + b_j^2}{b_i b_j} \right),$$

where $d_{ij}$ denotes the distance between the centers of the i-th and j-th circle, $d_{ij} = d_{ji}$, $r_i$ denotes the radius of the *i*-th circle, and $b_i = 1/r_i$ is the corresponding curvature, $i,j = 1,\ldots, 3$. Then the configuration matrix is:

$$f = \begin{bmatrix} -1 & \varphi_{12} & \varphi_{13} & \varepsilon_1 \\ \varphi_{12} & -1 & \varphi_{23} & \varepsilon_2 \\ \varphi_{13} & \varphi_{23} & -1 & \varepsilon_3 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & -1 \end{bmatrix}$$

where the terms $\varepsilon_n = \pm 1$ correspond to cases wher the *i*-th circle *is* or *is not* contained in the circle-solution, $i$ = 1, 2, 3. Finding the inverse matrix $F = f^{-1}$ and applying Theorem 3.3 gives the solution of the Apollonian problem. In particular, the quadratic equation for the curvatures is $\mathbf{b}^T F\, \mathbf{b} = 0$, where, as usual, $\mathbf{b} = [b_1, b_2, b_3, b_4]^T$. This needs to be solved for $b_4$, as $b_1$, $b_2$ and $b_3$ are known.

## 5. Multidimensional formulation

Consider the set $\Omega(\mathbf{E})$ of $(n-1)$-spheres in an $n$-dimensional Euclidean space $(\mathbf{E}, g_0)$, where $\mathbf{E} \cong \mathbf{R}^n$ and where $g_0$ represents the Euclidean metric. A sphere $C_r(\boldsymbol{p})$ of radius $r \in \mathbf{R}$ centered at $\mathbf{p} \in \mathbf{E}$ is given by equation

$$(\boldsymbol{x} - \boldsymbol{p})^2 = r^2 \tag{5.1}$$

Let $\mathbf{R}^{1,1}$ be a standard pseudo-Euclidean 2-dimensional space with metric $\omega$ given by

$$\omega = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$$

Define a Minkowski space $(\mathbf{M}, g)$ by these direct products:

$$\mathbf{M} = \mathbf{E} \oplus \mathbf{R}^{1,1} \cong \mathbf{R}^{1,n+1}, \quad g = g_0 \oplus \omega$$

**Definition 5.1:** Pedoe map $\pi$ sends $(n-1)$-spheres into rays of the (n+2)-dimensional Minkowski space $\mathbf{R}^{1,n+1} \cong \mathbf{R}^{1,1} \oplus \mathbf{R}^n$ (elements of projective space):

$$\pi\colon \Omega(\mathbf{R}^n) \to \mathrm{P}\mathbf{R}^{1,n+1}$$

where, for a sphere $C$ described in Eq. 8.1, the ray is spanned by vector

$$\pi(C) = \begin{cases} \text{span}\{\, [\,1,\ r^2 - \boldsymbol{p}^2, \boldsymbol{p}\,]^T \} & \text{if } C \text{ is not a plane, } r \neq 0 \\ \text{span}\{\, [\,0,\ c/2,\ \boldsymbol{q}\,]^T \} & \text{if } C \text{ is a plane } \boldsymbol{q} \cdot \boldsymbol{x} = c \\ \text{span}\{\, [\,1, -p^2,\ \boldsymbol{p}\,]^T \} & \text{if } C \text{ is a point } \boldsymbol{p} \text{ (improper sphere)} \end{cases}$$

In the standard basis, the Minkowski metric $g$ in $\mathbf{M}$ and its inverse are represented by (n+3)×(n+3) matrices:

$$g = \left[\begin{array}{cc|ccc} & 1/2 & & & \\ 1/2 & & & & \\ \hline & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{array}\right] \qquad G = g^{-1} = \left[\begin{array}{cc|ccc} & 2 & & & \\ 2 & & & & \\ \hline & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{array}\right]$$

**Special Pedoe map** sends proper spheres to normed future-oriented vectors

$$\dot{\pi} : \Omega(\mathbf{E}) \to \mathbf{M}$$

so that (1) $|\dot{\pi}(C)|^2 = -1$, Vector $\mathbf{C} = \dot{\pi}(C)$ is called the **Pedoe vector** of sphere $S$. Im($\dot{\pi}$) lies in a hyperboloid $H \subset \mathbf{R}^{1,n+1}$ of space-like unit vectors.

Let $A = [\,\dot{\pi}(C_1) \mid \dot{\pi}(C_2) \mid \ldots \mid \dot{\pi}(C_{n+3})\,]$ be an $(n+2)\times(n+2)$ matrix whose columns are Pedoe vectors of a set of $n$+2 distinct $(n-1)$-spheres. The central result is:

**Theorem 5.2:** Define the **configuration matrix** for a system of n+3 spheres to be: $f = A^T g A$. If $f$ is invertible, then $AFA^T = G$, where $F = f^{-1}$ and $G = g^{-1}$. In particular, the curvatures and positions of $n$+1 spheres determine the remaining sphere(s) of the configuration by the quadratic relation:

$$\boldsymbol{v}_i^T F \boldsymbol{v}_j = G_{ij}\,,$$

where we define $n$+2 vectors $\mathbf{v}_i$ , $i$ = 1,…,n+2, as follows:

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{b} = [b_1,\ldots,b_n]\,, && \text{where } \mathbf{b_i} = 1/r_i \text{ is the curvature of the i-th sphere;}\\
\mathbf{v}_2 &= \bar{\boldsymbol{b}} = [\,\bar{b}_1,\ldots,\bar{b}_{n+2}\,], && \text{where } \bar{b}_i \text{ is the co-curvature of the i-th sphere;}\\
\mathbf{v}_{2+k} &= \dot{\boldsymbol{x}}_k = [\,\dot{x}_{k1},\ldots,\dot{x}_{k\,n+2}\,], && \text{where } \dot{x}_{ki} = x_{ki}/r_i \text{ and } x_{ki} \text{ is the } k^{\text{th}} \text{ coordinate of the center}\\
& && \text{of the } i^{\text{th}} \text{ sphere.}
\end{aligned}$$

**Proof:** Follow the same few steps as in the proof of Theorem 3.1. □

Note that the second entry of the co-curvature vector are determined by the remaining entries by the condition of normalization $\dot{\pi}(C) = -1$.

**Example 5.3:** For instance, in case of the Descartes configuration, i.e., a system of $n$+3 mutually externally tangent $n$-spheres, the corresponding Pedoe vectors form in $\mathbf{R}^{1,\,n+1}$ a system of linearly independent vectors satisfying

$$\langle \mathbf{C}_i{\cdot}\mathbf{C}_j\rangle \;=\; \begin{cases} -1 & \textit{if } i=j \\ \;\;1 & \textit{otherwise.}\end{cases}$$

The configuration matrix f is an (n+2)×(n+2) matrix with –1 on the diagonal and 1's everywhere else, that is

$$f \;=\; N - 2\,I,$$

where $N$ is a matrix with 1 in every entry, and $I$ is the unit matrix. Its inverse is $F = (1/2n)\,(N - nI)$, thus, after multiplying both sides of (…) by 2$n$, we get:

$$\left[\begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & \cdots \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 & \bar{b}_4 & \cdots \\ \hdashline \dot{x}_1 & \dot{x}_2 & \dot{x}_3 & \dot{x}_4 & \cdots \\ \dot{y}_1 & \dot{y}_2 & \dot{y}_3 & \dot{y}_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right]
\left[\begin{array}{ccccc} 1-n & 1 & 1 & 1 & \cdots \\ 1 & 1-n & 1 & 1 & \cdots \\ 1 & 1 & 1-n & 1 & \cdots \\ 1 & 1 & 1 & 1-n & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right]
\left[\begin{array}{cc:ccc} b_1 & \bar{b}_1 & \dot{x}_1 & \dot{y}_1 & \cdots \\ b_2 & \bar{b}_2 & \dot{x}_2 & \dot{y}_2 & \cdots \\ b_3 & \bar{b}_3 & \dot{x}_3 & \dot{y}_3 & \cdots \\ b_4 & \bar{b}_4 & \dot{x}_4 & \dot{y}_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right]
\;=\; 2n \left[\begin{array}{cc:ccc} & 2 & & & \\ 2 & & & & \\ \hdashline & & -1 & & \\ & & & -1 & \\ & & & & \ddots \end{array}\right]$$

In particular, for the curvatures only, this results in

$$(b_1 + b_2 + \ldots + b_{n+2})^2 \;\;=\;\; n\,(b_1^{\,2} + b_2^{\,2} + \ldots + b_{n+2}^{\;\;2}\,),$$

which agrees with the formula discovered by Soddy [Sod] for $n$ = 3, and generalized to arbitrary dimension by Gossett [Gos].

**Remark 5.4 (on bends, circles and disks):** In order to account for different cases of Descartes configuration, it is customary to introduce the notion of a *bend* of a circle: the curvature with a sign that is negative if the circle contains the other three circles (see e.g., [LMW]). We propose an alternative solution: replace circles by generalized disks. Any circle ($n$-sphere) is a boundary of two discs, bounded and unbounded (see Figure 5.1). The radius of unbounded is negative. Then each case of Descartes configurations may be represented in terms *external* tangencies, as in Figure 5.2 (although not necessarily). The definition of the Pedoe map $\dot{\pi}$ extends to such discs: if $D$ and $D'$ are mutual complements, then we define $\dot{\pi}(D') = -\dot{\pi}(D)$ .

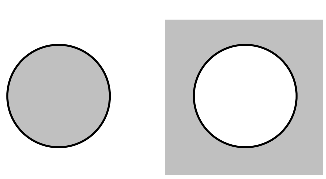


**Figure 5.1:** Bounded and unbounded disc

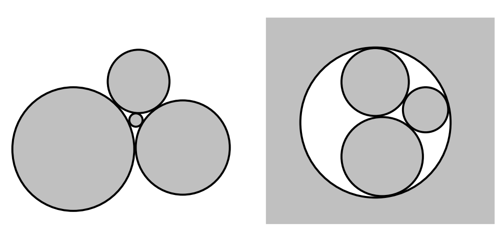


**Figure 5.2:** Descartes configurations reconsidered

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