NORM PRINCIPLES FOR FORMS OF HIGHER DEGREE PERMITTING COMPOSITION

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Abstract. Let F be a field of characteristic 0 or greater than d. Scharlau's norm principle holds for finite field extensions K over F, for certain forms φ of degree d over F which permit composition.

Introduction

Let $d \geq 2$ be an integer and let F be a field of characteristic 0 or $> d$. Let $\varphi : V \to F$ be a form of degree d on an F-vector space V of dimension n (i.e., after suitable identification, φ is a homogeneous polynomial of degree d in n indeterminates). Let K/F be a finite field extension of degree m. Scharlau's norm principle (SNP) says that if a is a similarity factor of φ_K, then $N_{K/F}(a)$ is a similarity factor of φ. Knebusch's norm principle (KNP) states that if a is represented by φ_F, then $N_{K/F}(a)$ is a product of m elements represented by φ, hence lies in the subgroup of $F^\times$ generated by $D_F(\varphi)$. Both norm principles were proved for nondegenerate quadratic forms over fields of characteristic not 2 (cf. [Sch, II.8.6] or [L, p. 205, p. 206]). For finite extensions of semi-local regular rings containing a field of characteristic 0, Knebusch's norm principle (for quadratic forms) was proved in [Z] and for finite étale extensions of semi-local Noetherian domains with infinite residue fields of characteristic different from 2 in [O-P-Z]. Barquero and Merkurjev [B1,2], [B-M] generalized the norm principle to algebraic groups.

We prove Scharlau's norm principle for certain nondegenerate forms φ of degree $d \geq 3$ which permit composition. Scharlau's and Knebusch's norm principle "coincide" for these forms, since they permit composition in the sense of Schafer [S] and thus satisfy $D_K(\varphi) = G_K(\varphi)$ for all field extensions K/F. We explicitly compute the norms of some similarity factors, if φ is the norm of an étale algebra over F or of a central simple algebra.

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1. Preliminaries

A *form of degree* d over F is a map $\varphi : V \to F$ on a finite-dimensional vector space V over F such that $\varphi(av) = a^d\varphi(v)$ for all $a \in F$, $v \in V$ and such that the map $\theta : V \times \cdots \times V \to F$ (d-copies) defined by

$$\theta(v_1, \dots, v_d) = \frac{1}{d!} \sum_{1\leq i_1<\cdots<i_l\leq d} (-1)^{d-l}\varphi(v_{i_1} + \cdots + v_{i_l})$$

(with $1 \leq l \leq d$) is F-multilinear and invariant under all permutations of its variables. The *dimension* of φ is defined as $\dim\varphi = \dim V$. φ is called *nondegenerate*, if $v = 0$ is the only vector such that $\theta(v, v_2, \dots, v_d) = 0$ for all $v_i \in V$. We will only study nondegenerate forms. Forms of degree d on V are in obvious one-one correspondence with homogeneous polynomials of degree d in $n = \dim V$ variables. If φ is represented by $a_1x_1^d + \ldots + a_mx_m^d$ ($a_i \in F^\times$), we use the notation $\varphi = \langle a_1, \dots, a_n\rangle$ and call φ *diagonal.*

Two forms (V_i, φ_i) of degree d, $i = 1, 2$, are called *isomorphic* (written $(V_1, \varphi_1) \cong (V_2, \varphi_2)$ or just $\varphi_1 \cong \varphi_2$) if there exists a bijective linear map $f : V_1 \to V_2$ such that $\varphi_2(f(v)) = \varphi_1(v)$ for all $v \in V_1$.

Let (V, φ) be a form over F of degree d in n variables over F. An element $a \in F$ is *represented by* φ if there is an $v \in V$ such that $\varphi(v) = a$. An element $a \in F^\times$ such that $\varphi \cong a\varphi$ is called a *similarity factor* of φ. Write $D_F(\varphi) = \{a \in F^\times \mid \varphi(x) = a \text{ for some } x \in V\}$ for the set of non-zero elements represented by φ over F and $G_F(\varphi) = \{a \in F^\times \mid \varphi \cong a\varphi\}$ for the group of similarity factors of φ over F. The subscript F is omitted if it is clear from the context that φ is a form over the base field F. φ is called *round* if $D(\varphi) \subset G(\varphi)$.

A nondegenerate form $\varphi(x_1, \dots, x_n)$ of degree d in n variables *permits composition* if $\varphi(x)\varphi(y) = \varphi(z)$ where x, y are systems of n indeterminates and where each z_l is a bilinear form in x, y with coefficients in F. In this case the vector space $V = F^n$ admits a bilinear map $V \times V \to V$ which can be viewed as the multiplicative structure of a nonassociative F-algebra and $\varphi(vw) = \varphi(v)\varphi(w)$ holds for all $v, w \in V$. Note that the form φ here is nondegenerate if and only if the underlying (automatically alternative) F-algebra is separable (Schafer [S]). For instance, every norm of a central simple algebra or of a separable finite field extension over F is nondegenerate and permits composition.

Remark 1. (i) There are two types of forms φ of degree d over F for which SNP trivially holds:
(a) if $G_F(\varphi) = F^\times$;
(b) if $G_K(\varphi) = K^{\times d}$ for every field extension K over F.
(ii) Let φ be a diagonal form over F of degree $d \geq 3$. If $\dim\varphi = 1$ or $\dim\varphi \in \{sd + 1, sd - 1\}$ for some integer $s \geq 1$, then $G_K(\varphi) = K^{\times d}$ for every finite field

extension K over F [Pu1, Proposition 1 (i)]. Hence φ trivially satisfies SNP for all field extensions K over F by (i). Moreover, every form $\langle a, a, \dots, a\rangle$ of degree $d \geq 3$ satisfies $G_K(\varphi) = K^{\times d}$ for all field extensions K over F [Pu1, Lemma 9 (ii)], hence SNP.
(iii) If φ is the determinant of the d-by-d matrices over F, then $G_K(\varphi) = K^\times$ for all field extensions K over F, hence SNP holds for all field extensions of F by (i).
(iv) The cubic norm φ of a reduced Freudenthal algebra $J = H_3(C, \Gamma)$, C a composition algebra over F or 0 [KMRT, p. 516], trivially satisfies SNP for all field extensions K of F, because $D_K(\varphi) = G_K(\varphi) = K^\times$.
(v) Suppose the base field F has characteristic 0 or greater than $d+1$. Let $\varphi_0 : V \to F$ be a form of degree d, then the form $\varphi(a + u) = a\varphi_0(u)$, $a \in F$, $u \in V$ of degree $d + 1$ satisfies $G_K(\varphi) = D_K(\varphi) = K^\times$ for all field extensions K over F, hence SNP.

Remark 2. (i) Let φ be a form of degree d over F. Let K/F be a finite field extension. Suppose we have $a\varphi_K \cong \varphi_K$ for some $a \in K^\times$.
(a) If $[K : F(a)] = dm$ then a straightforward calculation shows that $N_{K/F}(a) \in F^{\times d} \subset G(\varphi)$.
(b) If $a \in F$ then trivially $N_{K/F}(a) \in F^{\times d} \subset G(\varphi)$.
(ii) Let φ be a form of prime degree p over F. Then SNP holds for φ for all field extensions of degree p^r for some integer $r > 0$ by (a).

2. Forms satisfying Scharlau's norm principle

2.1. **Norms of étale algebras.** Let R be a unital commutative ring. Suppose that A is a finitely generated unital commutative associative R-algebra which is free as an R-module. For $a \in A$ we define the norm $N_{A/R}(a)$ to be the determinant of the regular representation $x \to ax$. If B is a finitely generated unital commutative associative A-algebra which is free as an A-module, then B is a finitely generated commutative R-algebra which is free as an R-module and

$$N_{B/R} = N_{A/R} \circ N_{B/A}. \tag{1}$$

This transitivity of norms follows from the general transitivity of determinants, see for instance [J, p. 406] or [Bou, p. 548].

In this subsection, let F be a field of arbitrary characteristic (that is, we drop our standing assumptions on char(F)).

Theorem 1. *Let L be an étale algebra over F and its norm $\varphi = N_{L/F}$ of degree d. Suppose that K/F is a finite field extension. If $e \in K^\times$ is represented by φ_K, then $N_{K/F}(e)$ is represented by φ and thus*

$$N_{K/F}(G_K(\varphi_K)) \subset G_F(\varphi).$$

Proof. Since L is an étale algebra over F, there are finite separable field extensions $K_1, \ldots, K_r$ of F such that
$$L \cong K_1 \times \cdots \times K_r.$$
For all field extensions K/F, $D_K(\varphi_K) = G_K(\varphi_K)$ [Pu2, Proposition 6]. Set $L_K = K \otimes_F L$, and note that $\varphi_K = N_{L_K/K}$ [Bou, p. 544]. Let $u_1, \ldots, u_d$ be an F-basis of L. If $e\varphi_K \cong \varphi_K$, then $e = \varphi_K(z_1, z_2, \ldots, z_d)$ with $z_i \in K$ and using equation (1) we obtain
$$\begin{aligned}
&N_{K/F}(\varphi_K(z_1, z_2, \ldots, z_d)) = \\
&N_{K/F}(N_{L_K/K}(z_1 \otimes u_1 + z_2 \otimes u_2 + \cdots + z_d \otimes u_d))) = \\
&N_{L/F}(N_{L_K/L}(z_1 \otimes u_1 + z_2 \otimes u_2 + \cdots + z_d \otimes u_d)) = \\
&N_{L/F}(a_1u_1 + a_2u_2 + \cdots + a_du_d) = \\
&\varphi(a_1, a_2, \ldots, a_d) \in G_F(\varphi)
\end{aligned}$$
for suitable $a_i \in F$. □

This simple trick which even gives an explicit identity for $N_{K/F}(e)$ in terms of the a_i's, was used in [F] to compute norms for the quadratic form $\langle 1, 1\rangle$.

Corollary 1. *Let $\widetilde{F} = F(\alpha)$ be a field extension of F of degree d and $\varphi = N_{\widetilde{F}/F}$. Suppose that K/F is a finite field extension. If $e \in K^\times$ is represented by φ_K, then $N_{K/F}(e)$ is represented by φ and thus*
$$N_{K/F}(G_K(\varphi_K)) \subset G_F(\varphi).$$

2.2. **Norms of central simple algebras.** We now turn to the (reduced) norm forms of central simple algebras over F. Let $\varphi = N_{A/F}$ be the norm of a central simple algebra A of degree d over F. Then SNP holds for all finite separable field extension [B-M, 3.1]. For the split central simple algebra $A \cong \mathrm{Mat}_d(F)$, φ trivially satisfies SNP for all field extensions of F by Remark 1 (iii).

If A is a division algebra then SNP holds for all finite field extensions:

Let K/F be a finite field extension of degree n. For $\alpha \in F$, $\rho_\alpha : K \to K$, $\rho_\alpha(x) = \alpha x$ is left multiplication with α. Fix a basis $B = \{w_1, w_2, \ldots, w_n\}$ of K/F. Let $\rho(\alpha)$ be the matrix representation of ρ_α with respect to B. The map $\rho : K \to M_n(F)$ is an injective ring homomorphism and the norm is given by $N_{K/F}(\alpha) = \det \rho(\alpha)$.

Let A be a central simple algebra over F. Pick $\Delta = \sum_{i=1}^n \alpha_i w_i$, where $\alpha_i \in A$ and so $\Delta \in \bar{A} = A \otimes K$. Again, $\rho_\Delta : \bar{A} \to \bar{A}$ is left multiplication and $\rho(\Delta)$ is the matrix, with entries in A, of ρ_Δ with respect to B. For the proof of the next theorem we need the following observation:

Lemma 1. $\rho(\Delta) = \sum_{i=1}^n \alpha_i \rho(w_i)$.

Proof. Let $a \in \bar{A}$. Then $\rho_\Delta(a) = \sum \alpha_i w_i a = \sum \alpha_i \rho_{w_i}(a)$. Hence $\rho_\Delta = \sum \alpha_i \rho_{w_i}$ and for matrices $\rho(\Delta) = \sum \alpha_i \rho(w_i)$. □

Let A be a central simple division algebra over F with basis $\epsilon_1, \ldots, \epsilon_m$. Let $A^\times$ be the invertible elements in A and $C(A^\times) = [A^\times, A^\times]$ be the commutator subgroup. Put $\bar{A} = A \otimes_F K$. Let $\det : \mathrm{GL}_n A \to A^\times / C(A^\times)$ be the Dieudonné determinant. There is a polynomial $G \in F[x_1, \ldots, x_m]$ such that for any extension L/F the norm from $A \otimes L \to L$ is given by

$$N(\sum_{i=1}^{m} l_i \epsilon_i) = G(l_1, \ldots, l_m).$$

We write

$$G(* \, l_k *) \text{ for } G(l_1, \ldots, l_k, \ldots, l_m).$$

Theorem 2. *Let A be a central simple division algebra over F. Let K/F be a finite extension (which need not be separable). Then*

$$N_{K/F}(N_{\bar{A}/K}(\Delta)) = N_{A/F}(\det \rho(\Delta)).$$

Proof. The matrices $\rho(w_1), \rho(w_2), \ldots, \rho(w_n)$ commute and so have a common eigenvector. A simple induction argument shows that there is a matrix P, over the algebraic closure $\bar{F}$, such that each $P^{-1}\rho(w_i)P$ is upper triangular. Let the diagonal entries of $P^{-1}\rho(w_i)P$ be denoted by d_{ij}, $1 \leq j \leq n$.

We compute both sides starting with the right-hand side: By Lemma 1,

$$P^{-1}\rho(\Delta)P = \begin{pmatrix} \sum_i \alpha_i d_{i1} & & & \\ & \sum_i \alpha_i d_{i2} & & * \\ 0 & & \ddots & \\ & & & \sum_i \alpha_i d_{in} \end{pmatrix}.$$

Now Dieudonné's determinant [P, p. 308] satisfies $\det(P^{-1}MP) = \det M$ and the determinant of an upper triangular matrix is the product of the diagonal elements (in [A, p. 163], the first is consequence h), the second follows from [A, Theorem 4.4]). Hence

$$\det \rho(\Delta) = \prod_{j=1}^{n} \left(\sum_{i=1}^{n} \alpha_i d_{ij} \right).$$

Write $\alpha_i = \sum_{k=1}^{m} a_{ik}\epsilon_k$ where $a_{ik} \in F$. For the right-hand side we know that

$$\begin{aligned} \det \rho(\Delta) &= \prod_{j=1}^{n} \sum_{k=1}^{m} \left(\sum_{i=1}^{n} a_{ik} d_{ij} \right) \epsilon_k, \\ N_{A/F}(\det \rho(\Delta)) &= \prod_{j=1}^{n} G(* \sum_{i=1}^{n} a_{ik} d_{ij} \, *). \end{aligned}$$

For the left-hand side we have

$$\begin{aligned}\Delta &= \sum_{k=1}^{m}\left(\sum_{i=1}^{n} a_{ik}w_i\right)\epsilon_k,\\ N_{\bar{A}/K}(\Delta) &= G(*\sum_{i=1}^{n} a_{ik}w_i\,*).\end{aligned}$$

As ρ is a ring homomorphism, $\rho(G(*\,u_k*)) = G(*\,\rho(u_k)*)$. Thus

$$N_{K/F}(N_{\bar{A}/K}(\Delta)) = \det G(*\,\sum_{i=1}^{n} a_{ik}\rho(w_i)\,*).$$

Conjugation by P is also a ring homomorphism, so

$$N_{K/F}(N_{\bar{A}/K}(\Delta)) = \det G(*\,\sum_{i=1}^{n} a_{ik}P^{-1}\rho(w_i)P\,*).$$

We conclude that $G(*\,\sum_{i=1}^{n} a_{ik}P^{-1}\rho(\beta)^iP\,*) =$

$$G\left(*\quad\begin{pmatrix}\sum_i a_{ik}d_{i1} & & & \\ & \sum_i a_{ik}d_{i2} & & * \\ & 0 & \ddots & \\ & & & \sum_i a_{ik}d_{in}\end{pmatrix}\quad *\right) =$$

$$\begin{pmatrix}G(*\,\sum_i a_{ik}d_{i1}\,*) & & & \\ & G(*\,\sum_i a_{ik}d_{i2}\,*) & & * \\ 0 & & \ddots & \\ & & & G(*\,\sum_i a_{ik}d_{in}\,*)\end{pmatrix}.$$

Hence

$$N_{K/F}(N_{\bar{A}/K}(\Delta)) = \prod_{j=1}^{n} G(*\,\sum_{i=1}^{n} a_{ik}d_{ij}\,*),$$

the same as the right-hand side, proving the identity. □

Theorem 3. *Let φ be the norm of a central simple division algebra A over F. Then SNP holds for all finite field extensions of F.*

Proof. The proof is analogous to the one given in [F, Lemma 2.1] for the norms of a quaternion division algebra: Let $\epsilon_1, \dots, \epsilon_m$ be a basis for A as a F-vector space (where $m = d^2$ if d is the degree of A). For $z_i \in K$ and $z = \epsilon_1 z_1 + \epsilon_2 z_2 + \cdots + \epsilon_m z_m$, we have

$$\begin{aligned}&N_{K/F}(\varphi_K(z)) =\\ &N_{K/F}(N_{\overline{A}/K}(z)) =\\ &N_{A/F}(\det(\rho(z))) =\\ &N_{A/F}(\epsilon_1 a_1 + \epsilon_2 a_2 + \cdots + \epsilon_m a_m)\end{aligned}$$

for suitable $a_i \in F$. (The second equality holds by Theorem 2.) □

Corollary 2. *Let φ be the norm of a central simple algebra A over F of prime degree. Then SNP holds for all finite field extensions of F.*

Remark 3. Let $K = F(\sqrt{c})$ be a quadratic field extension and A a division algebra over F of degree d. Let $z_i = u_i + v_i\sqrt{c} \in K$ and $z = z_1\epsilon_1 + z_2\epsilon_2 + \cdots + z_{d^2}\epsilon_{d^2}$, then $z = x + y\sqrt{c}$ with $x = u_1\epsilon_1 + u_2\epsilon_2 + \cdots + u_{d^2}\epsilon_{d^2}$ and $y = v_1\epsilon_1 + v_2\epsilon_2 + \cdots + v_{d^2}\epsilon_{d^2}$. We obtain, more explicitly than above (similar as in [F, 2.2]):

$$N_{K/F}(\varphi_K(z)) = N_{A/F}(\det(\rho(z))) = N_{A/F}(y(xy^{-1}x - cy)) \in D_F(N_{A/F}).$$

In particular, if A has degree 3, then we can also write

$$N_{K/F}(\varphi_K(z)) = \frac{1}{N_{A/F}(y)} N_{A/F}(xy^\sharp x - cN_{A/F}(y)y)$$

with $x^\sharp = x^2 - T_{A/F}(x)x + S_{A/F}(x)1_A$ [KMRT, p. 470].

2.3. Some construction methods.

Remark 4. Suppose there are $f, g \in F[X_1, \dots, X_n]$ such that $f(X_1, \dots, X_n)^m = g(X_1, \dots, X_n)^m$. Then, by unique factorization in $F[X_1, \dots, X_n]$, there is an mth root of unity μ in F such that $f(X_1, \dots, X_n) = \mu g(X_1, \dots, X_n)$.

Lemma 2. *Let $\varphi_1 \in F[X_1, \dots, X_n]$ be a form of degree d_1 which satisfies SNP for all finite field extensions. Put $\varphi(X) = \varphi_1(X)^m$ for some integer $m \geq 2$. Then φ satisfies SNP for all finite field extensions.*

Proof. Let $a\varphi_K \cong \varphi_K$ for some finite field extension K/F. Then there is an invertible $n \times n$ matrix M over F such that $a\varphi_{1,K}(X)^m = \varphi_{1,K}(MX)^m$. Let x be an anisotropic vector, then $a = (\varphi_{1,K}(Mx)/\varphi_{1,K}(x))^m$ is an mth power in K, hence write $a = b^m$ for some $b \in K^\times$. From $b^m\varphi_{1,K}^m \cong \varphi_{1,K}^m$ we conclude that $\mu b\varphi_{1,K} \cong \varphi_{1,K}$ for some mth root of unity μ in K (Remark 4). As φ_1 satisfies SNP, $N_{K/F}(\mu b) \in G_F(\varphi_1)$. Thus $N_{K/F}(\mu b)^m = N_{K/F}(a) \in G_F(\varphi)$. □

Lemma 3. *(i) Let $\varphi_i : V_i \to F$ be two forms over F of degree d_i which satisfy SNP for all finite field extensions K/F. Put $\varphi : V_1 \oplus V_2 \to k$, $\varphi(u) = \varphi_1(u_1)\varphi_2(u_2)$ for $u = u_1 + u_2$, $u_i \in V_i$. If $D_K(\varphi_i) = G_K(\varphi_i)$ for all finite field extensions K/F, then φ satisfies SNP for all finite field extensions.*
(ii) Let F'/F be a finite separable field extension and $\varphi_0 : V \to F'$ be a form over F'. Let $\varphi = N_{F'/F}(\varphi_0)$. Suppose that $(\varphi_0)_{L'}$ is a round form for all finite field extensions L' of F' and that SNP holds for φ_0 for all finite field extensions L' of F'. Then $\varphi = N_{F'/F}(\varphi_0)$ satisfies SNP for all finite field extensions K of F which are linearly disjoint with F' over F.

Proof. (i) By [Pu1], φ_K is a round form. Let $a\varphi_K \cong \varphi_K$. Then $a = \varphi_{1,K}(w_1)\varphi_{2,K}(w_2)$ and by assumption, $N_{K/F}(\varphi_{i,K}(w_i)) \in G_F(\varphi_i)$ for $i = 1, 2$. This immediately yields

$N_{K/F}(\varphi_1(w_1))N_{K/F}(\varphi_2(w_2)) = N_{K/F}(\varphi_1(w_1)\varphi_2(w_2)) = N_{K/F}(a) \in G_F(\varphi)$.
(ii) Let K be a finite field extension of F which is linearly disjoint with F' over F. Then

$$\varphi_K = N_{K'/K}((\varphi_0)_{K'})$$

with $K' = F' \cdot K$ the composite of F' and K (i.e., the homogeneous polynomials defining the forms are equal). Since $(\varphi_0)_{K'}$ is round by assumption, $D_{K'}((\varphi_0)_{K'}) = G_{K'}((\varphi_0)_{K'})$, and φ_K is a round form by [Pu1].
Let $a\varphi_K \cong \varphi_K$. Since φ_K is round, $a = N_{K'/K}((\varphi_0)_{K'}(z_0))$ for some $z_0 \in K'$. As $(\varphi_0)_{K'}$ is round, we have

$$((\varphi_0)_{K'}(z_0))(\varphi_0)_{K'} \cong (\varphi_0)_{K'}. \tag{2}$$

φ_0 satisfies SNP for all field extensions of F' by assumption, hence

$$N_{K'/F'}((\varphi_0)_{K'}(z_0))\varphi_0 \cong \varphi_0$$

and so $N_{F'/F}(N_{K'/F'}((\varphi_0)_{K'}(z_0)))\varphi \cong \varphi$. Hence

$$N_{F'/F}(N_{K'/F'}((\varphi_0)_{K'}(z_0))) = N_{K/F}(N_{K'/K}((\varphi_0)_{K'}(z_0))) = N_{K/F}(a) \in G_F(\varphi).$$

□

Similarly, we obtain:

Theorem 4. *Let F'/F be a finite separable field extension and $\varphi_0 : V \to F'$ be a form over F' of prime degree p. Let $\varphi = N_{F'/F}(\varphi_0)$. Suppose that $(\varphi_0)_{L'}$ is a round form for all finite field extensions L' of F'. Then $\varphi = N_{F'/F}(\varphi_0)$ satisfies SNP for all field extensions K of F of degree p^r coprime to $[F' : F]$.*

Proof. Let K be a field extension of degree p^r which is coprime to $[F' : F]$ and set $K' = F' \cdot K$. Then $[K' : F'] = p^r$ and K' is linearly disjoint from F' over F. The proof of Lemma 3 (ii) holds up to (2). By Remark 2 (ii), SNP holds for φ_0 for all extensions K/F' of degree a power of p, in particular, for K'. So (2) yields $N_{K/F}(a) \in G_F(\varphi)$. □

Forms φ_0 over F' which satisfy the conditions of Theorem 4 are not only those permitting composition [Pu2, Proposition 6], but also forms permitting Jordan composition of prime degree over fields of characteristic 0 or greater than $2d$, e.g. the cubic norm of an Albert algebra [Pu2, Proposition 7].

Example 1. Let $\varphi_0 = \langle\langle a_1, \dots, a_r\rangle\rangle$ ($a_i \in F^\times$) be an anisotropic r-fold quadratic Pfister form. If $K = F(\sqrt{c})$ is a quadratic field extension, then

$$N_{K/F}(\varphi_0)(u_1, w_1, \dots, u_{2^r}, w_{2^r}) =$$
$$(\langle\langle a_1, \dots, a_r, c\rangle\rangle)^2(u_1, u_2, \dots, u_{2^r}, w_1, w_2, \dots, w_{2^r}) - 4c\varphi_0(u_1w_1, \dots, u_{2^r}w_{2^r})$$

is an anisotropic quartic form of dimension 2^{r+1} which satisfies SNP for all finite field extensions of F which are linearly disjoint with K over F.

If F contains a primitive third root of unity and $K = F(\sqrt[3]{c})$ is a cubic Kummer field extension, then

$$\begin{aligned}
&N_{K/F}(\varphi_0)(u_1, v_1, w_1, \dots, u_{2^r}, v_{2^r}, w_{2^r}) = \\
&(\langle\langle a_1, \dots, a_r, 2c\rangle\rangle)^3(u_1, \dots, u_{2^r}, v_1 w_1, \dots, v_{2^r} w_{2^r}) \\
&+c(c\langle\langle a_1, \dots, a_r\rangle\rangle \perp 2\langle\langle a_1, \dots, a_r\rangle\rangle)^3(w_1, \dots, w_{2^r}, u_1 v_1, \dots, u_{2^r} v_{2^r}) \\
&+c^2(\langle\langle a_1, \dots, a_r\rangle\rangle \perp 2\langle\langle a_1, \dots, a_r\rangle\rangle)^3(v_1, \dots, v_{2^r}, u_1 w_1, \dots, u_{2^r} w_{2^r}) \\
&-3c[(\langle\langle a_1, \dots, a_r, 2c\rangle\rangle(u_1, u_2, \dots, u_{2^r}, v_1 w_1, \dots, v_{2^r} w_{2^r})) \\
&\cdot((c\langle\langle a_1, \dots, a_r\rangle\rangle \perp 2\langle\langle a_1, \dots, a_r\rangle\rangle)(w_1, \dots, w_{2^r}, u_1 v_1, \dots, u_{2^r} v_{2^r})) \\
&\cdot((\langle\langle a_1, \dots, a_r\rangle\rangle \perp 2\langle\langle a_1, \dots, a_r\rangle\rangle)(v_1, \dots, v_{2^r}, u_1 w_1, \dots, u_{2^r} w_{2^r}))]
\end{aligned}$$

is an anisotropic form of degree 6 and dimension $3 \cdot 2^r$ which satisfies SNP for all finite field extensions of F which are linearly disjoint with K over F.

There exists a nondegenerate form φ of degree $d > 2$ permitting composition on a finite dimensional unital F-algebra A if and only if A is a separable alternative algebra and φ is one of the following forms, for some integers $s_1, \dots, s_r > 0$: write A as direct sum of simple ideals $A = A_1 \oplus \dots \oplus A_r$ with the center of each A_i a separable field extension F_i of F. Any $a \in A$ can be written uniquely as $a = a_1 + \ldots + a_r$, $a_i \in A_i$ and any nondegenerate form φ on A permitting composition can be written as

$$\varphi(a) = N_1(a_1)^{s_1} \cdots N_r(a_r)^{s_r},$$

where $d = d_1 s_1 + \ldots + d_r s_r$, and where N_i is the generic norm of the F-algebra A_i of degree d_i [S]. If SNP holds for all N_i then it holds for φ (Lemma 2, 3).

Theorem 5. *If φ is a nondegenerate cubic form over F which permits composition, then SNP holds for all finite field extensions of F.*

Proof. We have either $\varphi \cong \langle 1\rangle$, φ is the norm of a cubic field extension, of a central simple F-algebra of degree 3 or $\varphi(a+x) = aN_C(x)$ for $a \in F$, $x \in C$, C a composition algebra over F. In all cases SNP holds for all finite field extensions of F by Corollary 1, Theorem 3 and Remark 1, (iii) and (v). □

Remark 5. Let $\varphi(x) = N_{F'/F}(N_C(x))$ with N_C the quadratic norm of a composition algebra over F', F' a quadratic field extension of F. φ is a form of degree 4 permitting composition. If C has dimension greater than 1 then φ satisfies SNP for all field extensions of odd degree (Lemma 3 (ii)). If C has dimension 1 then φ satisfies SNP for all finite field extensions (Lemma 2). Thus, by invoking Lemma 1, Theorem 3 and [B-M, 3.1], for any form of degree 4 permitting composition, SNP holds for all odd degree separable field extensions.

We conclude pointing out that already for cubic forms (which do not permit composition), it might not be enough any more to investigate if $a\varphi_K \cong \varphi_K$ implies that $N_{K/F}(a)\varphi \cong \varphi$. It might also be interesting to know if and when $N_{K/F}(a)^2\varphi \cong \varphi$ holds.

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