A Baire Category Approach to Besicovitch's Theorem

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Online Logic Seminar - September 2025

Outline

Hausdorff Measure - General Setting

Let X be a metric space, with d denoting its metric. If $U \subseteq X$ is nonempty, then we define the **diameter** of U by

$$|U| = \sup\{d(x,y) : x, y \in U\}.$$

Suppose $E \subseteq X$ is a set and $\delta > 0$ is a real number. If $\{U_i\}_{i\in\omega}$ is a countable collection of nonempty sets, we say that $\{U_i\}_{i\in\omega}$ is a **δ-cover of E** if:

1. $E \subseteq \bigcup_{i\in\omega} U_i$
2. $0 \leq |U_i| \leq \delta$ for all $i \in \omega$

Hausdorff Measure - General Setting

Suppose $E \subseteq X$ is a set and s is a nonnegative number. For any $\delta > 0$, we define the ***s*-dimensional Hausdorff δ-measure of *E*** by

$$\mathcal{H}^s_\delta(E) = \inf\left\{\sum_{i\in\omega} |U_i|^s : \{U_i\}_{i\in\omega} \text{ is a } \delta\text{-cover of } E\right\}.$$

We then define the (overall) ***s*-dimensional Hausdorff measure of *E*** by

$$\mathcal{H}^s(E) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(E).$$

Note that the quantity $\mathcal{H}^s_\delta(E)$ is monotonically increasing in δ, so the above limit always exists (although it may be infinite).

Hausdorff Measure - Cantor Space

Much of the study of Hausdorff measure has been over Euclidean space, but in this talk we will focus mainly on Cantor Space, or 2^ω.

Due to the topological properties of this space, we can make some simplifications in the definition of $\mathcal{H}^s_\delta(E)$:

1. It suffices to consider δ-covers comprised of clopen cylinders, rather than arbitrary sets U_i (follows from how the metric on 2^ω is defined).
2. If E is closed, it suffices to consider finite covers by cylinders, rather than countable (follows from 2^ω being compact).

Note: This means that for closed sets, the infimum only involves a *countable* search.

Besicovitch's Theorem

Unlike Lebesgue measure, the Hausdorff measure of a set may be infinite. Many results involving the local structure of fractal sets only apply to those with positive, finite Hausdorff measure.

The following theorem from Besicovitch gives us a means of "reducing" to this nicer case.

Theorem 2.1 (Besicovitch, 1952)

Let F be a closed subset of Euclidean space with $\mathcal{H}^s(F) = \infty$. Then for any positive number d, there exists a compact set $E \subseteq F$ with $\mathcal{H}^s(E) = d$.

Later the same year, this result was extended by Davies to the case when F is analytic (and thus includes all Borel sets).

Complexity of Besicovitch's Proof

If we want to study the "complexity" of Besicovitch's argument, we might ask:

1. What weak subsystems of second-order arithmetic can this theorem be proven in? ($\mathrm{RCA}_0, \mathrm{ACA}_0$, etc.) Can we obtain a reversal to prove that such a subsystem is optimal?
2. How complex is the witnessing subset F compared to E? For example, if E is Π^0_1 in some oracle X, how many jumps of X are necessary to compute F?

Complexity of Besicovitch's Proof

Examining the structure of the construction, we see that the witnessing set F is defined as $\bigcap_{k \in \omega} F_k$, where $E = F_0 \supseteq F_1 \supseteq F_2 \supseteq \ldots$ are defined inductively.

- This suggests that to compute F, we need to be able to compute each of the F_i uniformly

To obtain F_{k+1} from F_k, we need to check its s-dimensional Hausdorff δ-measure, as well as potentially construct a subset with a specific δ-measure

- These operations are not computable from F_k, since they require checking infinitely many potential δ-covers
- This suggests that each F_k might need an additional jump to compute, meaning our final set F would require infinitely many jumps of our original E - doesn't bode well for ACA_0!

Baire Category Approach

To simplify the construction, we can reframe the argument in terms of the Baire Category Theorem, a famously "simple" result from a reverse mathematics perspective.

First, recall two of the standard "Big 5" axiom systems of second order arithmetic:

- RCA_0 : basic axioms of Peano arithmetic, plus Σ^0_1-induction and Δ^0_1-comprehension
- ACA_0: all axioms of RCA_0, plus arithmetic comprehension

Baire Category - Standard Version

To formalize Baire Category in second-order arithmetic, we need a way to code open sets in Cantor space. We do this by associating an open set $U \subseteq 2^\omega$ with a subset $V \subseteq 2^{<\omega}$, such that U is the union of all cylinders corresponding to strings in V.
(Note that V can be coded as a second-order object.)

With that coding, consider the following formulation of Baire Category:

Theorem 3.1 (Baire Category Standard)

Let $\{U_n\}_{n\in\omega}$ be a sequence of dense open sets in 2^ω, with a respective sequence of open codes $\{V_n\}_{n\in\omega}$. Then the set $\bigcap_{n\in\omega} U_n$ is dense in $2^{<\omega}$.

Baire Category - Standard Version

Theorem 3.2

Baire Category Standard is provable in RCA_0.

Proof (Informal):

- Given an arbitrary $\tau \in 2^{<\omega}$, construct a sequence $\tau \subsetneq \sigma_0 \subsetneq \sigma_1 \subsetneq \sigma_2 \subsetneq \dots$ inductively
- With σ_{n-1} defined, enumerate through its proper extensions until we find a string with a prefix in V_n - this must happen eventually since each U_n is dense
- Define $X \in 2^{\omega}$ as the limit of the sequence $\{\sigma_n\}_{n\in\omega}$

Remark: Formally, this argument uses the primitive recursion and minimization principles provable in RCA_0.

Baire Category for Closed Sets

In classical analysis, the Baire Category Theorem can be applied to any complete metric space. This includes subspaces of complete metric spaces induced by closed sets. This motivates the following.

Definition 3.1

Let $F \subseteq 2^\omega$ be a closed set. We define a set U to be **dense in F** if for every open cylinder set $N \subseteq 2^\omega$ with $N \cap F \neq \emptyset$, there exists an element $X \in F \cap N \cap U$.

Theorem 3.3 (Baire Category Theorem - Closed Sets (BCTC))

Let $F \subseteq 2^\omega$ be a nonempty closed set, and let $\{U_n\}_{n\in\omega}$ be a sequence of open sets in 2^ω, each of which is dense in F. Then $\bigcap_{n\in\omega} U_n$ is dense in F.

Baire Category Theorem for Closed Sets - Proof

To formalize BCTC in second-order arithmetic, we can code the dense open sets as before, but now we must also code the closed set F as a tree $T \subseteq 2^{<\omega}$:

$$X \in F \iff \forall n \, X \restriction_n \in T.$$

We could try to adapt the proof of the standard version, only restricting our searches to strings $\sigma \in T$.

BUT at some point our inductive construction could place us on a "dead end," i.e. a string $\sigma \in T$ with no infinite extension in $[T] = F$. And we can't determine if a string has an infinite extension in a c.e. way.

Baire Category Theorem for Closed Sets - Proof

Fortunately, this issue is solved if we have arithmetic comprehension.

Theorem 3.4 (G. , 2025)

BCTC is provable in ACA_0.

Proof:
Given the tree T coding the closed set F, use arithmetic comprehension to define:

$$\tilde{T} = \{\sigma \in T : \forall n \geq |\sigma|, \exists \tau \supseteq \sigma, |\tau| = n \wedge \tau \in T\}.$$

Observe that:

- $[\tilde{T}] = [T] = F$
- For $\sigma \in T$: $\sigma \in \tilde{T} \iff \exists X \in F, \sigma \subseteq X \iff N_\sigma \cap F \neq \emptyset$

Now we can mimic the proof of the standard Baire Category theorem, but where we only consider the strings in $\tilde{T}$.

More on BCTC

Do we actually need ACA_0 to prove BCTC? As it turns out, yes:

Theorem 3.5 (G., 2025)

Over RCA_0, BCTC is equivalent to ACA_0.

Proof Idea: Given an arbitrary function $f : \omega \to \omega$, define a closed set F and a sequence of dense open sets $\{U_n\}_{n\in\omega}$ such that any $X \in F \cap \bigcap_{n\in\omega} U_n$ can compute the range of f.

Nonetheless, note that the proof of BCTC used relatively little of ACA_0's power; only one instance of Π^0_1 comprehension using the tree T.

In other words, given an oracle Z for computing T, we can compute a witnessing element $X \in F \cap \bigcap_{n\in\omega} U_n$ from Z'.

Topology on Closed Subsets

To use BCTC to prove Besicovitch's Theorem, we need to define a topology on the space of closed subsets of our given set $F \subseteq 2^\omega$.

So consider $T_F \subseteq 2^\omega$ as a tree representation for F.

Identify ω with $2^{<\omega}$ via the standard length-lexicographic enumeration. Then define a new tree $T_P \subseteq 2^{<\omega}$, computable from T_F, by setting $\nu \in T_P$ iff:

1. For all $\sigma < |\nu|$ with $\nu(\sigma) = 1$, we have $\nu(\tau) = 1$ for all $\tau \subseteq \nu$
2. For all $\sigma < |\nu|$ with $\nu(\sigma) = 1$, we have $\sigma \in T_F$.

Let $\mathcal{P}_F$ denote the set of infinite paths through T_P.

Topology on Closed Subsets

Consider $Z \in \mathcal{P}_F$, and associate a subset $T_Z \subseteq 2^{<\omega}$ by taking all σ with $Z(\sigma) = 1$. Then the conditions for T_Z guarantee that:

1. T_Z is a tree
2. T_Z is a subtree of T_F

Therefore, the closed set $E_Z \subseteq 2^\omega$ corresponding to T_Z will be a subset of F.

We can now work with the usual topology on 2^ω, but restricted to the elements in the closed set $\mathcal{P}_F$.

Coding Hausdorff Measure

We now want to associate subsets of F having particular Hausdorff δ-measures with certain open and closed subsets of $\mathcal{P}_F$. Assume for the rest of the talk that $s \geq 0$ is fixed.

Suppose $n \in \omega$ and $d > 0$. Define

$$\mathcal{S}_n^d = \{Z \in \mathcal{P}_F : \mathcal{H}^s_{2^{-n}}(E_Z) \geq d\},$$

where $E_Z \subseteq F$ is the closed set associated to $Z \in \mathcal{P}_F$.

Coding Hausdorff Measure

Lemma 4.1

The set $\mathcal{S}^d_n$ is closed in 2^ω, and the corresponding tree is computable in T_F, s, and d.

Proof Sketch:

- By definition, we have $\mathcal{H}^s_{2^{-n}}(E_Z) \geq d$ if and only if

$$\sum |N_i|^s \geq d$$

for all 2^{-n}-covers of E_Z by cylinders.

- By compactness, we only need to verify this bound for finite covers, so we only have a countable collection of covers to check.

Coding Hausdorff Measure

- For any cover $\{N_i\}$ of E_Z, we will determine that it is valid using only finitely many bits of the code Z, since eventually all strings in T_Z of a fixed length will be covered by some cylinder N_i
- We can use this idea to define a tree coding $\mathcal{S}_n^d$: take all $\nu \in T_P$ such that for all 2^{-n}-covers $\{N_i\}$ which are valid for the tree "approximated" by ν, we have

$$\sum |N_i|^s \geq d.$$

Remark: This also means that the complement $\mathcal{U}_n^d = 2^\omega \setminus \mathcal{S}_n^d$ is open and is generated by the strings NOT in the above tree.

Baire Category Outline

Theorem 5.1 (G., 2025)

Over RCA_0, Besicovitch's Theorem is provable from BCTC.

Recall that we start with a set $F \subseteq 2^\omega$ with $\mathcal{H}^s(F) = \infty$, and we want to find a closed subset $E \subseteq F$ satisfying $\mathcal{H}^s(E) = d$. The high-level overview:

1. Choose $n_0 \in \omega$ large enough that $\mathcal{H}^s_{2^{-n_0}}(F) \geq d$; then the corresponding closed set $\mathcal{S}^d_{n_0}$ will be nonempty
2. Prove that each open set $\mathcal{U}^{d_n}_n$ is dense in $\mathcal{S}^d_{n_0}$, where (d_n) is a strictly decreasing sequence converging downward to d
3. Use BCTC to find an element $Z \in \mathcal{S}^d_{n_0} \cap \bigcap_{n\in\omega} \mathcal{U}^{d_n}_n$

Baire Category Outline

Let's check: does the closed set E_Z from Step 3 satisfy $\mathcal{H}^s(E_Z) = d$?

First, $Z \in \mathcal{S}^d_{n_0} \implies \mathcal{H}^s_{2^{-n_0}}(E_Z) \geq d \implies \mathcal{H}^s(E_Z) \geq d$.

Then for any $\epsilon > 0$, $d_n < d + \epsilon$ for all n sufficiently large:

- $Z \in \mathcal{U}^{d_n}_n \implies \mathcal{H}^s_{2^{-n}}(E_Z) < d_n < d + \epsilon$
- Since this applies to all sufficiently large n: $\mathcal{H}^s(E_Z) \leq d + \epsilon$
- Since ϵ was arbitrary: $\mathcal{H}^s(E_Z) \leq d$.

Density of the open sets $\mathcal{U}_n^{d_n}$

Proving the density of the relevant sets is the trickiest step. We wish to show the following:

Lemma 5.2

Let $0 < d < c$, and let $n_0 \in \omega$ be such that $\mathcal{S}_{n_0}^d$ is nonempty. Then for all $n \in \omega$, $\mathcal{U}_n^c$ is dense in $\mathcal{S}_{n_0}^d$.

We can prove this result by induction on n. Note that it suffices to start with n_0 as the base case, since any element $Z \in \mathcal{S}_{n_0}^d \cap \mathcal{U}_{n_0}^c$ is also in $\mathcal{U}_k^c$ for $k < n_0$.

Density Argument - Base Case

Claim: $\mathcal{U}^c_{n_0}$ is dense in $\mathcal{S}^d_{n_0}$.

Proof (Informal): Fairly straightforward, if technical. In this case, we are only working with a single Hausdorff δ-measure, i.e. $\mathcal{H}^s_{2^{-n_0}}$.

Given a code $Z \in \mathcal{S}^d_{n_0}$, we can just start "shaving off" sufficiently small cylinder sets until the measure of the remaining set E_Y satisfies $d \leq \mathcal{H}^s_{2^{-n_0}}(E_Y) < c$, giving a code $Y \in \mathcal{S}^d_{n_0} \cap \mathcal{U}^c_{n_0}$.

Remark: The code Y is computable from Z, since the changes made can be specified with finite instructions. However, the computability is not uniform in c.

Density Argument - Inductive Step

Assume now that $\mathcal{U}_n^c$ is dense in $\mathcal{S}_{n_0}^d$ for some $n \geq n_0$.

This means that given any $Z \in \mathcal{S}_{n_0}^d$ and $\tau \subseteq Z$, we can find $Y \supseteq \tau$ with $Y \in \mathcal{S}_{n_0}^d \cap \mathcal{U}_n^c$.

In particular, there is some 2^{-n}-cover $\{N_i\}$ of E_Y with

$$\sum_i |N_i|^s < c.$$

Note that if all cylinders in the cover have diameter $< 2^{-(n+1)}$, then that would witness $\mathcal{H}^s_{2^{-(n+1)}}(E_Y) < c$ putting $Y \in \mathcal{U}_{n+1}^c$, and we'd be done.

Thin Elements

If that isn't true for this cover, then the next best case would be to know that any cylinders of diameter 2^{-n} were unnecessary.

That is, they could be replaced with cylinders of smaller diameter, still covering the appropriate portion of E_Y, but without significantly increasing the overall cover's weight.

Capturing this idea, define $Z \in \mathcal{P}_F$ to be n-**thin** if for all $\sigma \in 2^n$:

$$\mathcal{H}^s_{2^{-(n+1)}}(E_Z \cap N_\sigma) \leq 2^{-sn}.$$

Let $\mathcal{T}_n$ denote the n-thin elements of $\mathcal{P}_F$.

Finding Thin Elements

Claim: If $Z \in \mathcal{U}_n^c \cap \mathcal{T}_n$, then $Z \in \mathcal{U}_{n+1}^c$

Proof: Straightforward - essentially outlined above.

While the element Y from our inductive assumption may not be n-thin itself, it turns out that n-thin elements are sufficiently numerous within $\mathcal{S}_{n_0}^d$:

Lemma 5.3

If $\mathcal{S}_{n_0}^d$ is nonempty, then for all $n \geq n_0$ we have $\mathcal{T}_n$ dense in $\mathcal{S}_{n_0}^d$.

Density of Thin Elements - Proof Sketch

If we have an element $Z \in \mathcal{S}^d_{n_0}$ which is NOT n-thin, we want to trim it down "as little as possible."

For each cylinder N of diameter 2^{-n}:

- If $\mathcal{H}^s_{2^{-(n+1)}}(E_Z \cap N) \leq 2^{-ns}$, do nothing
- Otherwise, replace $E_Z \cap N$ with a subset $E' \subseteq E_Z \cap N$ satisfying $\mathcal{H}^s_{2^{-(n+1)}}(E') = 2^{-ns}$
(**Note:** This step uses another application of BCTC!)

One can show that with these adjustments, the new set is n-thin (by definition) but is still an element of $\mathcal{S}^d_{n_0}$.

Finishing the Proof

Now to complete the inductive step:

1. Let $Z \in \mathcal{S}^d_{n_0}$ and let $\tau \subseteq Z$. By the inductive hypothesis, there is some $Y \supseteq \tau$ with $Y \in \mathcal{S}^d_{n_0} \cap \mathcal{U}^c_n$.
2. Let $\tau' \subseteq Y$ be sufficiently long that $\tau' \supseteq \tau$ and $N_{\tau'} \subseteq \mathcal{U}^c_n$ (possible since $\mathcal{U}^c_n$ is open)
3. Choose an element $W \supseteq \tau'$ with $W \in \mathcal{S}^d_{n_0} \cap \mathcal{T}_n$. Then since $W \in \mathcal{U}^c_n$, it follows that $W \in \mathcal{U}^c_{n+1}$ as desired.

What's the Difference?

Ultimately, this argument uses a lot of Besicovitch's original ideas. Each of the closed sets in his inductive construction is essentially obtained by trimming down the previous set to be appropriately "thin".

However, without showing that the "thin" elements are dense, one needs to keep track of each sequence in the construction and apply a new jump each time.

With the Baire Category argument, one only needs to apply a jump once: to determine which strings in the tree for $\mathcal{S}^d_{n_0}$ have infinite extensions. After that point, the construction can proceed in a computable manner.

Refining the Strength

We've now shown that Besicovitch's Theorem is provable from BCTC, and that BCTC is equivalent to ACA_0. But is Besicovitch's Theorem *itself* equivalent to ACA_0/BCTC?

- From Kolmogorov Complexity considerations, we know that Besicovitch's Theorem is not provable in RCA_0
- Also, an analog to Besicovitch's Theorem for Lebesgue measure is sufficient to prove $WWKL_0$
- Current conjecture: uncertain - if a proof of ACA_0 from BCTC exists, it appears nontrivial

Main References

[1] Falconer K. J. (1985) The Geometry of Fractal Sets, Cambridge University Press, Cambridge.

[2] Simpson, S. G. (2009). Subsystems of second order arithmetic (Vol. 1). Cambridge University Press.