Martin's conjecture in the enumeration degrees

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Martin's Conjecture

Martin's Conjecture (informal version)

Every *"natural"* function in the Turing degrees is either constant, the identity or an iterate of the Turing jump (almost everywhere).

What does "natural" mean?

Natural functions in the Turing degrees should come from constructions of (incomputable) sets such that:

- They are definable
- They relativize
- They are Turing-invariant

Martin's Cone Theorem

The cone above x is the set $\nabla_x = \{y \in 2^\omega : x \leq_T y\}$

Theorem (Martin, 1968)

Assume AD. Let $\mathcal{A} \subseteq 2^\omega$ be closed under Turing equivalence. Either $\mathcal{A}$ contains a cone or $2^\omega \setminus \mathcal{A}$ contains a cone.

We can define a countably additive measure in $\mathcal{D}_T$:

$$\mu(\mathcal{A}) = \begin{cases} 1 & \text{if } \mathcal{A} \text{ contains a cone} \\ 0 & \text{otherwise} \end{cases}$$

Theorem (Martin's Cone Theorem 2.0)

If $\mathcal{A} \subseteq \mathcal{D}_T$ is cofinal, then $\mathcal{A}$ contains a cone.

Working on a cone

We need to work on a cone to avoid getting counterexamples to Martin's conjecture by *Frankensteining* functions.

The moral of Martin's cone theorem is: if you glue together countably many Turing-invariant functions, one prevails on a cone.

Comparing functions on a cone

Definition

Let $f, g : 2^\omega \to 2^\omega$. We say that

- $f \leq_T^\nabla g$ if $f(x) \leq_T g(x)$ for all x on some cone.
- f is *constant on a cone* if there is $y \in 2^\omega$ such that $f(x) \equiv_T y$ for all x on some cone.
- f is *increasing on a cone* if $x \leq_T f(x)$ for all x on some cone.

Martin's Conjecture

Martin's Conjecture

Assume $ZF + AD + DC$.

1. Let $f : 2^\omega \to 2^\omega$ Turing-invariant. If f is not constant on a cone, then f is increasing on a cone.
2. The relation $\leq_T^\nabla$ prewell orders the Turing-invariant functions $\leq_T^\nabla$-above the identity.
   Moreover, if $\mathrm{rank}_T^\nabla(f) = \alpha$, then $\mathrm{rank}_T^\nabla(f') = \alpha + 1$.

Where f' is defined by $f'(x) = f(x)'$, for all $x \in 2^\omega$.

Partial Results

- Part I and II for uniformly Turing-invariant functions. (Steel, 1982; Slaman and Steel, 1988)
- Part I for regressive functions. (Slaman and Steel, 1988)
- Part I for order-preserving functions. (Lutz and Siskind, 2025)
- Part II for Borel order-preserving functions. Moreover, if f is such a function, there is $\alpha < \omega_1^{CK}$ such that

  $$f(x) = x^{\alpha} \text{ on a cone}$$

  (Slaman and Steel, 1988)
- Functions from many-one degrees to Turing degrees (Kihara and Montalbán, 2018)
- The conjecture is false in the arithmetic degrees. (Slaman and Steel, 2016^*)

Uniform Martin's Conjecture

Definition

A function $f : 2^\omega \to 2^\omega$ is uniformly Turing-invariant if there is $u : \omega^2 \to \omega^2$ such that for any $x, y \in 2^\omega$

$$x \equiv_T y \, \mathrm{via}(i,j) \text{ implies that } f(x) \equiv_T f(y) \, \mathrm{via} \, u(i,j)$$

Theorem (Slaman and Steel 1988 ; Steel 1982)

1. *Let $f : 2^\omega \to 2^\omega$ uniformly Turing-invariant. If f is not constant on a cone, then f is increasing on a cone.*
2. *The relation $\leq_T^\nabla$ prewell orders the uniformly Turing-invariant functions $\leq_T^\nabla$-above the identity.*
   Moreover, if $\mathrm{rank}_T^\nabla(f) = \alpha$, then $\mathrm{rank}_T^\nabla(f') = \alpha + 1$.

Here f' is defined by $f'(x) = f(x)'$, for all $x \in 2^\omega$

The Case for the Uniformity Assumption

Steel's Conjecture

Under AD, if f is Turing-invariant, then there is an uniformly Turing-invariant function g such that $f \equiv_T^\nabla g$.

Notice that Steel's conjecture implies Martin's conjecture.

Montalbán argues that all the philosophical motivation behind Martin's conjecture also holds for the uniform Martin's conjecture.

Enumeration Reduction

Definition (Friedberg and Rogers, 1959)

Let $A, B \subseteq \mathbb{N}$. We say $A \leq_e B$ (via e) if

$$n \in A \quad \text{if and only if} \quad \langle n, D \rangle \in \Gamma_e \ \&\ D \subseteq B$$

where Γ_e is the eth c.e. set.

Observation

$A \leq_e B$ means that using positive information about B, we can compute positive information about A. In contrast, $A \leq_T B$ means that using positive and negative information about B, we can compute positive and negative information about A.

Enumeration Degrees

Definition

We say that A is *enumeration equivalent* to B, denoted by $A \equiv_e B$, if $A \leq_e B$ and $B \leq_e A$.

The *Enumeration Degrees* are the following structure:

$$\mathcal{D}_e = \left(\mathcal{P}(\mathbb{N})/\equiv_e, \leq\right)$$

- Upper semilattice with a least element $\mathbf{0}_e$ that consist of the c.e. sets.
- The least upper bound is given by the join operator

$$A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$$

Total Degrees

Theorem

For any $A, B \in \mathcal{P}(\mathbb{N})$

$$A \leq_T B \text{ if and only if } A \oplus \overline{A} \leq_e B \oplus \overline{B}$$

This means that the Turing degrees embed into the enumeration degrees via

$$\iota(A) = A \oplus \overline{A}$$

Total and Cototal degrees

Definitions

- A set A is total if $\overline{A} \leq_e A$.
- An enumeration degree is total if it contains a total set.
- A set A is cototal if $A \leq_e \overline{A}$.
- An enumeration degree is cototal if it contains a cototal set.

- Total degrees are exactly the degrees in the range of ι.
- Every total degree is cototal.
- There is a cototal degree that is not total.
- Not every degree is cototal.

Jump and Skip

Definition

- The *enumeration jump* is the map $A \mapsto K^A \oplus \overline{K^A} = A'$
- The *enumeration skip* is the map $A \mapsto \overline{K^A} = A^\diamond$

Theorem (AGKLMSS, 2019)

- *$A <_e A^\diamond$ if and only if $\deg_e(A)$ is cototal. Another way to say this, $\deg_e(A)$ is cototal iff $A' = A^\diamond$.*
- *There is some A such that $A = (A^\diamond)^\diamond$*

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The skip is a uniformly enumeration-invariant function that is neither increasing nor constant on any cone!

No cone theorem

Martin's cone theorem completely fails in the enumeration degrees.

- The total, nontotal cototal, and the noncototal degrees are 3 disjoint cofinal classes of enumeration degrees.
- You can *frankenstein* different functions along this classes and all of them prevail in a cone.
- For example,

$$f(A) = \begin{cases} A & \text{if } A \text{ has cototal degree} \\ A^{\diamond} & \text{otherwise} \end{cases}$$

Martin's conjecture fails catastrophically

Theorem (N. C.)

Given any countable family $\{f_n\}_{n\in\mathbb{N}}$ of uniformly enumeration-invariant functions, there is a uniformly invariant function f that is not comparable with any f_n on a cone.

Theorem (Jacobsen-Grocott)

There is a Borel uniformly enumeration-invariant function $f : 2^\omega \to 2^\omega$ such that for every X there are continuum many B and pairwise disjoint cofinal sets C_B with

$$f(Y) = B \quad \textit{for all } Y \in C_X(B)$$

A Local Approach

Bard (2020) gave a new proof of Part 1 of the uniform Martin's conjecture using a local approach.

Theorem (Bard, 2020)

Assume ZF. Let $x \in 2^\omega$ and $f : \deg_T(x) \to 2^\omega$ be uniformly Turing-invariant. Then, either $x \leq_T f(x)$ or f is constant (literally!).

A Local Approach

Theorem (Bard, 2020)

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Corollary (Bard, 2020)

Under $ZF + TD$, part I of the uniform Martin's conjecture holds.

Turing Determinacy (TD) is the statement "every set of Turing degrees either contains a cone, or is disjoint from a cone".

Local Result in the enumeration degrees

Lemma

Every uniformly enumeration-invariant function has a computable uniformity function.

Theorem (N. C.)

Let $A \in \mathcal{P}(\mathbb{N})$. If $f : \deg_e(A) \to \mathcal{P}(\mathbb{N})$ is uniformly enumeration-invariant and non-constant, then

$$A \leq_e f(A) \quad \text{or} \quad A^\diamond \leq_e f(A).$$

Consequences

Corollary

Part 1 of Martin's Conjecture holds for Turing-to-enumeration uniformly invariant functions.

Theorem (N. C.)

There is a Borel enumeration-invariant function $f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ that is not uniformly enumeration-invariant on any upper cone.

$\mathcal{K}$-pairs

$\mathcal{K}$-pairs are a powerful tool in the enumeration degrees. They were introduced by Kalimullin as a generalization of semicomputable sets.

Definition (Kalimullin)

The pair $\{A, B\}$ forms a (nontrivial) $\mathcal{K}_U$*-pair* if for all $X \geq_e U$

$$\deg_e(X) = \deg_e(A \oplus X) \wedge \deg_e(B \oplus X)$$

They have been used in many structural results about the enumeration degrees

- Definability of the jump. (Kalimullin, 2003)
- Definability of the total degrees. (CGLMS, 2016)
- Every degree is either *almost total* or half of some nontrivial $\mathcal{K}_U$-pair. (GKMS, 2022)

Properties of $\mathcal{K}$-pairs

Theorem (Kalimullin 2003)

Let $\{A, B\}$ be a nontrivial $\mathcal{K}_U$-pair. Then

- $A \oplus U$ *and* $B \oplus U$ *are quasiminimal covers of* U. *That is, if* $X \leq_e A \oplus U$ *is total, then* $X \leq_e U$.
- *The set of* $\mathcal{K}_U$*-partners of* A *form an enumeration-ideal.*
- $A \leq_e B^\diamond \oplus U$ *and* $A^\diamond \leq_e B \oplus U^\diamond$

Non-uniformly invariance

Lemma

For any enumeration degrees $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{u}$ such that $\{\boldsymbol{a}, \boldsymbol{b}\}$ is a nontrivial $\mathcal{K}_{\boldsymbol{u}}$-pair, every uniformly e-invariant function $f : \boldsymbol{a} \to \boldsymbol{b}$ is constant.

Assume towards a contradiction that f is uniformly e-invariant and not constant. By the local theorem, $\boldsymbol{a} \leq \boldsymbol{b}$ or $\boldsymbol{a}^\diamond \leq \boldsymbol{b}$.

1. If $\boldsymbol{a} \leq \boldsymbol{b}$, then $\boldsymbol{u} = (\boldsymbol{a} \vee \boldsymbol{u}) \wedge (\boldsymbol{b} \vee \boldsymbol{u}) = \boldsymbol{a} \vee \boldsymbol{u}$. This means that $\boldsymbol{a} \leq \boldsymbol{u}$, so $\{\boldsymbol{a}, \boldsymbol{b}\}$ is a trivial $\mathcal{K}_{\boldsymbol{u}}$-pair.
2. If instead $\boldsymbol{a}^\diamond \leq \boldsymbol{b}$, then

$$\boldsymbol{u} < \boldsymbol{u}' = \boldsymbol{u}^\diamond \vee \boldsymbol{u} \leq \boldsymbol{a}^\diamond \vee \boldsymbol{u} \leq \boldsymbol{b} \vee \boldsymbol{u}.$$

   Since $\boldsymbol{u}'$ is total, we have contradicted the fact that $\boldsymbol{b} \vee \boldsymbol{u}$ is a quasiminimal cover of $\boldsymbol{u}$.

From Local to Global

Theorem (N. C.)

There is a Borel enumeration-invariant function $f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ that is not uniformly enumeration-invariant on any upper cone.

- You would like to map $A \mapsto B$ whenever $\{A, B\}$ are a nontrivial $\mathcal{K}_U$-pair. However, for each A there are many choices for U and B.
- If U is total, any nontrivial $\mathcal{K}$-pair can be extended to a maximal $\mathcal{K}_U$-pair.
- If you are a nontrivial $\mathcal{K}_U$-pair relative to a total U and you are strictly above U, the choice of U is unique!
- So, for A half of a maximal $\mathcal{K}_U$-pair, map it to B in the other half, ensuring that some $C \equiv_e A$ goes to $D \equiv_e B$ but $D \neq B$.

References I

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