The logics of closures and kernels

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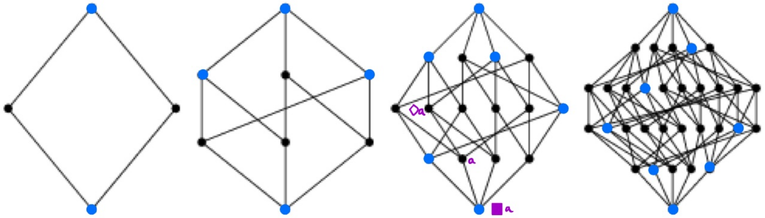


Online Logic Seminar

Thursday, May 8th, 2025

Boolean approximations

Designated objects: propositions, subsets or properties of a system that are directly accessible (e.g. observable, or definable in a fixed language).

We approximate Boolean objects by meets and joins of designated elements above/below it.

Let $\mathcal{A}$ a complete Boolean algebra and $D \subset \mathcal{A}$. Define

$$\Diamond_D(a) := \bigwedge\{d \in D \,|\, a \leq d\}$$
$$\blacksquare_D(a) := \bigvee\{d \in D \,|\, d \leq a\}.$$

Boolean approximations

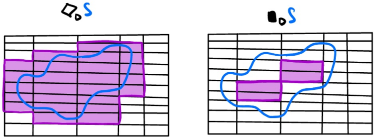
Examples:

- $\mathfrak{M} = (M, \{R_i^{\mathfrak{M}}\}_{i\in I})$ a first-order model, $\mathcal{A} = \mathcal{P}(M)$ and $D = \mathrm{Def}(\mathcal{M})$ the definable subsets.
- Let $\mathcal{A}$ an algebra of propositions, $\mathcal{D}$ the *observation statements.* Let Γ a theory.

$\Diamond_D\Gamma$ is the strongest observable prediction;

$\blacksquare_D\Gamma$ is the minimal sufficient evidence for Γ.

- Knowledge representation, databases, rough sets, abductive reasoning.



- Uniform interpolants
- Non-objectual quantification: distributive properties [Bassett].

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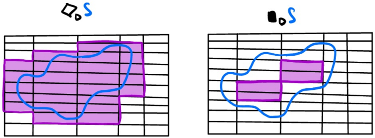

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Approximation models

Let $\mathcal{L}$ given by $\varphi, \psi ::= p \mid \varphi \wedge \psi \mid \varphi \wedge \psi \mid \neg\varphi \mid \Diamond\varphi \mid \blacksquare\varphi$.

An *approximation model* is a structure $(\mathcal{A}, D, \llbracket\cdot\rrbracket)$ with $\mathcal{A}$ a complete Boolean algebra, $D \subset \mathcal{A}$, and a valuation $\llbracket\cdot\rrbracket : \mathcal{L} \to \mathcal{A}$ satisfying

$$\llbracket\Diamond\varphi\rrbracket = \Diamond_D(\llbracket\varphi\rrbracket)$$
$$\llbracket\blacksquare\varphi\rrbracket = \blacksquare_D(\llbracket\varphi\rrbracket).$$

A formula φ holds in $(\mathcal{A}, D, \llbracket\cdot\rrbracket)$ if and only if $\llbracket\varphi\rrbracket = \top_{\mathcal{A}}$: we then write

$$(\mathcal{A}, D, \llbracket\cdot\rrbracket) \models \varphi.$$

We are interested in structures $(\mathcal{A}, \Diamond_D, \blacksquare_D)$ where $D \subseteq \mathcal{A}$.

What is the complete logic of approximation structures $(\mathcal{A}, D)$?

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Class of substructures D	Class of algebras	Corresponding modal logic
$\mathcal{C}$	$\mathbb{K}_{\mathcal{C}}$	$\mathrm{Log}(\mathbb{K}_{\mathcal{C}})$

$$\mathbb{K}_{\mathcal{C}} := \left\{ (\mathcal{A}, \mathsf{c}, \mathsf{k}) \mid \exists D \in \mathcal{C},\ \mathsf{c} = \Diamond_D \text{ and } \mathsf{k} = \blacksquare_D \right\}$$

$$\mathrm{Log}(\mathbb{K}) := \{\varphi \in \mathcal{L} \mid \mathfrak{M} \models \varphi \text{ for all } \mathfrak{M} \in \mathbb{K}\}$$

What is the complete logic of approximation structures $(\mathcal{A}, D)$?

Class of substructures D	Class of algebras	Corresponding modal logic
$\mathcal{C}$	$\mathbb{K}_{\mathcal{C}}$	$\mathrm{Log}(\mathbb{K}_{\mathcal{C}})$
arbitrary sets	?	?
sublattices	?	?
subalgebras	?	?
complete sublattices	?	?
complete subalgebras	?	?
	(representation theorem)	*(completeness theorem)*

$$\mathbb{K}_{\mathcal{C}} := \{(\mathcal{A}, \mathrm{c}, \mathrm{k}) \mid \exists D \in \mathcal{C},\ \mathrm{c} = \Diamond_D \text{ and } \mathrm{k} = \blacksquare_D\}$$

$$\mathrm{Log}(\mathbb{K}) := \{\varphi \in \mathcal{L} \mid \mathfrak{M} \models \varphi \text{ for all } \mathfrak{M} \in \mathbb{K}\}$$

Closure-kernel algebras

A *CK algebra* is a structure $(\mathcal{A}, \mathrm{c}, \mathrm{k})$ with c a closure operator and k a kernel operator with $\mathrm{c}(\bot) = \bot$ and $\mathrm{k}(\top) = \top$.

Closure operator: a map $\mathrm{c} : \mathcal{A} \to \mathcal{A}$ such that, for all $a, b \in \mathcal{A}$:

- $a \leq \mathrm{c}(a)$;
- $\mathrm{c}(\mathrm{c}(a)) = \mathrm{c}(a)$;
- if $a \leq b$ then $\mathrm{c}(a) \leq \mathrm{c}(b)$.

Kernel operator: map $\mathrm{k} : \mathcal{A} \to \mathcal{A}$ such that, for every $a, b \in \mathcal{A}$:

- $\mathrm{k}(a) \leq a$;
- $\mathrm{k}(\mathrm{k}(a)) = \mathrm{k}(a)$;
- if $a \leq b$ then $\mathrm{k}(a) \leq \mathrm{k}(b)$.

Let $D \subseteq \mathcal{A}$. Then $\Diamond_D$ is a closure operator and $\blacksquare_D$ a kernel operator.

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- $a \leq \mathsf{c}(a)$; (T) $\varphi \to \Diamond\varphi$
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Kernel operator: map $\mathsf{k} : \mathcal{A} \to \mathcal{A}$ such that, for every $a, b \in \mathcal{A}$:

- $\mathsf{k}(a) \leq a$; $(\mathrm{T}_\blacksquare)$ $\blacksquare\varphi \to \varphi$
- $\mathsf{k}(\mathsf{k}(a)) = \mathsf{k}(a)$; $(4_\blacksquare)$ $\blacksquare\varphi \to \blacksquare\blacksquare\varphi$
- if $a \leq b$ then $\mathsf{k}(a) \leq \mathsf{k}(b)$. $\dfrac{\varphi \to \psi}{\blacksquare\varphi \to \blacksquare\psi}\ (\mathrm{M}_\blacksquare)$

Let $D \subseteq \mathcal{A}$. Then $\Diamond_D$ is a closure operator and $\blacksquare_D$ a kernel operator.

Inference rules:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ (\mathbf{MP}) \qquad \frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}\ (\mathrm{M}_\Diamond) \qquad \frac{\varphi \to \psi}{\blacksquare\varphi \to \blacksquare\psi}\ (\mathrm{M}_\blacksquare)$$

Axioms:

- (E) Propositional tautologies and dual axioms:
  (Du$_\Diamond$) $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
  (Du$_\blacksquare$) $\blacklozenge\varphi \leftrightarrow \neg\blacksquare\neg\varphi$

- Axioms for the $\Diamond$ modality:
  (N) $\Box\top$
  (T) $\varphi \to \Diamond\varphi$
  (4) $\Diamond\Diamond\varphi \to \Diamond\varphi$

- Axioms for the $\blacksquare$ modality:
  (N$_\blacksquare$) $\blacksquare\top$
  (T$_\blacksquare$) $\blacksquare\varphi \to \varphi$
  (4$_\blacksquare$) $\blacksquare\varphi \to \blacksquare\blacksquare\varphi$

All approximation operators validate EMNT4 $\oplus$ EMNT4.

Approximation operators from complete sublattices

Let $\mathcal{D}$ a complete sublattice of $\mathcal{A}$ (with $\top_{\mathcal{A}}, \bot_{\mathcal{A}} \in \mathcal{D}$).

- $\Diamond_D$ is completely $\bigvee$-distributive:

$$\Diamond_D(\bigvee X) = \bigvee\{\Diamond_D(x) \mid x \in X\}$$

- $\blacksquare_D$ is completely $\bigwedge$-distributive:

$$\blacksquare_D(\bigwedge X) = \bigwedge\{\blacksquare_D(x) \mid x \in X\}$$

- We have

$$\Diamond_D \blacksquare_D (a) = \blacksquare_D (a)$$

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Logic for complete sublattices?

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- Interaction axioms:

  ($\mathrm{F}_\Diamond$) $\Diamond\blacksquare\varphi \leftrightarrow \blacksquare\varphi$

  ($\mathrm{F}_\blacksquare$) $\blacksquare\Diamond\varphi \leftrightarrow \Diamond\varphi$

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  (N) $\Box\top$

  (T) $\varphi \to \Diamond\varphi$

  (4) $\Diamond\Diamond\varphi \to \Diamond\varphi$

  ($\mathrm{C}_\Diamond$) $\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$

- Axioms for the $\blacksquare$ modality:

  ($\mathrm{N}_\blacksquare$) $\blacksquare\top$

  ($\mathrm{T}_\blacksquare$) $\blacksquare\varphi \to \varphi$

  ($4_\blacksquare$) $\blacksquare\varphi \to \blacksquare\blacksquare\varphi$

  ($\mathrm{C}_\blacksquare$) $(\blacksquare\varphi \wedge \blacksquare\psi) \to \blacksquare(\varphi \wedge \psi)$

S4t is sound for the class of closure-kernel algebras generated by complete sublattices.

Complete Sublattices: a duality

Tarski [1955] proved the following duality result for complete meet-semilattices:

$$
\begin{tikzcd}
(\mathcal{A}, \mathrm{c}) \arrow[rd, bend left, "\mathcal{FP}"] \arrow[d, phantom, "\shortparallel"] & \\
(\mathcal{A}, \Diamond_{\mathrm{fp}(\mathrm{c})}) & (\mathcal{A}, \mathrm{fp}(\mathrm{c})) \arrow[l, bend left, "\mathcal{G}"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & (\mathcal{A}, \mathrm{fp}(\Diamond_D)) \arrow[d, phantom, "\shortparallel"] \\
(\mathcal{A}, \Diamond_D) \arrow[ru, bend left, "\mathcal{FP}"] & (\mathcal{A}, D) \arrow[l, bend left, "\mathcal{G}"]
\end{tikzcd}
$$

Complete Sublattices: a duality

Extend Tarski's result to a duality between pairs of operators and complete sublattices:

$$
\begin{tikzcd}
(\mathcal{A}, \mathrm{c}, \mathrm{k}) \arrow[d, phantom, "\shortparallel"] & \\
(\mathcal{A}, \Diamond_{\mathrm{fp(c)}}, \blacksquare_{\mathrm{fp(c)}}) \arrow[r, bend left=60, "\mathcal{FP}"] & (\mathcal{A}, \mathrm{fp(c)}) \arrow[l, bend left=60, "\mathcal{G}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\mathcal{A}, \mathrm{fp}(\Diamond_D)) \arrow[d, phantom, "\shortparallel"] \\
(\mathcal{A}, \Diamond_D, \blacksquare_D) \arrow[ur, bend left, "\mathcal{FP}"] & (\mathcal{A}, D) \arrow[l, bend left, "\mathcal{G}"]
\end{tikzcd}
$$

This yields a representation result:

A CK algebra $(\mathcal{A}, \mathrm{c}, \mathrm{k})$ is representable by a complete sublattice iff we have $\mathrm{c} \circ \mathrm{k} = \mathrm{k}$, $\mathrm{k} \circ \mathrm{c} = \mathrm{c}$, and

$$\mathrm{c}(\bigvee X) = \bigvee \{\mathrm{c}(x) \mid x \in X\}$$
$$\mathrm{k}(\bigwedge X) = \bigwedge \{\mathrm{k}(x) \mid x \in X\}$$

Complete sublattices and tense S4

A *tense algebra* is a Boolean algebra $\mathcal{A}$ with two normal[1] and additive operators p, f and their respective duals defined as $h(x) := \neg p(\neg x)$ and $g(x) := \neg f(\neg x)$, which satisfy

$$\text{for all } a \in \mathcal{A},\ a \leq g(p(a)) \text{ and } a \leq h(f(a)). \tag{1}$$

An S4 *tense algebra* is a tense algebra where the operators p and f are both inflationary and idempotent.

Proposition

The class of CK algebras representable by complete sublattices corresponds exactly to complete S4 *tense algebras.*

[1] An operator m is normal if $m(\bot) = \bot$.

Complete sublattices and tense S4

Theorem

The logic of approximation operators generated by complete sublattices is the tense logic S4t:

$$\mathsf{Log}(\mathsf{CSLat}) = \mathsf{S4t}$$

Representable closure-kernel algebras

What is the logic of approximation operators for arbitrary sets?

A CK algebra $(\mathcal{A}, \mathrm{c}, \mathrm{k})$ *representable* if there exists some $\mathcal{D} \subseteq \mathcal{A}$ such that $\Diamond_D = \mathrm{c}$ and $\blacksquare_D = \mathrm{k}$.

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Example: Let (X, τ) a T_1 topological space. Then $(\mathcal{P}(X), \mathrm{cl}_\tau, \mathrm{id})$ is a representable CK algebra: let D consist of the closed sets.

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Example: Let (X, τ) a T_1 topological space. Then $(\mathcal{P}(X), \mathrm{cl}_\tau, \mathrm{id})$ is a representable CK algebra: let D consist of the closed sets.

Non-example: Let (X, τ) an non-trivial, connected topological space. Consider the algebra $(\mathcal{P}(X), \mathrm{cl}, \mathrm{int})$ with cl and int the closure and interior operators generated by the topology τ.

Representable closure-kernel algebras

Proposition (Representability Criterion)

Let $(\mathcal{A}, \mathrm{c}, \mathrm{k})$ *a CK algebra. The following are equivalent:*

(1) $(\mathcal{A}, \mathrm{c}, \mathrm{k})$ *is representable;*

(2) *Any fixpoint is expressible by common fixpoints:*

$$\mathrm{fp}(\mathrm{c}) = \mathbb{M}[\mathrm{fp}(\mathrm{c}) \cap \mathrm{fp}(\mathrm{k})]$$
$$\mathrm{fp}(\mathrm{k}) = \mathbb{J}[\mathrm{fp}(\mathrm{c}) \cap \mathrm{fp}(\mathrm{k})]$$

Notation: $\mathbb{M}[F] := \{\bigwedge S \mid S \subseteq F\}$ for the set of F-meets and $\mathbb{J}[F] := \{\bigvee S \mid S \subseteq F\}$ for the F-joins.

If c and k are representable, then taking $D := \mathrm{fp}(\mathrm{c}) \cap \mathrm{fp}(\mathrm{k})$ generates c and k.

General duality for representable algebras

$$
\begin{tikzcd}
(\mathcal{A}, \mathrm{c}, \mathrm{k}) \arrow[dr, bend left, "\mathcal{FP}"] \arrow[d, phantom, "\shortparallel"] & \\
(\mathcal{A}, \Diamond_{\mathcal{F}(\mathrm{c},\mathrm{k})}, \blacksquare_{\mathcal{F}(\mathrm{c},\mathrm{k})}) & (\mathcal{A}, \mathcal{F}(\mathrm{c}, \mathrm{k})) \arrow[l, bend left, "\mathcal{G}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\mathcal{A}, \mathcal{F}(\Diamond_D, \blacksquare_D)) \arrow[d, phantom, "\cup\!\shortmid"] \\
(\mathcal{A}, \Diamond_D, \blacksquare_D) \arrow[ur, bend left, "\mathcal{FP}"] & (\mathcal{A}, D) \arrow[l, bend left, "\mathcal{G}"]
\end{tikzcd}
$$

Dualities: here, $\mathcal{F}(\mathrm{c}, \mathrm{k}) := \mathrm{fp}(\mathrm{c}) \cap \mathrm{fp}(\mathrm{k})$ is the set of shared fixpoints.

Representable closure-kernel algebras

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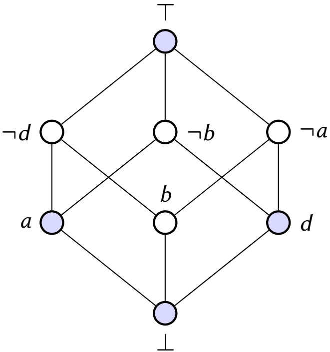
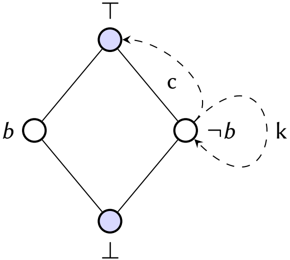


(a) $(\mathcal{A}, \mathrm{c}, \mathrm{k})$ with $\mathrm{fp}(\mathrm{c}) \cap \mathrm{fp}(\mathrm{k})$ shaded.



(b) A non-representable subalgebra of $\mathcal{A}$.

Subnormal logic: the distributivity formula $\mathrm{M}(\varphi \to \psi) \to (\mathrm{M}\varphi \to \mathrm{M}\psi)$ is not valid on representable CK algebras for $\mathrm{M} \in \{\square, \blacksquare\}$.

Representable closure-kernel algebras

The logic of CK algebras is EMNT4 $\oplus$ EMNT4.

Axioms for the $\Diamond$ modality:

(N) $\Box\top$

(T) $\varphi \to \Diamond\varphi$

(4) $\Diamond\Diamond\varphi \to \Diamond\varphi$

Axioms for the $\blacksquare$ modality:

($N_{\blacksquare}$) $\blacksquare\top$

($T_{\blacksquare}$) $\blacksquare\varphi \to \varphi$

($4_{\blacksquare}$) $\blacksquare\varphi \to \blacksquare\blacksquare\varphi$

Does representability force any new validities?

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Does representability force any new validities? No.

Theorem

The logic $\mathsf{EMNT4} \oplus \mathsf{EMNT4}$ *is sound and complete with respect to representable CK algebras.*

Proof idea: showing that every complete CK algebra is isomorphically embeddable into a representable CK algebra.

Representable closure-kernel algebras

Embedding Lemma:

Every complete CK algebra is isomorphically embeddable into a representable CK algebra.

- c-*defective* elements $N_c := \mathrm{fp}(\mathrm{c}) \setminus \mathbb{M}[\mathcal{F}(c, k)]$
- k-*defective* elements $N_{\mathrm{k}} := \mathrm{fp}(\mathrm{k}) \setminus \mathbb{J}[\mathcal{F}(c, k)]$

Take the direct product $\mathcal{A} \otimes \mathcal{A}$ and let

$$D := \big\{(x, x) \,|\, x \in \mathcal{F}(c, k)\big\} \cup \big\{(x, \top), (\top, x) \,\big|\, x \in N_{\mathrm{c}}\big\} \\ \cup \big\{(x, \bot), (\bot, x) \,\big|\, x \in N_{\mathrm{k}}\big\}.$$

The diagonal map $\theta : a \mapsto (a, a)$ embeds the full algebra $(\mathcal{A}, \mathrm{c}, \mathrm{k})$ into the representable algebra $(\mathcal{A} \otimes \mathcal{A}, \Diamond_D, \blacksquare_D)$.

Completeness for arbitrary set approximations

Theorem

The logic EMNT4 $\oplus$ EMNT4 *is sound and complete with respect to representable CK algebras.*

Step 1: For any non-theorem, get a countermodel $\mathfrak{M}$ by taking a canonical *neighbourhood* model (routine).

Step 2: Switching perspectives, the canonical model gives us a complete CK algebra $\mathcal{A}_{\mathfrak{M}}$. So we have a countermodel based on a CK-algebra.

Step 3: By the embedding lemma, the embedding $\mathcal{A}_{\mathfrak{M}} \otimes \mathcal{A}_{\mathfrak{M}}$ with the obvious valuation gives a countermodel based on a representable CK-algebra.

Sublattice-based approximation operators

Now consider the case where D is a sublattice.

- The fusion S4 $\oplus$ S4 is sound with respect to sublattice-generated approximation structures.

Axioms for the $\Diamond$ modality:

(N) $\Box\top$

(T) $\varphi \to \Diamond\varphi$

(4) $\Diamond\Diamond\varphi \to \Diamond\varphi$

($C_\Diamond$) $\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$

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($C_\blacksquare$) $(\blacksquare\varphi \wedge \blacksquare\psi) \to \blacksquare(\varphi \wedge \psi)$

Sublattices: finite algebraic models?

Consider

$$\varphi = \Diamond p \wedge \neg \blacksquare \Diamond p$$

Note that $\Diamond p \to \blacksquare \Diamond p$ is valid on any complete sublattice model. But any sublattice of a finite lattice is complete sublattice, so φ has no sublattice-countermodel based on any finite algebra.

Yet φ is satisfiable: take $(\mathcal{P}(\mathbb{R}), \mathcal{O})$ where $\mathcal{O}$ are the standard opens, and set $V(p) = [0, 1]$.

Observation

The logic $\mathsf{S4} \oplus \mathsf{S4}$ does not have the finite model property with respect to sublattice-based CK algebras.

Sublattices: finite algebraic models?

A slightly less trivial example: the convergence formula $\Diamond\blacksquare\varphi \to \blacksquare\Diamond\varphi$.

For sublattice-based approximations, this holds in any *finite* approximation structure.

But it fails for sublattice-based approximations in general. Consider the structure $(\mathcal{P}(\mathbb{N}), D)$ where

$$D := \mathrm{COF}[1] \cup \mathcal{P}_{<\omega}(\mathbb{N} \setminus \{1, 2\})$$

D is the sublattice of $\mathcal{P}(\mathbb{N})$ consisting of all cofinite sets containing 1 and all finite sets containing neither 1 nor 2. Now let $A := \{2n \mid n \in \mathbb{N}\}$.

$$\Diamond\blacksquare A = (A \setminus \{2\}) \cup \{1\}$$

$$\blacksquare\Diamond A = A \setminus \{2\}$$

Sublattices: a topological connection

A bitopological space is a structure (X, τ_1, τ_2) where both τ_1 and τ_2 are topologies on X. The representability criterion gives us:

Lemma

A bitopological space (X, τ_1, τ_2) is pairwise zero-dimensional if and only if the algebra $(\mathcal{P}(X), \mathrm{cl}_{\tau_1}, \mathrm{int}_{\tau_2})$ is sublattice-representable.

The algebra $(\mathcal{P}(X), \mathrm{cl}_{\tau_1}, \mathrm{int}_{\tau_2})$ is representable exactly if:

$$\mathrm{fp}(\mathrm{cl}_{\tau_1}) = \mathbb{M}\big(\mathrm{fp}(\mathrm{cl}_{\tau_1}) \cap \mathrm{fp}(\mathrm{int}_{\tau_2})\big) \qquad \mathcal{C}_1 = \mathbb{M}(\mathcal{C}_1 \cap \mathcal{O}_2)$$

$$\mathrm{fp}(\mathrm{int}_{\tau_2}) = \mathbb{J}\big(\mathrm{fp}(\mathrm{cl}_{\tau_1}) \cap \mathrm{fp}(\mathrm{int}_{\tau_2})\big) \qquad \mathcal{O}_2 = \mathbb{J}(\mathcal{C}_1 \cap \mathcal{O}_2)$$

i.e., τ_1 admits a basis of τ_2-closed sets, and vice-versa.[2]

- For Alexandrov bitopologies generated by a Kripke frame, this means $\mathcal{C}_1 = \mathcal{O}_2$. For $\mathsf{S4} \oplus \mathsf{S4}$ frames, this means $R_1 = R_2^{-1}$. So we cannot obtain completeness via canonical Kripke frames.

[2] Notion introduced by I. L Reilly [1973].

Sublattices: a topological connection

Let τ the standard subspace topology on $\mathbb{Q}$ inherited from $\mathbb{R}$.

The *horizontal-vertical bitopology* on $\mathbb{Q} \times \mathbb{Q}$ is given by the *horizontal* topology $\mathcal{H}$ generated by the basis $\left\{ B \times \{q\} \,\middle|\, q \in \mathbb{Q},\ B \in \tau \right\}$ and the *vertical* topology $\mathcal{V}$ generated by $\left\{ \{q\} \times B \,\middle|\, q \in \mathbb{Q},\ B \in \tau \right\}$.

Theorem (Van Benthem, Bezhanishvili, Sarenac, ten Cate 2006)

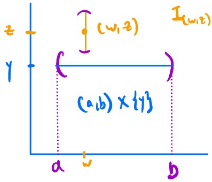
S4 $\oplus$ S4 *is topologically complete with respect to the space* $\mathbb{Q} \times \mathbb{Q}$ *equipped with the horizontal-vertical bitopology.*

Sublattices: a topological connection

Theorem

S4 ⊕ S4 is complete with respect to sublattice-representable CK algebras.

From any topological countermodel on $(\mathbb{Q} \times \mathbb{Q}, \mathcal{H}, \mathcal{V})$ we obtain an algebraic countermodel $\big(\mathcal{P}(\mathbb{Q} \times \mathbb{Q}), \mathrm{cl}_{\mathcal{H}}, \mathrm{int}_{\mathcal{V}}\big)$. To show this sublattice-representable, enough to observe that $(\mathbb{Q} \times \mathbb{Q}, \mathcal{H}, \mathcal{V})$ is pairwise zero-dimensional.



$I_{(w,z)} = \{w\} \times (z-\delta, z+\delta)$

$\delta = \begin{cases} 1/2\,|z-y| \ ; & z \neq y \\ 1 & \text{o/w} \end{cases}$

$A \in \mathcal{H} \Rightarrow A = \bigcup_{(x,y) \in A} I_{(x,y)}$

Corollary

S4 ⊕ S4 is complete with respect to pairwise zero-dimensional bitopological spaces.

Subalgebras: the collapse of modalities

When D is a subalgebra of $\mathcal{A}$, the operators $\Diamond_D$ and $\blacksquare_D$ become duals:

$$\Diamond_D(a) = \neg \blacksquare_D(\neg a)$$

(1) S5 is the complete logic of approximation structures $(\mathcal{A}, D)$ where D is a complete subalgebra of $\mathcal{A}$.

(2) The complete logic of subalgebra-generated approximation operators is monomodal S4.

(1): immediate via (taking the complex algebras of) Kripke frames.

(2): by completeness of S4 with respect to $\mathbb{Q}$.

S4 and S5 can be seen as logics of approximation through subalgebras.

Summary of results

Class of $(\mathcal{A}, D)$ structures	**Duality?**	**Representation for $(\mathcal{A}, \mathrm{c}, \mathrm{k})$**	**Corresponding modal logic**
[No duality with $(\mathcal{A}, D)$ structures]	no	c closure, k kernel, $\mathrm{c}(\bot) = \bot$, $\mathrm{k}(\top) = \top$	$\mathsf{EMNT4}_\Diamond \oplus \mathsf{EMNT4}_\blacksquare$
All	quasi	as above, plus $\mathrm{fp}(\mathrm{c}) \subseteq \mathbb{M}[\mathrm{fp}(\mathrm{c}) \cap \mathrm{fp}(\mathrm{k})]$ $\mathrm{fp}(\mathrm{k}) \subseteq \mathbb{J}[\mathrm{fp}(\mathrm{c}) \cap \mathrm{fp}(\mathrm{k})]$ [*representable algebras*]	$\mathsf{EMNT4}_\Diamond \oplus \mathsf{EMNT4}_\blacksquare$
D is a sublattice with $\bot, \top \in D$	quasi	representable additive CK algebras	$\mathsf{S4} \oplus \mathsf{S4}$ (add distributivity axioms)
D is a subalgebra	quasi	as above, plus $\mathrm{k}(a) = \neg\mathrm{c}(\neg a)$	$\mathsf{S4}$ ($\mathsf{S4}_\Diamond$ with $\blacksquare\varphi \leftrightarrow \neg\Diamond\neg\varphi$)
D is a complete sublattice with $\bot, \top \in D$	$\cong$	CK algebras with $\mathrm{c} \circ \mathrm{k} = \mathrm{k}$, $\mathrm{k} \circ \mathrm{c} = \mathrm{c}$, $\mathrm{c}(\bigvee X) = \bigvee_{x \in X} \mathrm{c}(x)$, $\mathrm{k}(\bigwedge X) = \bigwedge_{x \in X} \mathrm{k}(x)$	$\mathsf{S4t} = (\mathsf{S4}_\Diamond \oplus \mathsf{S4}_\blacksquare) \oplus \{\Diamond\blacksquare\varphi \leftrightarrow \blacksquare\varphi, \blacksquare\Diamond\varphi \leftrightarrow \Diamond\varphi\}$
D is a complete subalgebra	$\cong$	as above, plus $\mathrm{k}(a) = \neg\mathrm{c}(\neg a)$	$\mathsf{S5}$ ($\mathsf{S5}_\Diamond$ with $\blacksquare\varphi \leftrightarrow \neg\Diamond\neg\varphi$)

Conclusion

$$
\begin{tikzcd}
(\mathcal{A}, \mathrm{c}, \mathrm{k}) \arrow[d, phantom, "\parallel"] \arrow[rr, bend left, "\mathcal{FP}"] & & \\
(\mathcal{A}, \Diamond_{\mathcal{F}(\mathrm{c},\mathrm{k})}, \blacksquare_{\mathcal{F}(\mathrm{c},\mathrm{k})}) & & (\mathcal{A}, \mathcal{F}(\mathrm{c}, \mathrm{k})) \arrow[ll, bend left, "\mathcal{G}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& (\mathcal{A}, \mathcal{F}(\Diamond_D, \blacksquare_D)) \arrow[d, phantom, "\cup\!\mid"] \\
(\mathcal{A}, \Diamond_D, \blacksquare_D) \arrow[ur, bend left, "\mathcal{FP}"] & (\mathcal{A}, D) \arrow[l, bend left, "\mathcal{G}"]
\end{tikzcd}
$$

We determined the modal logics of approximation for the most salient types of approximation structures: in each case, we identified the corresponding class of algebras.

- The emergence of various modal laws traced to simple structural features of the generating set;
- We recovered several well-known bimodal logics as logics of approximation;
- Interplay of algebraic and topological methods in modal logic:
  - Duality between CK algebras and approximation structures;
  - Bitopological spaces.

Other directions?

- Multimodal case: algebra of substructures.

  $\Diamond_D$, $\blacksquare_D$ $(D \in \mathbb{D})$ given by a class $\mathbb{D}$ of substructures $D \subseteq \mathcal{A}$, itself equipped with its own algebraic operations for merging or combining approximating sets. E.g. operations on partitions. Interaction laws between operators $\Diamond_{\Pi_1 \wedge \Pi_2}$, $\blacksquare_{\Pi_1 \vee \Pi_2}$, etc.

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- Uniform Interpolation

  Take the Lindenbaum algebra of a locally tabular logic. Let $\mathcal{D}(\Lambda)$ the collection of (equivalence classes of) formulas in the sublanguage generated by $\Lambda = \{q_1, \ldots, q_n\}$. Then $\Diamond_{\mathcal{D}(\Lambda)}([\varphi]) = \bigwedge \{[\psi] \in \mathcal{D}(\Lambda) \mid \vdash \varphi \to \psi\}$ is the (equivalence class of) the *uniform post-interpolant* of φ, and $\blacksquare_{\mathcal{D}(\Lambda)}$ yields the *uniform pre-interpolant.*

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- Stone-type dualities:

  There is a natural duality between complete-sublattice-representable CK algebras (i.e. tense S4 algebras) and bicompact, strongly pairwise-zero-dimensional bitopologies [Bezhanishvili et al., 2010]. Is there a natural Stone duality for sublattice-representable CK algebras?

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Thank you!

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Modal systems for closures and kernels

Inference rules:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ (\mathbf{MP}) \qquad \frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}\ (\mathrm{M}_{\Diamond}) \qquad \frac{\varphi \to \psi}{\blacksquare\varphi \to \blacksquare\psi}\ (\mathrm{M}_{\blacksquare})$$

Axioms:

- (E) Propositional tautologies and dual axioms:
  (Du$_{\Diamond}$) $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
  (Du$_{\blacksquare}$) $\blacklozenge\varphi \leftrightarrow \neg\blacksquare\neg\varphi$
- Axioms for the $\Diamond$ modality:
  (N) $\Box\top$
  (T) $\varphi \to \Diamond\varphi$
  (4) $\Diamond\Diamond\varphi \to \Diamond\varphi$
  (C$_{\Diamond}$) $\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$
- Axioms for the $\blacksquare$ modality:
  (N$_{\blacksquare}$) $\blacksquare\top$
  (T$_{\blacksquare}$) $\blacksquare\varphi \to \varphi$
  (4$_{\blacksquare}$) $\blacksquare\varphi \to \blacksquare\blacksquare\varphi$
  (C$_{\blacksquare}$) $(\blacksquare\varphi \wedge \blacksquare\psi) \to \blacksquare(\varphi \wedge \psi)$
- Interaction axioms:
  (F$_{\Diamond}$) $\Diamond\blacksquare\varphi \leftrightarrow \blacksquare\varphi$
  (F$_{\blacksquare}$) $\blacksquare\Diamond\varphi \leftrightarrow \Diamond\varphi$

Systems:

- S4t: all rules and axioms above.
- $\mathrm{S4} \oplus \mathrm{S4} = \mathrm{ECMNT4} \oplus \mathrm{ECMNT4} = \mathrm{S4t} \setminus \{\mathrm{F}_{\Diamond}, \mathrm{F}_{\blacksquare}\}$