Computable type: an overview

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Inria Nancy (France)

Online Logic Seminar

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Computable type

A compact set $K \subseteq \mathbb{R}^n$ is:

- **Computable** if the set of rational balls intersecting K is decidable,
- **Semicomputable** if the set of rational balls that are disjoint from K is computably enumerable (c.e.).

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Example

There is a semicomputable disk in $\mathbb{R}^2$ which is not computable.
Center $(0, 0)$, radius $1 - \sum_{n \in \text{halting set}} 2^{-n}$.

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Question

Is there a semicomputable *circle* which is not computable?

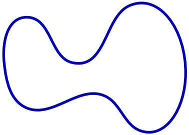
Computable type

Spheres

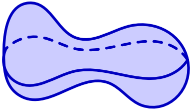
Theorem ([Miller 2002])

If $X \subseteq \mathbb{R}^m$ is homeomorphic to an n-dimensional sphere, then

$$X \text{ is semicomputable} \iff X \text{ is computable.}$$



1-sphere



2-sphere

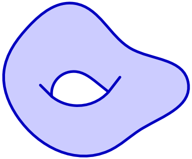
Computable type

Manifolds

Theorem ([Iljazović 2013])

If $X \subseteq \mathbb{R}^m$ is a closed manifold, then

$$X \text{ is semicomputable} \iff X \text{ is computable.}$$



Torus

Computable type

Other spaces?

Definition

A compact space X has **computable type** if for every set $K \subseteq \mathbb{R}^m$ that is homeomorphic to X,

$$K \text{ is semicomputable} \iff K \text{ is computable.}$$

Computable type

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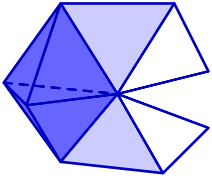
We give a characterization of the simplicial complexes that have computable type.

Computable type

Simplicial complexes

A simplicial complex is made of simplices of any dimensions.

- 1-simplex = edge
- 2-simplex = triangle,
- 3-simplex = tetrahedron,
- Etc.



Main question

Which finite simplicial complexes X have computable type?

Computable type

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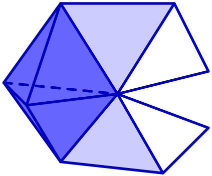
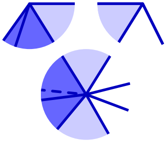


Main question

Which finite simplicial complexes X have computable type?

Answer

Look at the stars...

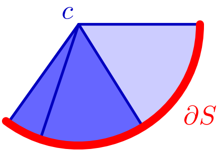


Computable type

Simplicial complexes

A star S consists of:

- A center c,
- A boundary ∂S,
- Rays from c to ∂S.



Definition

A star $(S, \partial S)$ has the **surjection property** if every continuous function $f : S \to S$ such that $f|_{\partial S} = \mathrm{id}_{\partial S}$ is surjective.

Computable type

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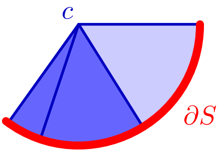

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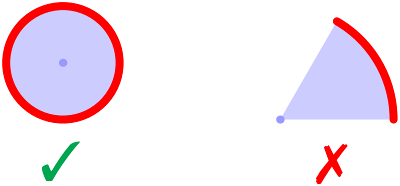
Examples



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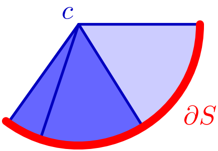


Definition

A star $(S, \partial S)$ has the **surjection property** if every continuous function $f : S \to S$ such that $f|_{\partial S} = \mathrm{id}_{\partial S}$ is surjective.

Theorem ([Amir, H. 2021])

A finite simplicial complex has computable type $\iff$ every star has the surjection property.

Computable type

The surjection property

Topological invariants

Products

The surjection property

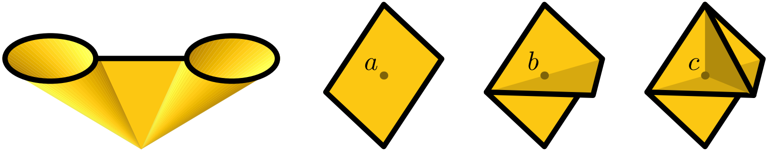
How to determine whether a star $(S, \partial S)$ has the surjection property?

- A partial characterization using (homology) cycles,
- A complete characterisation using homotopy.

The surjection property

Cycles

The boundary of a 2-dimensional star is a graph.



The surjection property

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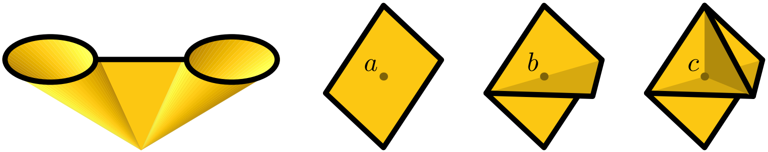

Theorem ([Amir, H. 2021])

A 2-dimensional star has the surjection property $\iff$ its boundary is a union of cycles.

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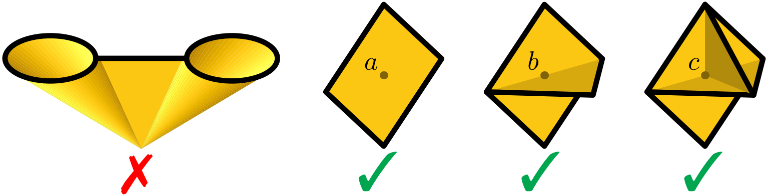
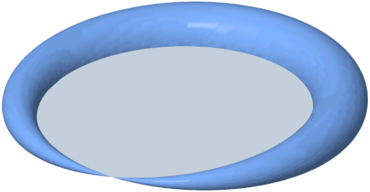


Theorem ([Amir, H. 2021])

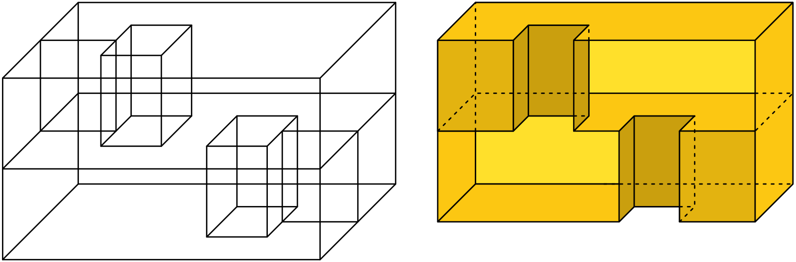
A 2-dimensional star has the surjection property $\iff$ *its boundary is a union of cycles.*

The surjection property

Cycles



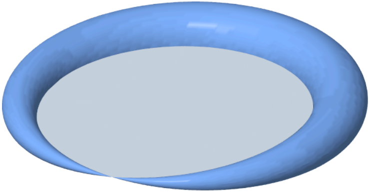
(a) Crescent



(b) House with two rooms + half-cut

The surjection property

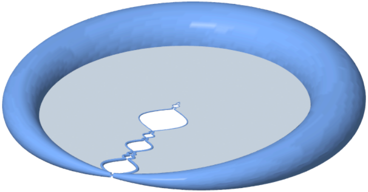
Cycles



Corollary

The "crescent" does not have computable type.

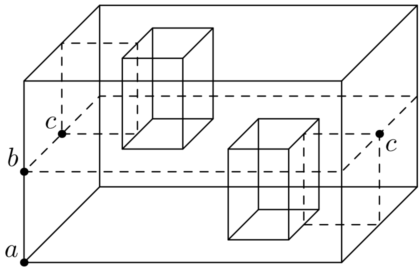
Indeed, this star does not have the surjection property:

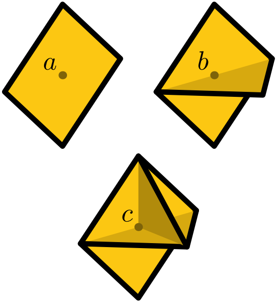


Corollary

The "house with two rooms" has computable type.

Indeed, all the stars have the surjection property:



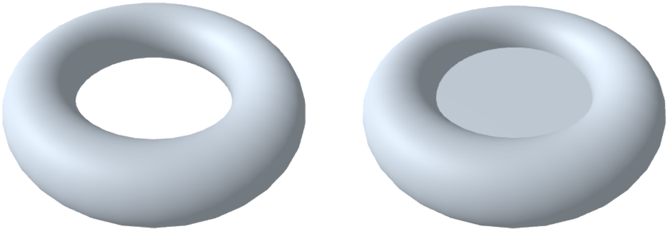


Higher dimensions

Theorem ([Amir, H. 2021])

A pure n-dimensional star has the surjection property $\iff$ its boundary is a union of $(n-1)$-dimensional cycles.

Higher-dimensional cycles are described by homology.



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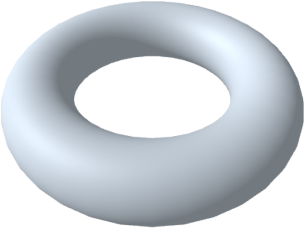
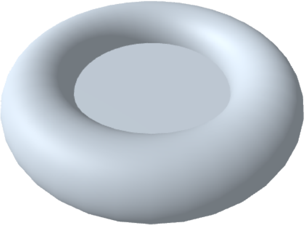


✓ Union of cycles

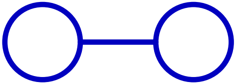


✗ Not a union of cycles

The surjection property

Homotopy

- Let X be a space,



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Homotopy

- Let X be a space,
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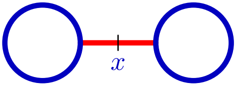

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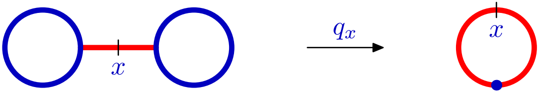

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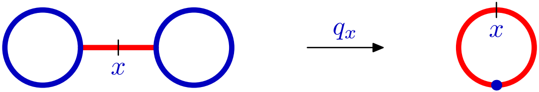

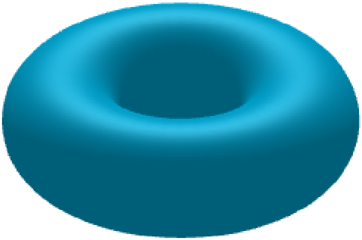

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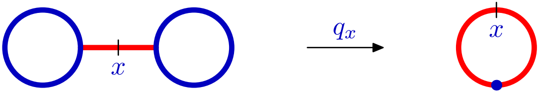

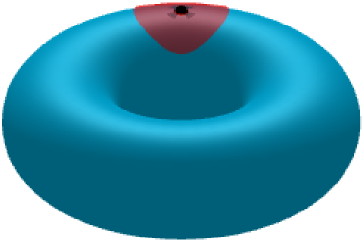

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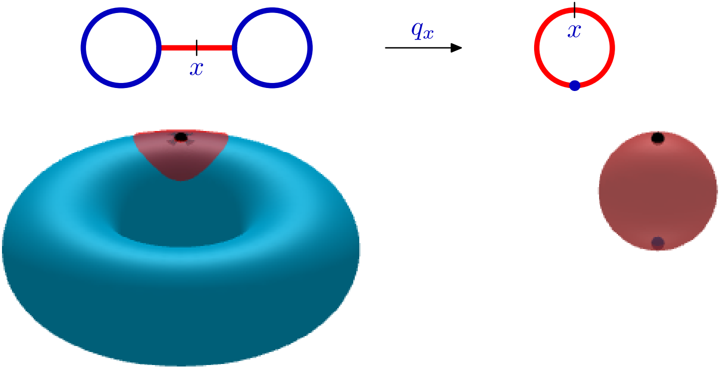

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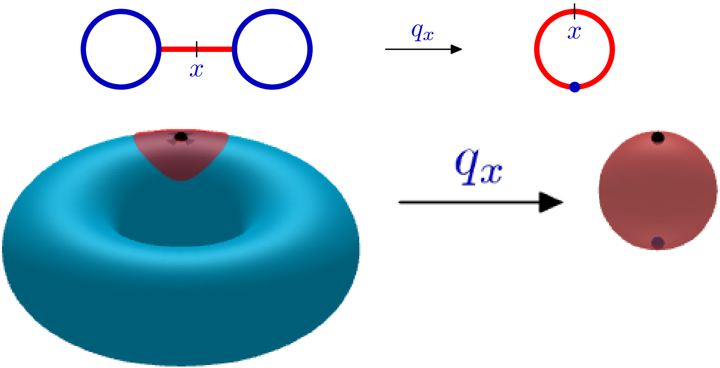

The surjection property

Homotopy

Theorem

A star $(S, \partial S)$ *has the surjection property* $\iff \forall x \in \partial S$, *the quotient map* $q_x : \partial S \to \mathbb{S}_n$ *is not homotopic to a constant.*

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Corollary

Whether a finite simplicial complex has computable type is decidable.

Proof.

The article [1] shows that homotopy between maps is decidable. □

[1] M. Filakovský, L. Vokrínek, **Are two given maps homotopic? An algorithmic viewpoint**, Found. Comput. Math. 20 (2) (2020) 311–330

Computable type

The surjection property

Topological invariants

Products

Topological invariants

Let X be a compact space and $\mathcal{P}$ a topological invariant.

Theorem ([Amir, H. 2021])

If X is minimal satisfying $\mathcal{P}$ and $\mathcal{P}$ is Σ_2^0, then X has computable type.

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- A rational ball B intersects $K \iff K \setminus B \notin \mathcal{P}_n$,
- As $\mathcal{P}_n$ is Π_1^0, the latter can be effectively tested. □

Topological invariants

Miller and Iljazović proved that spheres, more generally closed manifolds, have computable type. Are they minimal for some Σ_2^0 invariant?

Topological invariants

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Definition (The invariant $\mathcal{H}_n$)

A space X satisfies $\mathcal{H}_n$ iff $\exists f : X \to \mathbb{S}_n$ which is not homotopic to a constant.

- $\mathcal{H}_n$ is Σ^0_2,
- Every closed n-manifold is minimal satisfying $\mathcal{H}_n$.

Products

Question [Čelar, Iljazović 2021]

If X and Y both have computabe type, does $X \times Y$ have computable type?

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Answer [Amir, H. 2023]

No. There exists X that has computable type, but $X \times \mathbb{S}_1$ does not.

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Answer [Amir, H. 2023]

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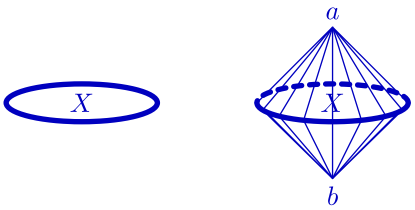
Again, we reduce the problem to homotopy of maps.

Suspension

The **suspension** of a space X is the space ΣX obtained as follows:

- Add two points a, b to X,
- For each $x \in X$, add a segment from x to a, and a segment from x to b.



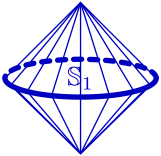
(a) X (b) ΣX

Suspension

The suspension of a sphere is a sphere:

$$\Sigma\mathbb{S}_n = \mathbb{S}_{n+1}.$$

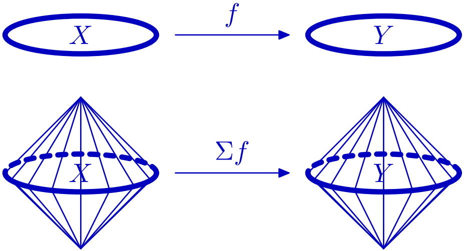
(a) $\mathbb{S}_1$



(b) $\Sigma\mathbb{S}_1 \cong \mathbb{S}_2$

Suspension

The **suspension** of a function $f : X \to Y$ is $\Sigma f : \Sigma X \to \Sigma Y$.

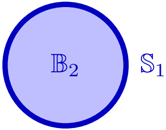


Suspension

When X is a simplicial complex, we obtain a further characterization of the X's such that ΣX has computable type.

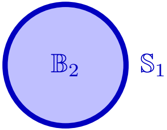
A family of spaces

- The boundary of the ball $\mathbb{B}_{n+1}$ is $\mathbb{S}_n$.



A family of spaces

- The boundary of the ball $\mathbb{B}_{n+1}$ is $\mathbb{S}_n$.
- Let $f : \mathbb{S}_n \to \mathbb{S}_p$. We attach $\mathbb{B}_{n+1}$ to $\mathbb{B}_{p+1}$ along their boundaries using f: each $x \in \mathbb{S}_n$ is glued to $f(x) \in \mathbb{S}_p$.



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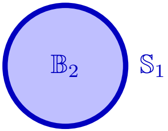

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- We obtain the space $X_f = \mathbb{B}_{p+1} \cup_f \mathbb{B}_{n+1}$ *(click on the picture below to launch animation, the file ajunction.mp4 should be stored in the same folder as the slides)*

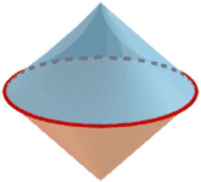

Figure: X_f where $f : \mathbb{S}_1 \to \mathbb{S}_1$ is the doubling map

A family of spaces

Theorem

ΣX_f has computable type $\iff$ Σf is not homotopic to a constant.

$\Sigma X_f \times \mathbb{S}_1$ has computable type $\iff$ $\Sigma^2 f$ is not homotopic to a constant.

A family of spaces

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- $\Sigma f : \mathbb{S}_8 \to \mathbb{S}_4$ is not homotopic to a constant,
- $\Sigma^2 f : \mathbb{S}_9 \to \mathbb{S}_5$ is homotopic to a constant.

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Corollary

ΣX_f and $\mathbb{S}_1$ have computable type, but $\Sigma X_f \times \mathbb{S}_1$ does not.

Summary

We give topological characterization of computable type, in terms of:

- the surjection property,
- homology,
- homotopy.

We derive applications: computable type is:

- visually decidable for 2-dimensional spaces,
- decidable,
- not preserved by products.

An open question

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If X is minimal satisfying some Σ_2^0 invariant, then X has computable type.

The converse implication fails in general. Does it hold for finite simplicial complexes?

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Thank you for your attention!