Spectral Theory and Descriptive Combinatorics for Borel PMP Graphs

Cecelia Higgins (joint with Pieter Spaas, Alexander Tenenbaum, and Sofia Torelli)

Rutgers University, New Brunswick

Southern Illinois University Logic Seminar

Motivation

- The goal of classical spectral graph theory:

Motivation

- The goal of classical spectral graph theory: Use linear algebra to extract combinatorial information about finite graphs.

Motivation

- The goal of classical spectral graph theory: Use linear algebra to extract combinatorial information about finite graphs.
- **Our goal:** Use operator theory to extract descriptive combinatorial information about infinite (bounded-degree Borel pmp) graphs.

The Basics

Proposition (Greedy Bound)

$\chi(\mathcal{G}) \leq \Delta(\mathcal{G}) + 1.$

The Basics

Proposition (Greedy Bound)

$\chi(\mathcal{G}) \leq \Delta(\mathcal{G}) + 1.$

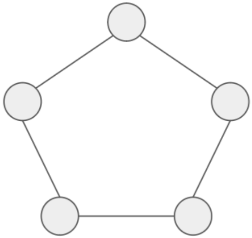
This is best possible:

The Basics

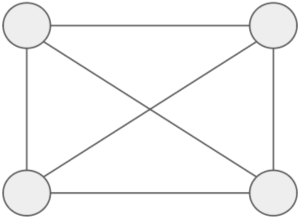
Proposition (Greedy Bound)

$\chi(\mathcal{G}) \leq \Delta(\mathcal{G}) + 1.$

This is best possible:

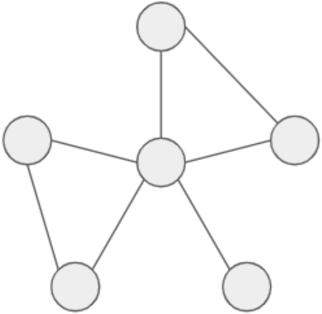


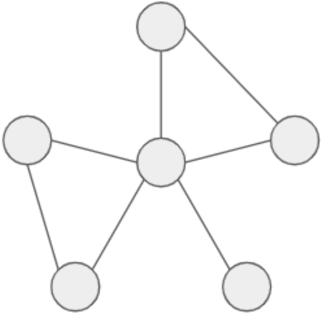
Odd cycles



Complete graphs

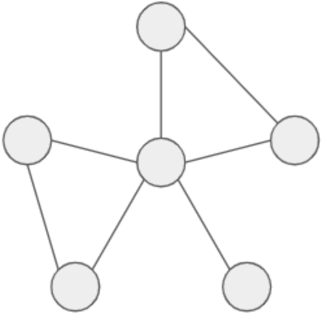
The Basics





Here $\chi(\mathcal{G}) = 3$, but the greedy bound only gives $\chi(\mathcal{G}) \leq 6$.

The Basics



Here $\chi(\mathcal{G}) = 3$, but the greedy bound only gives $\chi(\mathcal{G}) \leq 6$.
The one high-degree vertex strongly influences the greedy bound.

Classical Spectral Graph Theory

Let $|X| = n$.

Classical Spectral Graph Theory

Let $|X| = n$.

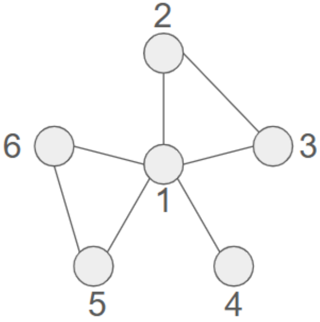
- **Adjacency matrix:** The $n \times n$ matrix $A(\mathcal{G})$ such that $A(\mathcal{G})_{ij} = 1$ if vertices i and j are $\mathcal{G}$-adjacent, 0 otherwise.

Classical Spectral Graph Theory

Let $|X| = n$.

- **Adjacency matrix:** The $n \times n$ matrix $A(\mathcal{G})$ such that $A(\mathcal{G})_{ij} = 1$ if vertices i and j are $\mathcal{G}$-adjacent, 0 otherwise.
- By the **spectral theorem,** all eigenvalues of the adjacency matrix are real.

Classical Spectral Graph Theory

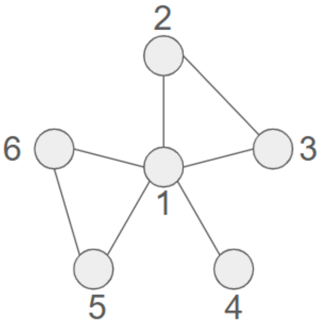


$\mathcal{G}$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$A(\mathcal{G})$

Classical Spectral Graph Theory



$\mathcal{G}$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$A(\mathcal{G})$

$$\sigma(A(\mathcal{G})) = \{2.7, 1, 0.2, -1, -1, -1.9\}$$

Classical Spectral Graph Theory

We write $\sigma(A(\mathcal{G})) = \{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n\}$.

We write $\sigma(A(\mathcal{G})) = \{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n\}$.

Proposition

- $\mu_1 \leq \Delta(\mathcal{G})$.

Classical Spectral Graph Theory

We write $\sigma(A(\mathcal{G})) = \{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n\}$.

Proposition

- $\mu_1 \leq \Delta(\mathcal{G})$. If $\mathcal{G}$ is connected, then equality holds if and only if $\mathcal{G}$ is regular.

Classical Spectral Graph Theory

We write $\sigma(A(\mathcal{G})) = \{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n\}$.

Proposition

- $\mu_1 \leq \Delta(\mathcal{G})$. If $\mathcal{G}$ is connected, then equality holds if and only if $\mathcal{G}$ is regular.
- $\deg_{av}(\mathcal{G}) = \frac{1}{n}\sum_{x \in X} \deg_{\mathcal{G}}(x) \leq \mu_1$.

Classical Spectral Graph Theory

We write $\sigma(A(\mathcal{G})) = \{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n\}$.

Proposition

- $\mu_1 \leq \Delta(\mathcal{G})$. If $\mathcal{G}$ is connected, then equality holds if and only if $\mathcal{G}$ is regular.
- $\deg_{av}(\mathcal{G}) = \frac{1}{n}\sum_{x \in X} \deg_{\mathcal{G}}(x) \leq \mu_1$.
- If $E(\mathcal{G}) \neq \emptyset$, then $\mu_1 > 0$ and $\mu_n < 0$.

Classical Spectral Graph Theory

We write $\sigma(A(\mathcal{G})) = \{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n\}$.

Proposition

- $\mu_1 \leq \Delta(\mathcal{G})$. If $\mathcal{G}$ is connected, then equality holds if and only if $\mathcal{G}$ is regular.
- $\deg_{av}(\mathcal{G}) = \frac{1}{n}\sum_{x \in X} \deg_{\mathcal{G}}(x) \leq \mu_1$.
- If $E(\mathcal{G}) \neq \emptyset$, then $\mu_1 > 0$ and $\mu_n < 0$.
- (Perron–Frobenius) $\mu_1 \geq |\mu_i|$ for all $i \leq n$.

Wilf's Theorem

It is possible to give a spectral upper bound on the chromatic number.

Wilf's Theorem

It is possible to give a spectral upper bound on the chromatic number.

Theorem [Wilf]

$\chi(\mathcal{G}) \leq \lfloor \mu_1 \rfloor + 1.$

Wilf's Theorem

It is possible to give a spectral upper bound on the chromatic number.

Theorem [Wilf]

$\chi(\mathcal{G}) \leq \lfloor \mu_1 \rfloor + 1.$

When $\mathcal{G}$ is regular: We recover the greedy bound.

Wilf's Theorem

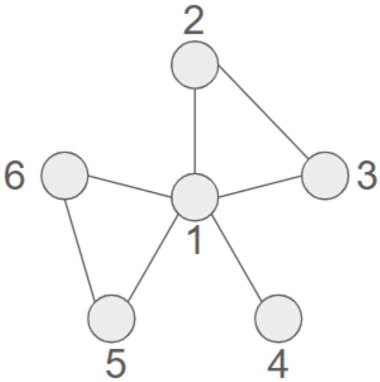
It is possible to give a spectral upper bound on the chromatic number.

Theorem [Wilf]

$\chi(\mathcal{G}) \leq \lfloor \mu_1 \rfloor + 1.$

When $\mathcal{G}$ is regular: We recover the greedy bound.
When $\mathcal{G}$ is non-regular: We can improve on the greedy bound.

Wilf's Theorem



$\chi(\mathcal{G}) = 3$, $\mu_1 \approx 2.7$. Greedy: $\chi(\mathcal{G}) \leq 6$, Wilf: $\chi(\mathcal{G}) \leq 3$.

Hoffman's Theorem

Analogously, there are spectral *lower* bounds on the chromatic number.

Hoffman's Theorem

Analogously, there are spectral *lower* bounds on the chromatic number.

Theorem [Hoffman]

$\chi(\mathcal{G}) \geq \lceil 1 - \frac{\mu_1}{\mu_n} \rceil.$

Hoffman's Theorem

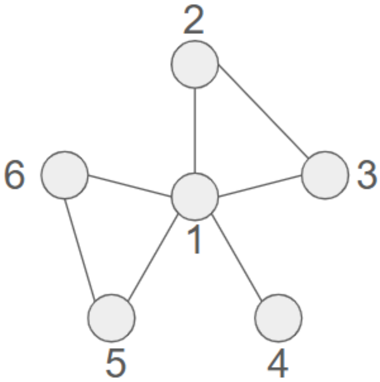
Analogously, there are spectral *lower* bounds on the chromatic number.

Theorem [Hoffman]

$\chi(\mathcal{G}) \geq \lceil 1 - \frac{\mu_1}{\mu_n} \rceil$.

Combining Wilf's theorem and Hoffman's theorem, we can in some cases exactly compute the chromatic number.

Hoffman's Theorem



$\chi(\mathcal{G}) = 3$, $\mu_1 \approx 2.7$, $\mu_6 \approx -1.9$. Greedy: $\chi(\mathcal{G}) \leq 5$, Wilf: $\chi(\mathcal{G}) \leq 3$, Hoffman: $\chi(\mathcal{G}) \geq 3$.

- **Borel graph:** A graph $\mathcal{G} = (X, E)$ such that X is a standard Borel space and E is a Borel relation.

- **Borel graph:** A graph $\mathcal{G} = (X, E)$ such that X is a standard Borel space and E is a Borel relation.
- **Borel pmp graph:** A Borel graph $\mathcal{G} = (X, E)$ such that X has a Borel probability measure μ and $E_{\mathcal{G}}$ is μ-pmp.

- **Borel graph:** A graph $\mathcal{G} = (X, E)$ such that X is a standard Borel space and E is a Borel relation.
- **Borel pmp graph:** A Borel graph $\mathcal{G} = (X, E)$ such that X has a Borel probability measure μ and $E_{\mathcal{G}}$ is μ-pmp.
- **Example:** Let Γ be a countable discrete group with finite generating set S, and let $\cdot : \Gamma \times X \to X$ be a Borel measure-preserving group action, i.e.:

$$\forall \gamma \in \Gamma, \forall A \subseteq X \text{ measurable}, \mu(\gamma \cdot A) = \mu(A).$$

- **Borel graph:** A graph $\mathcal{G} = (X, E)$ such that X is a standard Borel space and E is a Borel relation.
- **Borel pmp graph:** A Borel graph $\mathcal{G} = (X, E)$ such that X has a Borel probability measure μ and $E_{\mathcal{G}}$ is μ-pmp.
- **Example:** Let Γ be a countable discrete group with finite generating set S, and let $\cdot : \Gamma \times X \to X$ be a Borel measure-preserving group action, i.e.:

$$\forall \gamma \in \Gamma, \forall A \subseteq X \text{ measurable}, \mu(\gamma \cdot A) = \mu(A).$$

Then the Schreier graph $\mathcal{G}(\Gamma, X, S)$ is a Borel pmp graph.

We work in the setting of *bounded-degree Borel pmp graphs.*

We work in the setting of *bounded-degree Borel pmp graphs.*
Why?

Spectral Graph Theory and Descriptive Combinatorics

We work in the setting of *bounded-degree Borel pmp graphs.*
Why? So we can use the theory of bounded self-adjoint operators!

We work in the setting of *bounded-degree Borel pmp graphs.*
Why? So we can use the theory of bounded self-adjoint operators!

Definition

The *adjacency operator* $T_{\mathcal{G}} : L^2(X, \mu) \to L^2(X, \mu)$ of $\mathcal{G}$ is given by

$$(T_{\mathcal{G}} f)(x) = \sum_{yEx} f(y).$$

Spectral Graph Theory and Descriptive Combinatorics

We work in the setting of *bounded-degree Borel pmp graphs.*
Why? So we can use the theory of bounded self-adjoint operators!

Definition

The *adjacency operator* $T_{\mathcal{G}} : L^2(X, \mu) \to L^2(X, \mu)$ of $\mathcal{G}$ is given by

$$(T_{\mathcal{G}} f)(x) = \sum_{yEx} f(y).$$

Proposition

1. If $\mathcal{G}$ has bounded degree, then $T_{\mathcal{G}}$ is bounded and self-adjoint.

Spectral Graph Theory and Descriptive Combinatorics

We work in the setting of *bounded-degree Borel pmp graphs.*
Why? So we can use the theory of bounded self-adjoint operators!

Definition

The *adjacency operator* $T_{\mathcal{G}} : L^2(X, \mu) \to L^2(X, \mu)$ of $\mathcal{G}$ is given by

$$(T_{\mathcal{G}} f)(x) = \sum_{yEx} f(y).$$

Proposition

1. If $\mathcal{G}$ has bounded degree, then $T_{\mathcal{G}}$ is bounded and self-adjoint.
2. $\sigma(\mathcal{G}) = \{\lambda \in \mathbb{C} : T_{\mathcal{G}} - \lambda I \text{ is not invertible}\}$ is a compact set of real numbers,

Spectral Graph Theory and Descriptive Combinatorics

We work in the setting of *bounded-degree Borel pmp graphs.* Why? So we can use the theory of bounded self-adjoint operators!

Definition

The *adjacency operator* $T_{\mathcal{G}} : L^2(X, \mu) \to L^2(X, \mu)$ of $\mathcal{G}$ is given by

$$(T_{\mathcal{G}} f)(x) = \sum_{yEx} f(y).$$

Proposition

1. If $\mathcal{G}$ has bounded degree, then $T_{\mathcal{G}}$ is bounded and self-adjoint.
2. $\sigma(\mathcal{G}) = \{\lambda \in \mathbb{C} : T_{\mathcal{G}} - \lambda I \text{ is not invertible}\}$ is a compact set of real numbers, and the residual spectrum is empty.

We work in the setting of *bounded-degree Borel pmp graphs.*
Why? So we can use the theory of bounded self-adjoint operators!

Definition

The *adjacency operator* $T_{\mathcal{G}} : L^2(X, \mu) \to L^2(X, \mu)$ of $\mathcal{G}$ is given by

$$(T_{\mathcal{G}} f)(x) = \sum_{yEx} f(y).$$

Proposition

1. If $\mathcal{G}$ has bounded degree, then $T_{\mathcal{G}}$ is bounded and self-adjoint.
2. $\sigma(\mathcal{G}) = \{\lambda \in \mathbb{C} : T_{\mathcal{G}} - \lambda I \text{ is not invertible}\}$ is a compact set of real numbers, and the residual spectrum is empty.

Write $M(T_{\mathcal{G}}) = \max(\sigma(\mathcal{G}))$ and $m(T_{\mathcal{G}}) = \min(\sigma(\mathcal{G}))$.

- **Measurable chromatic number:** The least number of colors needed for a μ-measurable coloring.

- **Measurable chromatic number:** The least number of colors needed for a μ-measurable coloring.
- **Approximate measurable chromatic number:** The least number k such that, for each $\varepsilon > 0$, there is a Borel set A with $\mu(A) > 1 - \varepsilon$ such that $\mathcal{G}[A]$ has a μ-measurable k-coloring.

- **Measurable chromatic number:** The least number of colors needed for a μ-measurable coloring.
- **Approximate measurable chromatic number:** The least number k such that, for each $\varepsilon > 0$, there is a Borel set A with $\mu(A) > 1 - \varepsilon$ such that $\mathcal{G}[A]$ has a μ-measurable k-coloring.

Theorem [Conley–Kechris]

Let $\mathbb{F}_n$ be the free group on n generators, and let $\mathcal{G}_n$ be the Schreier graph of the shift action $\mathbb{F}_n \times 2^{\mathbb{F}_n} \to 2^{\mathbb{F}_n}$. Then

$$\frac{n + \sqrt{2n-1}}{\sqrt{2n-1}} \leq \chi_\mu^{ap}(\mathcal{G}_n) \leq 2n.$$

- **Measurable chromatic number:** The least number of colors needed for a μ-measurable coloring.
- **Approximate measurable chromatic number:** The least number k such that, for each $\varepsilon > 0$, there is a Borel set A with $\mu(A) > 1 - \varepsilon$ such that $\mathcal{G}[A]$ has a μ-measurable k-coloring.

Theorem [Conley–Kechris]

Let $\mathbb{F}_n$ be the free group on n generators, and let $\mathcal{G}_n$ be the Schreier graph of the shift action $\mathbb{F}_n \times 2^{\mathbb{F}_n} \to 2^{\mathbb{F}_n}$. Then

$$\frac{n + \sqrt{2n-1}}{\sqrt{2n-1}} \leq \chi_\mu^{ap}(\mathcal{G}_n) \leq 2n.$$

For $n = 2$: $3 \leq \chi_\mu^{ap}(\mathcal{G}_2) \leq 4$.

Spectral Graph Theory and Descriptive Combinatorics

- **Measurable chromatic number:** The least number of colors needed for a μ-measurable coloring.
- **Approximate measurable chromatic number:** The least number k such that, for each $\varepsilon > 0$, there is a Borel set A with $\mu(A) > 1 - \varepsilon$ such that $\mathcal{G}[A]$ has a μ-measurable k-coloring.

Theorem [Conley–Kechris]

Let $\mathbb{F}_n$ be the free group on n generators, and let $\mathcal{G}_n$ be the Schreier graph of the shift action $\mathbb{F}_n \times 2^{\mathbb{F}_n} \to 2^{\mathbb{F}_n}$. Then

$$\frac{n + \sqrt{2n-1}}{\sqrt{2n-1}} \leq \chi_\mu^{ap}(\mathcal{G}_n) \leq 2n.$$

For $n = 2$: $3 \leq \chi_\mu^{ap}(\mathcal{G}_2) \leq 4$. Thornton later showed $\chi_\mu^{ap}(\mathcal{G}_2) = 3$.

Approximate Measurable Wilf's Theorem

We obtain an approximate measurable version of Wilf's theorem:

Approximate Measurable Wilf's Theorem

We obtain an approximate measurable version of Wilf's theorem:

Theorem [H.–Spaas–Tenenbaum]

$\chi_\mu^{ap}(\mathcal{G}) \leq \lfloor M(T_\mathcal{G}) \rfloor + 1.$

Approximate Measurable Wilf's Theorem

We obtain an approximate measurable version of Wilf's theorem:

Theorem [H.–Spaas–Tenenbaum]

$\chi_\mu^{ap}(\mathcal{G}) \leq \lfloor M(T_\mathcal{G}) \rfloor + 1.$

Proof sketch.

1. Show that $\deg_{av}(\mathcal{G}) \leq \lfloor M(T_\mathcal{G}) \rfloor$.

Approximate Measurable Wilf's Theorem

We obtain an approximate measurable version of Wilf's theorem:

Theorem [H.–Spaas–Tenenbaum]

$\chi_\mu^{ap}(\mathcal{G}) \leq \lfloor M(T_\mathcal{G}) \rfloor + 1.$

Proof sketch.

1. Show that $\deg_{av}(\mathcal{G}) \leq \lfloor M(T_\mathcal{G}) \rfloor$.
2. Show that $A_0 = \{x \in X : \deg_\mathcal{G}(x) \leq \lfloor M(T_\mathcal{G}) \rfloor\}$ has "large" measure. Then remove A_0.

Approximate Measurable Wilf's Theorem

We obtain an approximate measurable version of Wilf's theorem:

Theorem [H.–Spaas–Tenenbaum]

$\chi_\mu^{ap}(\mathcal{G}) \leq \lfloor M(T_\mathcal{G}) \rfloor + 1.$

Proof sketch.

1. Show that $\deg_{av}(\mathcal{G}) \leq \lfloor M(T_\mathcal{G}) \rfloor$.
2. Show that $A_0 = \{x \in X : \deg_\mathcal{G}(x) \leq \lfloor M(T_\mathcal{G}) \rfloor\}$ has "large" measure. Then remove A_0.
3. Repeat (2) until $\mu(A_0 \cup \cdots \cup A_k) > 1 - \varepsilon$.

Approximate Measurable Wilf's Theorem

We obtain an approximate measurable version of Wilf's theorem:

Theorem [H.–Spaas–Tenenbaum]

$\chi_\mu^{ap}(\mathcal{G}) \leq \lfloor M(T_\mathcal{G}) \rfloor + 1.$

Proof sketch.

1. Show that $\deg_{av}(\mathcal{G}) \leq \lfloor M(T_\mathcal{G}) \rfloor$.
2. Show that $A_0 = \{x \in X : \deg_\mathcal{G}(x) \leq \lfloor M(T_\mathcal{G}) \rfloor\}$ has "large" measure. Then remove A_0.
3. Repeat (2) until $\mu(A_0 \cup \cdots \cup A_k) > 1 - \varepsilon$.
4. List color A_k, then A_{k-1}, then A_{k-2}, ...

Approximate Measurable Colorings of Directed Graphs

Proposition

Suppose $\deg^+(x) \leq n$ for all $x \in X$. Then

$$\chi(\tilde{\mathcal{D}}) \leq 2n + 1.$$

Approximate Measurable Colorings of Directed Graphs

Proposition

Suppose $\deg^+(x) \leq n$ for all $x \in X$. Then

$$\chi(\tilde{\mathcal{D}}) \leq 2n + 1.$$

If $\tilde{\mathcal{D}}$ is bounded-degree and pmp, then we obtain the following:

Approximate Measurable Colorings of Directed Graphs

Proposition

Suppose $\deg^+(x) \leq n$ for all $x \in X$. Then

$$\chi(\tilde{\mathcal{D}}) \leq 2n+1.$$

If $\tilde{\mathcal{D}}$ is bounded-degree and pmp, then we obtain the following:

Theorem [H.–Spaas–Tenenbaum]

Suppose $\deg^+(x) \leq n$ for all $x \in X$. Then

$$\chi_\mu^{ap}(\tilde{\mathcal{D}}) \leq 2n+1.$$

Approximate Measurable Colorings of Directed Graphs

Proposition

Suppose $\deg^+(x) \leq n$ for all $x \in X$. Then

$$\chi(\tilde{\mathcal{D}}) \leq 2n + 1.$$

If $\tilde{\mathcal{D}}$ is bounded-degree and pmp, then we obtain the following:

Theorem [H.–Spaas–Tenenbaum]

Suppose $\deg^+(x) \leq n$ for all $x \in X$. Then

$$\chi_\mu^{ap}(\tilde{\mathcal{D}}) \leq 2n + 1.$$

This is best possible.

Approximate Measurable Colorings of Directed Graphs

Theorem [H.–Spaas–Tenenbaum]

Suppose $\deg^+(x) \leq n$ for all $x \in X$. Then

$$\chi_\mu^{ap}(\tilde{\mathcal{D}}) \leq 2n + 1.$$

Proof sketch.

1. Use mass transport to show that $\deg^+_{av}(\mathcal{D}) = \deg^-_{av}(\mathcal{D})$.

Approximate Measurable Colorings of Directed Graphs

Theorem [H.–Spaas–Tenenbaum]

Suppose $\deg^+(x) \leq n$ for all $x \in X$. Then

$$\chi_\mu^{ap}(\tilde{\mathcal{D}}) \leq 2n + 1.$$

Proof sketch.

1. Use mass transport to show that $\deg^+_{av}(\mathcal{D}) = \deg^-_{av}(\mathcal{D})$.
2. Show that $A_0 = \{x \in X : \deg_{\tilde{\mathcal{D}}}(x) \leq 2n\}$ has "large" measure. Then remove A_0.

Approximate Measurable Colorings of Directed Graphs

Theorem [H.–Spaas–Tenenbaum]

Suppose $\deg^+(x) \leq n$ for all $x \in X$. Then

$$\chi_\mu^{ap}(\tilde{\mathcal{D}}) \leq 2n + 1.$$

Proof sketch.

1. Use mass transport to show that $\deg_{av}^+(\mathcal{D}) = \deg_{av}^-(\mathcal{D})$.
2. Show that $A_0 = \{x \in X : \deg_{\tilde{\mathcal{D}}}(x) \leq 2n\}$ has "large" measure. Then remove A_0.
3. Repeat (2) until $\mu(A_0 \cup \cdots \cup A_k) > 1 - \varepsilon$.

Approximate Measurable Colorings of Directed Graphs

Theorem [H.–Spaas–Tenenbaum]

Suppose $\deg^+(x) \leq n$ for all $x \in X$. Then

$$\chi_\mu^{ap}(\tilde{\mathcal{D}}) \leq 2n + 1.$$

Proof sketch.

1. Use mass transport to show that $\deg_{av}^+(\mathcal{D}) = \deg_{av}^-(\mathcal{D})$.
2. Show that $A_0 = \{x \in X : \deg_{\tilde{\mathcal{D}}}(x) \leq 2n\}$ has "large" measure. Then remove A_0.
3. Repeat (2) until $\mu(A_0 \cup \cdots \cup A_k) > 1 - \varepsilon$.
4. List color A_k, then A_{k-1}, then A_{k-2}, ...

Approximate Measurable Hoffman's Theorem

We obtain an approximate measurable version of Hoffman's theorem, which improves on earlier theorems of Conley–Kechris and Lyons–Nazarov:

Approximate Measurable Hoffman's Theorem

We obtain an approximate measurable version of Hoffman's theorem, which improves on earlier theorems of Conley–Kechris and Lyons–Nazarov:

Theorem [H.–Spaas–Tenenbaum]

$\chi_{\mu}^{ap}(\mathcal{G}) \geq \lceil 1 - \frac{M(T_{\mathcal{G}})}{m(T_{\mathcal{G}})} \rceil$.

Approximate Measurable Hoffman's Theorem

Theorem [H.–Spaas–Tenenbaum]

$\chi_\mu^{ap}(\mathcal{G}) \geq \lceil 1 - \frac{M(T_\mathcal{G})}{m(T_\mathcal{G})} \rceil$.

Proof sketch. **First,** to get just $\chi_\mu(\mathcal{G}) \geq \lceil 1 - \frac{M(T_\mathcal{G})}{m(T_\mathcal{G})} \rceil$:

Approximate Measurable Hoffman's Theorem

Theorem [H.–Spaas–Tenenbaum]

$\chi_\mu^{ap}(\mathcal{G}) \geq \lceil 1 - \frac{M(T_\mathcal{G})}{m(T_\mathcal{G})} \rceil$.

Proof sketch. **First,** to get just $\chi_\mu(\mathcal{G}) \geq \lceil 1 - \frac{M(T_\mathcal{G})}{m(T_\mathcal{G})} \rceil$:

1. Put $k = \chi_\mu(\mathcal{G})$, and partition $\mathcal{G}$ into measurable independent sets $A_1, \ldots, A_k$.

Approximate Measurable Hoffman's Theorem

Theorem [H.–Spaas–Tenenbaum]

$\chi_\mu^{ap}(\mathcal{G}) \geq \lceil 1 - \frac{M(T_\mathcal{G})}{m(T_\mathcal{G})} \rceil$.

Proof sketch. **First,** to get just $\chi_\mu(\mathcal{G}) \geq \lceil 1 - \frac{M(T_\mathcal{G})}{m(T_\mathcal{G})} \rceil$:

1. Put $k = \chi_\mu(\mathcal{G})$, and partition $\mathcal{G}$ into measurable independent sets $A_1, \ldots, A_k$. Block-decompose $T_\mathcal{G}$ over this partition:

Approximate Measurable Hoffman's Theorem

Theorem [H.–Spaas–Tenenbaum]

$\chi_\mu^{ap}(\mathcal{G}) \geq \lceil 1 - \frac{M(T_\mathcal{G})}{m(T_\mathcal{G})} \rceil$.

Proof sketch. **First,** to get just $\chi_\mu(\mathcal{G}) \geq \lceil 1 - \frac{M(T_\mathcal{G})}{m(T_\mathcal{G})} \rceil$:

1. Put $k = \chi_\mu(\mathcal{G})$, and partition $\mathcal{G}$ into measurable independent sets $A_1, \ldots, A_k$. Block-decompose $T_\mathcal{G}$ over this partition:

$$T_\mathcal{G} = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1k} \\ T_{21} & T_{22} & \cdots & T_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ T_{k1} & T_{k2} & \cdots & T_{kk} \end{bmatrix}.$$

Approximate Measurable Hoffman's Theorem

Theorem [H.–Spaas–Tenenbaum]

$\chi_\mu^{ap}(\mathcal{G}) \geq \lceil 1 - \frac{M(T_\mathcal{G})}{m(T_\mathcal{G})} \rceil$.

Proof sketch. **First,** to get just $\chi_\mu(\mathcal{G}) \geq \lceil 1 - \frac{M(T_\mathcal{G})}{m(T_\mathcal{G})} \rceil$:

1. Put $k = \chi_\mu(\mathcal{G})$, and partition $\mathcal{G}$ into measurable independent sets $A_1, \ldots, A_k$. Block-decompose $T_\mathcal{G}$ over this partition:

$$T_\mathcal{G} = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1k} \\ T_{21} & T_{22} & \cdots & T_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ T_{k1} & T_{k2} & \cdots & T_{kk} \end{bmatrix}.$$

2. Show by induction on k that

$$(k-1)m(T_\mathcal{G}) + M(T_\mathcal{G}) \leq \sum_{1 \leq i \leq n} M(T_{ii}) = 0.$$

Approximate Measurable Hoffman's Theorem

Proof sketch. **Second,** to get $k = \chi_{\mu}^{ap}(\mathcal{G}) \geq \lceil 1 - \frac{M(T_{\mathcal{G}})}{m(T_{\mathcal{G}})} \rceil$:

Approximate Measurable Hoffman's Theorem

Proof sketch. **Second,** to get $k = \chi_\mu^{ap}(\mathcal{G}) \geq \lceil 1 - \frac{M(T_\mathcal{G})}{m(T_\mathcal{G})} \rceil$:

1. For each $\varepsilon > 0$, partition $\mathcal{G}$ into measurable independent sets $A_1^\varepsilon, \ldots, A_k^\varepsilon$, together with a "bad set" B^ε of measure $< \varepsilon$,

Approximate Measurable Hoffman's Theorem

Proof sketch. **Second,** to get $k = \chi_\mu^{ap}(\mathcal{G}) \geq \lceil 1 - \frac{M(T_\mathcal{G})}{m(T_\mathcal{G})} \rceil$:

1. For each $\varepsilon > 0$, partition $\mathcal{G}$ into measurable independent sets $A_1^\varepsilon, \dots, A_k^\varepsilon$, together with a "bad set" B^ε of measure $< \varepsilon$, and block-decompose $T_\mathcal{G}$ over this partition:

Approximate Measurable Hoffman's Theorem

Proof sketch. **Second,** to get $k = \chi_\mu^{ap}(\mathcal{G}) \geq \lceil 1 - \frac{M(T_\mathcal{G})}{m(T_\mathcal{G})} \rceil$:

1. For each $\varepsilon > 0$, partition $\mathcal{G}$ into measurable independent sets $A_1^\varepsilon, \ldots, A_k^\varepsilon$, together with a "bad set" B^ε of measure $< \varepsilon$, and block-decompose $T_\mathcal{G}$ over this partition:

$$T_\mathcal{G} = \begin{bmatrix} S^\varepsilon & R_{12}^\varepsilon \\ R_{21}^\varepsilon & T_B^\varepsilon \end{bmatrix}, \text{with } S^\varepsilon = \begin{bmatrix} 0 & T_{12}^\varepsilon & \cdots & T_{1k}^\varepsilon \\ T_{21}^\varepsilon & 0 & \cdots & T_{2k}^\varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ T_{k1}^\varepsilon & T_{k2}^\varepsilon & \cdots & 0 \end{bmatrix}.$$

Approximate Measurable Hoffman's Theorem

Proof sketch. **Second,** to get $k = \chi_\mu^{ap}(\mathcal{G}) \geq \lceil 1 - \frac{M(T_\mathcal{G})}{m(T_\mathcal{G})} \rceil$:

1. For each $\varepsilon > 0$, partition $\mathcal{G}$ into measurable independent sets $A_1^\varepsilon, \ldots, A_k^\varepsilon$, together with a "bad set" B^ε of measure $< \varepsilon$, and block-decompose $T_\mathcal{G}$ over this partition:

$$T_\mathcal{G} = \begin{bmatrix} S^\varepsilon & R_{12}^\varepsilon \\ R_{21}^\varepsilon & T_B^\varepsilon \end{bmatrix}, \text{with } S^\varepsilon = \begin{bmatrix} 0 & T_{12}^\varepsilon & \cdots & T_{1k}^\varepsilon \\ T_{21}^\varepsilon & 0 & \cdots & T_{2k}^\varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ T_{k1}^\varepsilon & T_{k2}^\varepsilon & \cdots & 0 \end{bmatrix}.$$

2. Show that, as $\varepsilon \to 0$, we have $M(S^\varepsilon) \to M(T_\mathcal{G})$

Approximate Measurable Hoffman's Theorem

Proof sketch. **Second,** to get $k = \chi_\mu^{ap}(\mathcal{G}) \geq \lceil 1 - \frac{M(T_\mathcal{G})}{m(T_\mathcal{G})} \rceil$:

1. For each $\varepsilon > 0$, partition $\mathcal{G}$ into measurable independent sets $A_1^\varepsilon, \ldots, A_k^\varepsilon$, together with a "bad set" B^ε of measure $< \varepsilon$, and block-decompose $T_\mathcal{G}$ over this partition:

$$T_\mathcal{G} = \begin{bmatrix} S^\varepsilon & R_{12}^\varepsilon \\ R_{21}^\varepsilon & T_B^\varepsilon \end{bmatrix}, \text{with } S^\varepsilon = \begin{bmatrix} 0 & T_{12}^\varepsilon & \cdots & T_{1k}^\varepsilon \\ T_{21}^\varepsilon & 0 & \cdots & T_{2k}^\varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ T_{k1}^\varepsilon & T_{k2}^\varepsilon & \cdots & 0 \end{bmatrix}.$$

2. Show that, as $\varepsilon \to 0$, we have $M(S^\varepsilon) \to M(T_\mathcal{G})$ and that $(k-1)m(S^\varepsilon) + M(S^\varepsilon) \leq 0$, so that also $(k-1)m(T_\mathcal{G}) + M(T_\mathcal{G}) \leq 0$.

Approximate Measurable Bipartiteness

Theorem

Classically, if $\mathcal{G}$ is connected, then $\sigma(\mathcal{G})$ is symmetric if and only if $\chi(\mathcal{G}) \leq 2$ (i.e., $\mathcal{G}$ is bipartite).

Approximate Measurable Bipartiteness

Theorem

Classically, if $\mathcal{G}$ is connected, then $\sigma(\mathcal{G})$ is symmetric if and only if $\chi(\mathcal{G}) \leq 2$ (i.e., $\mathcal{G}$ is bipartite).

We prove an approximate measurable version of this equivalence:

Theorem [H.–Spaas–Tenenbaum]

If $\chi_{\mu}^{ap}(\mathcal{G}) \leq 2$, then $\sigma(\mathcal{G})$ is symmetric.

Approximate Measurable Bipartiteness

Theorem

Classically, if $\mathcal{G}$ is connected, then $\sigma(\mathcal{G})$ is symmetric if and only if $\chi(\mathcal{G}) \leq 2$ (i.e., $\mathcal{G}$ is bipartite).

We prove an approximate measurable version of this equivalence:

Theorem [H.–Spaas–Tenenbaum]

If $\chi_\mu^{ap}(\mathcal{G}) \leq 2$, then $\sigma(\mathcal{G})$ is symmetric.
Assume in addition that $\mathcal{G}$ is regular and ergodic with spectral gap. If $\sigma(\mathcal{G})$ is symmetric, then $\chi_\mu^{ap}(\mathcal{G}) \leq 2$.

Approximate Measurable Bipartiteness

Theorem [H.–Spaas–Tenenbaum]

If $\chi_\mu^{ap}(\mathcal{G}) \leq 2$, then $\sigma(\mathcal{G})$ is symmetric.

Proof sketch.

1. Take $\lambda \in \sigma(T_\mathcal{G})$, and let f be an approximate eigenfunction.

Approximate Measurable Bipartiteness

Theorem [H.–Spaas–Tenenbaum]

If $\chi_\mu^{ap}(\mathcal{G}) \leq 2$, then $\sigma(\mathcal{G})$ is symmetric.

Proof sketch.

1. Take $\lambda \in \sigma(T_\mathcal{G})$, and let f be an approximate eigenfunction.
2. Let A_1, A_2 be measurable independent sets, with $X \setminus (A_1 \cup A_2)$ of small measure.

Approximate Measurable Bipartiteness

Theorem [H.–Spaas–Tenenbaum]

If $\chi_\mu^{ap}(\mathcal{G}) \leq 2$, then $\sigma(\mathcal{G})$ is symmetric.

Proof sketch.

1. Take $\lambda \in \sigma(T_\mathcal{G})$, and let f be an approximate eigenfunction.
2. Let A_1, A_2 be measurable independent sets, with $X \setminus (A_1 \cup A_2)$ of small measure.
3. Flip the sign of f on A_2 to obtain g, and show that g is an approximate eigenfunction for $-\lambda$.

Approximate Measurable Bipartiteness

- **Spectral gap:** A d-regular graph $\mathcal{G}$ has spectral gap if there is $\varepsilon > 0$ such that $\sigma(T_{\mathcal{G}}) \subseteq [-d, d-\varepsilon] \cup \{d\}$.

Approximate Measurable Bipartiteness

- **Spectral gap:** A d-regular graph $\mathcal{G}$ has spectral gap if there is $\varepsilon > 0$ such that $\sigma(T_{\mathcal{G}}) \subseteq [-d, d-\varepsilon] \cup \{d\}$.

Theorem [H.–Spaas–Tenenbaum]

Assume that $\mathcal{G}$ is d-regular and ergodic with spectral gap. If $\sigma(\mathcal{G})$ is symmetric, then $\chi_{\mu}^{ap}(\mathcal{G}) \leq 2$.

Approximate Measurable Bipartiteness

- **Spectral gap:** A d-regular graph $\mathcal{G}$ has spectral gap if there is $\varepsilon > 0$ such that $\sigma(T_{\mathcal{G}}) \subseteq [-d, d-\varepsilon] \cup \{d\}$.

Theorem [H.–Spaas–Tenenbaum]

Assume that $\mathcal{G}$ is d-regular and ergodic with spectral gap. If $\sigma(\mathcal{G})$ is symmetric, then $\chi_{\mu}^{ap}(\mathcal{G}) \leq 2$.
In fact, if $-d \in \sigma(\mathcal{G})$, then $\chi_{\mu}^{ap}(\mathcal{G}) \leq 2$.

Approximate Measurable Bipartiteness

- **Spectral gap:** A d-regular graph $\mathcal{G}$ has spectral gap if there is $\varepsilon > 0$ such that $\sigma(T_{\mathcal{G}}) \subseteq [-d, d-\varepsilon] \cup \{d\}$.

Theorem [H.–Spaas–Tenenbaum]

Assume that $\mathcal{G}$ is d-regular and ergodic with spectral gap. If $\sigma(\mathcal{G})$ is symmetric, then $\chi_\mu^{ap}(\mathcal{G}) \leq 2$.
In fact, if $-d \in \sigma(\mathcal{G})$, then $\chi_\mu^{ap}(\mathcal{G}) \leq 2$.

Proof sketch.

1. Take a sequence (f_n) of approximate eigenfunctions for $-d$.
2. Show $(|f_n|)$ is a sequence of approximate d-eigenfunctions.
3. By combining ergodicity, regularity, and spectral gap, show that $(|f_n|)$ can be taken to be the constant functions 1.
4. Use the signs of a well-chosen f_n to approximately bipartition the vertex set.

Approximate Measurable Bipartiteness

Recently, we removed the regularity assumption.

Approximate Measurable Bipartiteness

Recently, we removed the regularity assumption.

Theorem [H.–Spaas–Torelli]

Assume that $\mathcal{G}$ is ergodic with spectral gap. If $-M(T_{\mathcal{G}}) \in \sigma(T_{\mathcal{G}})$, then $\chi_{\mu}^{ap}(\mathcal{G}) \leq 2$.

Approximate Measurable Bipartiteness

Recently, we removed the regularity assumption.

Theorem [H.–Spaas–Torelli]

Assume that $\mathcal{G}$ is ergodic with spectral gap. If $-M(T_{\mathcal{G}}) \in \sigma(T_{\mathcal{G}})$, then $\chi_{\mu}^{ap}(\mathcal{G}) \leq 2$.

Proof sketch. Same idea as before, except the functions $|f_n|$ are now equal to a strictly positive function h that may differ from 1.

Further Directions

- Can we remove the assumption of spectral gap in our characterization of approximate measurable bipartiteness?

Further Directions

- Can we remove the assumption of spectral gap in our characterization of approximate measurable bipartiteness?
- We prove that a condition analogous to a classical spectral condition of Brouwer and Haemers implies a measurable "Tutte-like" property. Can we improve this to say something about measurable perfect matchings?

Further Directions

- Can we remove the assumption of spectral gap in our characterization of approximate measurable bipartiteness?
- We prove that a condition analogous to a classical spectral condition of Brouwer and Haemers implies a measurable "Tutte-like" property. Can we improve this to say something about measurable perfect matchings?
- Can we use spectral theory (e.g., Hoffman's bound) to find an "interesting" graph $\mathcal{G}$ for which $\chi_\mu(\mathcal{G})$ is large but $\chi_{BM}(\mathcal{G})$ is small?

Further Directions

- Can we remove the assumption of spectral gap in our characterization of approximate measurable bipartiteness?
- We prove that a condition analogous to a classical spectral condition of Brouwer and Haemers implies a measurable "Tutte-like" property. Can we improve this to say something about measurable perfect matchings?
- Can we use spectral theory (e.g., Hoffman's bound) to find an "interesting" graph $\mathcal{G}$ for which $\chi_\mu(\mathcal{G})$ is large but $\chi_{BM}(\mathcal{G})$ is small?
- Can we say anything about the mcp (rather than pmp) setting?

Thank you!