Expansions of ordered Abelian groups by unary predicates

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OAGs and examples

An *ordered abelian group* (or OAG) is an abelian group $(G;+)$ endowed with a translation-invariant ordering $<$ (i.e. $x < y \Rightarrow x+z < y+z$).

Example: The additive group $(\mathbb{R};+,<)$ of real numbers is an OAG. It is **divisible**:

$$\forall x \in \mathbb{R}\, \forall n \in \mathbb{N} \setminus \{0\}\, \exists y \in \mathbb{R}\, [ny = x] .$$

After adding a constant symbol for 0 and a unary function symbol for $x \mapsto -x$, its complete theory eliminates quantifiers and is decidable.

Example: $(\mathbb{Z};+,<)$ is an OAG. Its complete theory is decidable and has q.e. after adding symbols for 0, $x \mapsto -x$, and unary predicates for divisibility by n for each n (Presburger, 1930)

Other interesting examples of OAGs include dense **non-divisible** subgroups of $(\mathbb{R};+,<)$, such as

$$\mathbb{Q}_{(p)} := \left\{ \frac{a}{b} \,:\, a, b \in \mathbb{Z} \text{ and } p \nmid b \right\} \qquad \text{where } p \text{ is prime.}$$

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The main question

Question

Suppose that $\mathfrak{G} = (G; 0, <, +, \ldots)$ is an ordered Abelian group, possibly with extra structure, $X \subseteq G$, and $\mathfrak{G}_P$ is the expansion by a unary predicate P naming the subset X.

1. *When is there a "nice" language in which $\mathfrak{G}_P$ eliminates quantifiers?*
2. *When is $\mathfrak{G}_P$ dependent (NIP), strong, or finite dp-rank?*

The definitions of "strong" and "dp-rank" will be defined soon.

Generally we seek a language for eliminating quantifiers before trying to compute dp-rank, so hopefully Question 1 is interesting even if you don't care about Question 2.

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Inp-patterns

Let T be a complete theory, $p(\overline{x})$ a partial type in T, $\overline{x}$ a finite tuple of variables. All parameters $\overline{a}_{i,j}$ live in a sufficiently saturated model of T.

Definition

(Shelah) The type $p(\overline{x})$ *has burden* $\geq \kappa$ if there is a $\kappa \times \omega$ array of formulas

$$\begin{array}{cccc} \varphi_0(\overline{x};\overline{a}_{0,0}) & \varphi_0(\overline{x};\overline{a}_{0,1}) & \varphi_0(\overline{x};\overline{a}_{0,2}) & \dots \\ \varphi_1(\overline{x};\overline{a}_{1,0}) & \varphi_1(\overline{x};\overline{a}_{1,1}) & \varphi_1(\overline{x};\overline{a}_{1,2}) & \dots \\ \vdots & \vdots & \vdots & \end{array}$$

and a sequence of natural numbers k_i such that:

1. for every function $\eta : \kappa \to \omega$, the partial type $p(\overline{x}) \cup \{\varphi_i(\overline{x};\overline{a}_{i,\eta(i)})\} \ : \ i < \kappa\}$ is consistent, and
2. for every $i < \kappa$, the row $\{\varphi_i(\overline{x};\overline{a}_{i,j}) \ : \ j < \omega\}$ is k_i-inconsistent.

An array of parameterized formulas above is called an *inp-pattern (of depth κ)*.

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Example: $(\mathbb{R}; <, +, \mathbb{Q})$

$T = Th(\mathbb{R}; <, +, \mathbb{Q})$

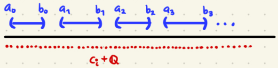
↑

(Divisible Ordered Abelian Group w/ unary predicate for $\mathbb{Q}$)

Inp-pattern of depth 2 in $x = x$:

Row 1: Pairwise disjoint intervals $a_i < x < b_i$

Row 2: Distinct cosets $x \in c_i + \mathbb{Q}$



Each row is 2-inconsistent, and each formula $a_i < x < b_i$ is consistent with every $x \in c_j + \mathbb{Q}$

... so burden$(x = x) \geq 2$.

In fact, using quantifier elimination for T, we have

$$\boxed{\text{burden}(x = x) = 2.}$$

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Fact (Adler): The theory T is NIP iff there is some cardinal κ such that the partial type $x = x$ does not have burden at least κ. If T is NIP, then for any partial type $p(\overline{x})$ in T, we have dp-rk$(p(\overline{x})) = \text{bd}(p(\overline{x}))$.

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Finite dp-rank theories and strong theories

Definition

Let T be a complete theory.

We say T *has dp-rank* κ if dp-rk$(x = x) = \kappa$ in T, where x is a single variable.

T is *strong* if it has no inp-pattern with infinitely many rows.

If dp-rk$(T) < \aleph_0$, we say T *has finite dp-rank*.

If dp-rk$(T) = 1$, we say T is *dp-minimal*.

Note that a type has dp-rank 0 iff it is algebraic, so all theories with infinite models have dp-rank at least 1.

Dp-rank is sub-additive (Kaplan, Onshuus, and Usvyatsov): dp-rk$(\bar{a}\bar{b}) \leq$ dp-rk$(\bar{a})$ + dp-rk$(\bar{b})$. Thus if T has finite dp-rank, then any type in finitely many variables has finite dp-rank.

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Dp-minimality: examples

All of the following theories are dp-minimal:

1. Any o-minimal theory, or any weakly o-minimal theory.
2. The field of p-adic numbers, or any finite extension of such a field.
3. Any theory which is strongly minimal, or even weakly minimal.

Any simple theory of SU-rank 1 is inp-minimal. For an example of an ordered abelian group which is inp-minimal but not dp-minimal, we may take $(\mathbb{R}, <, +)$ and expand with a "generic" unary predicate P in the manner of Chatzidakis and Pillay.

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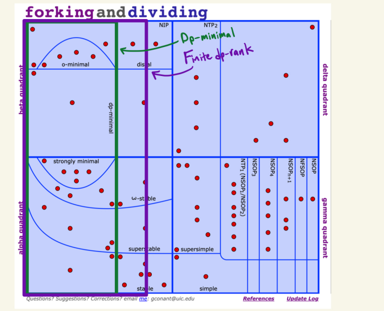
Conant's map of the universe



Source: https://www.forkinganddividing.com , by Gabriel Conant

dp-minimal	$\Rightarrow$	finite dp-rank	$\Rightarrow$	strongly NIP	$\Rightarrow$	NIP
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$
inp-minimal	$\Rightarrow$	finite burden	$\Rightarrow$	strong	$\Rightarrow$	NTP_2

(and none of the implications above are reversible)

Outline of the rest of the talk

Throughout, "DOAG" stands for "**Densely**[1] Ordered Abelian Group." I will address two main questions:

1. Suppose G is a densely ordered Abelian group and H is a subgroup. When does the structure $(G; +, <, H)$ have dp-rank 2?

2. What tameness conditions are satisfied by unary sets definable in DOAGs of dp-rank 1 or 2?

All new (and new-ish) results presented here are joint work with Alf Dolich (CUNY).

[1] I do not assume G is **divisible**, i.e. there may be $g \in G$, $n \in \mathbb{N}$ such that $nx = g$ has no solution.

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Pairs of OAGs

First question: If $(G; <, +)$ is an ordered Abelian group and H is a subgroup, how can we calculate the dp-rank of $(G; <, +, H)$ (adding a predicate for H)? When is its dp-rank 2?

For simplicity I will concentrate on the case when G is densely ordered, though the question is interesting even when G is discretely ordered (Erik Walsberg has some results on dp-minimal discrete OAGs).

If the subgroup H is nontrivial, the dp-rank is usually **at least** 2. Dp-minimal OAGs in the "pure" language of ordered groups are well understood (see the next slides).

I will **not** assume H is an elementary substructure of $(G; +, <)$, much less a "lovely pair." There has been a lot of interesting work on "tame pairs," e.g. van den Dries (pairs of o-minimal structures), Berenstien-Dolich-Onshuus (dense pairs), Block Gorman considered companionability of pairs, *inter alia*.

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I will **not** assume H is an elementary substructure of $(G; +, <)$, much less a "lovely pair." There has been a lot of interesting work on "tame pairs," e.g. van den Dries (pairs of o-minimal structures), Berenstien-Dolich-Onshuus (dense pairs), Block Gorman considered companionability of pairs, *inter alia*.

Pairs of OAGs

First question: If $(G; <, +)$ is an ordered Abelian group and H is a subgroup, how can we calculate the dp-rank of $(G; <, +, H)$ (adding a predicate for H)? When is its dp-rank 2?

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In the case of a **torsion-free** Abelian group $(G;+)$, her results imply that it is elementarily equivalent to a direct sum of copies of $\mathbb{Q}$ and

$$\mathbb{Q}_{(p)} = \left\{ \frac{a}{b} \,:\, a, b \in \mathbb{Z} \text{ and } p \nmid b \right\}$$

for various primes p.

To eliminate quantifiers in any Abelian group, it suffices to add unary predicates for $\exists y \left[p^k y \,|\, p^\ell x \right]$ for all primes p and k, ℓ. In the torsion-free case we only need predicates for divisibility by p^k.

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Eliminating Quantifiers in $(G; 0, +, -)$

Positive primitive formula:

$$\varphi(\bar{x}) = \exists \bar{y} \bigwedge_{i=1}^{k} \underbrace{t_i(\bar{x};\bar{y})}_{\text{terms in } \{0,+,-\}} = 0$$

Equivalently:

$$(*) \quad \varphi(\bar{x}) = \exists \bar{y}\, [A\bar{x} + B\bar{y} = \bar{0}]$$

A, B: matrices over $\mathbb{Z}$

p.p. formulae define subgroups of $G^{|\bar{x}|}$

Note: If $\varphi(\bar{x}_1, \bar{x}_2)$ is p.p. and $|\bar{b}| = |\bar{x}_2|$, $\varphi(\bar{x}_1, \bar{b})$ defines a coset of $\varphi(\bar{x}_1, \bar{0})$ or $\emptyset$.

To eliminate a quantifier "$\exists z$" from a formula

$\exists z$ (some Boolean comb. of p.p formulae),

it suffices to eliminate $\exists z$ in

$$\exists z \left(\varphi(\bar{x}; z) \wedge \bigwedge_{i=1}^{n} \neg \psi_i(\bar{x}; z)\right)$$

"Some coset of $\varphi(\bar{0}; G)$ is NOT covered by certain $\psi_i(\bar{0}; G)$-cosets"

Determined by indices of intersections of $\psi_i(\bar{0}; G)$ in each other, and for which I $\bigcap_{i \in I} \psi_i(\bar{x}; G)$ is empty.

Finally, find invertible U, V over $\mathbb{Z}$ s.t. UBV is diagonal (Smith normal form) to simplify $(*)$.

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Also, it was found that for **any** Abelian group G, the theory $\mathrm{Th}(G; +)$ is **stable.**

In fact, if H is any subgroup of an Abelian group G, then the theory of the unordered pair $\mathrm{Th}(G; +, H)$ is stable, by unpublished work of Fisher (1970's).

The dp-rank of "pure" Abelian groups can be calculated (Halevi and Palacín): it is the maximal κ such that there exist $\mathrm{acl}^{eq}(\emptyset)$-definable subgroups $(H_\alpha : \alpha < \kappa)$ such that for every $i < \kappa$,

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(Jahnke, Simon, and Walsberg) An ordered Abelian group $(G;+,<)$ is dp-minimal if and only if it has **no singular primes**, *i.e. no primes p such that $[G : pG] = \infty$.*

We can also characterize when an OAG $(G;+,<)$ has finite dp-rank (found independently by Halevi-Palacín, Farré, Dolich and G.):

Theorem

For an ordered Abelian group $(G;+,<)$, the following are equivalent:

1. *$(G;+,<)$ has finite dp-rank;*
2. *$(G;+,<)$ is strong;*
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Regular OAGs

Definition: An ordered Abelian group $(G; <, +)$ is *regular* if for every $n \in \mathbb{N}$, every interval in G which contains at least n elements contains at least one element which is divisible by n.

Examples: Any divisible OAG is regular, and $(\mathbb{Z}; <, +)$ is regular.

The direct product $(\mathbb{Z}; <, +) \times (\mathbb{Q}; <, +)$ with the lexicographic ordering is **not** regular: for any $q < r$ in $\mathbb{Q}$, the interval between $(1, q)$ and $(1, r)$ is infinite, but contains no elements which are divisible by 2.

Fact (folklore): An ordered Abelian group is regular if and only if it is elementarily equivalent to an ordered subgroup of $(\mathbb{R}; <, +)$, if and only if it eliminates quantifiers after adding symbols for $0, -$, and divsibility by each n.

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Now consider a *dense* OAG $(G; +, <)$ and a subgroup H of G.

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because $p\mathbb{Q}_{(p)}$ is definable in the dp-minimal structure $(\mathbb{Q}_{(p)}; <, +)$.

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Definable unary sets in dp-rank 1 or 2

Finally, I will mention some tameness results for unary sets definable in OAGs of dp-rank 1 or 2.

There is a **lot** I could say here. The general intuition is that in finite dp-rank OAGs, definable sets ought to be Boolean combinations of sets which are topologically "tame" (think of the o-miimal case) *plus cosets of subgroups*.

Example: $\mathbb{Q}_{(p)} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \text{ and } p \not| b \right\}$ is dp-minimal. It has a dense proper subgroup $p\mathbb{Q}_{(p)}$, so it is not o-minimal.

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Simon: If G is **divisible** and X is infinite, then X has nonempty interior.

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Theorem

(Dolich and G.) Say $\mathscr{G}$ is a dense OAG of burden 2 and is definably complete. Then there is a subgroup Z of $(G; <)$ such that:

1. $(G; <, +, Z) \equiv (\mathbb{R}; <, +, \mathbb{Z})$, *and*
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Note that this result is only on the unary sets definable in $\mathscr{G}$, and there could be more complicated structure definable in G^n.

For instance, $(\mathbb{R}; <, +, \sin)$ has dp-rank 2. (It is not o-minimal, but it is locally o-miniimal).

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A conjecture

Suppose that $(G; <, +, P)$ is a definably complete, Archimedean, dense OAG and P is a predicate for a **dense** subset of G.

Conjecure: If dp-rk$(G; <, +, P) \leq 2$, then there is a family of dense subgroups $(H_i : i \in I)$ of G such that P is definable (possibly with parameters) in the structure $(G; <, +, H_i : i \in I)$.

The intuition is that any dense set definable in a dp-rank 2 OAG ought to be "group-like." (Or at least definable from groups, e.g. a Boolean combination of cosets and convex sets.)

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Thank you!

¡Gracias por su atención!

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