When Logic meets Algebra: interpolation in Substructural Logics

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based on joint works with Sara Ugolini

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When Logic meets Algebra

Substructural logics are logics laking some or all the structural rules present in the Gentzen's sequent calculi for Classical and Intuitionistic Logic, that are

- Exchange: the order of assumptions doesen't matter.
- Contraction: an assumption can always be reused.
- Weakening: it is possible to add irrelevant assumptions.

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They cover many of the well-known nonclassical logics as for instance Lambek calculus for categorial grammar (no structural rules), linear logic (no weakening and contraction), Łukasiewicz's many valued logic (no contraction) and relevant logics (no weakening).

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Algebraic logics provide a uniform framework for the study of substructural logics, seeing them as particular algebraic structures called residuated lattices, offering algebraic tools in order to study logical properties.

For this talk...

- Connection between Logic and Algebra

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- How to study interpolation with algebraic tools

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- How to study interpolation with algebraic tools
- The failure of interpolation in the logic MTL via algebraic methods, solving a long standing open problem for this area.

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- A (propositional) **logic** $\vdash$ over $\mathcal{L}$ is a consequence relation on $\mathsf{Fm}_{\mathcal{L}}$,

$$\vdash \subseteq \mathcal{P}(\mathsf{Fm}_{\mathcal{L}}) \times \mathsf{Fm}_{\mathcal{L}},$$

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  i.e. *reflexive*, *transitive*, and *monotone*, that is also *substitution-invariant* and *finitary*.

Logical ingredients

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When is a logic $\mathcal{L}$ algebraizable ?

Block-Pigozzi algebraizability

Fix a type of algebras τ.

- An **equation** is a pair of terms $p \approx q$. An equation is valid in an algebra $\mathbf{A}$, if

$$p(a_1, \ldots, a_n) = q(a_1, \ldots, a_n) \quad \text{for every } a_1, \ldots, a_n \in A.$$

- A **quasiequation** is of the from

$$(p_1 \approx q_1) \;\&\; \cdots \;\&\; (p_m \approx q_m) \quad \Longrightarrow \quad p \approx q$$

and it is valid in an algebra $\mathbf{A}$ iff for any assignment of variables $a_1, \ldots, a_n$ in $\mathbf{A}$ if all $p_i \approx q_i$ are valid such assignment, then

$$p(a_1, \ldots, a_n) = q(a_1, \ldots, a_n)$$

.

Block-Pigozzi algebraizability

Let K be a class of algebras. Then it is

- a quasivariety iff it is the class of algebras in which a set of quasiequations is valid;
- a variety iff it is the class of algebras in which a set of equations is valid.

Block-Pigozzi algebrizability

Given a logic $\mathcal{L}$ two formulas φ and ψ are provably equivalent, $\psi \equiv \varphi$, iff

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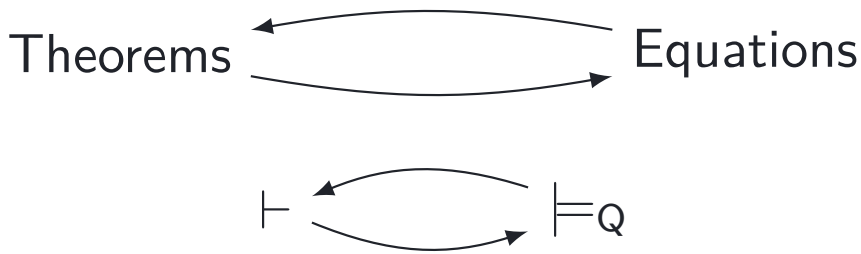
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$$\mathsf{Fm}_{\mathcal{L}} / \equiv$$

Let $\mathcal{Q}_{\mathcal{L}}$ be the quasivariety generated by $\mathsf{Fm}_{\mathcal{L}} / \equiv$.

Blok-Pigozzi algebrizability

A logic $\mathcal{L}$ is **algebrizable** with equivalent algebraic semantics $\mathbb{Q}_{\mathcal{L}}$ iff



Blok-Pigozzi algebraizability

Classical logic is algebrizable, and Boolean algebras are its equivalent algebraic semantics.

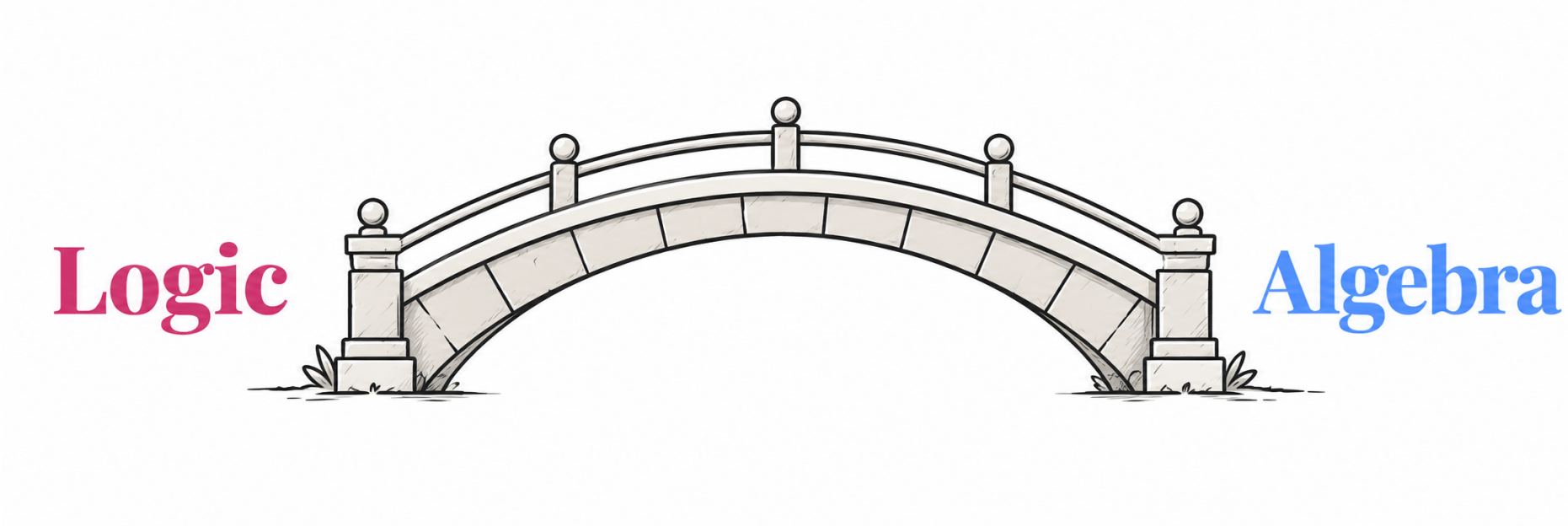
Theorems	$\Leftrightarrow$	**Equations**
$\vdash_{\mathrm{CPC}} x \vee \neg x$		$\models_{\mathsf{BA}} x \vee \neg x \approx 1$
$\vdash_{\mathrm{CPC}} \neg(x \wedge y) \leftrightarrow \neg x \vee \neg y$		$\models_{\mathsf{BA}} \neg(x \wedge y) \approx \neg x \vee \neg y$
Deductions	$\Leftrightarrow$	**Quasi-equations**
$\neg\neg x \vdash_{\mathrm{CPC}} x$		$\models_{\mathsf{BA}} (\neg\neg x \approx 1) \Rightarrow (x \approx 1)$

Examples

Examples of algebraizable logics:

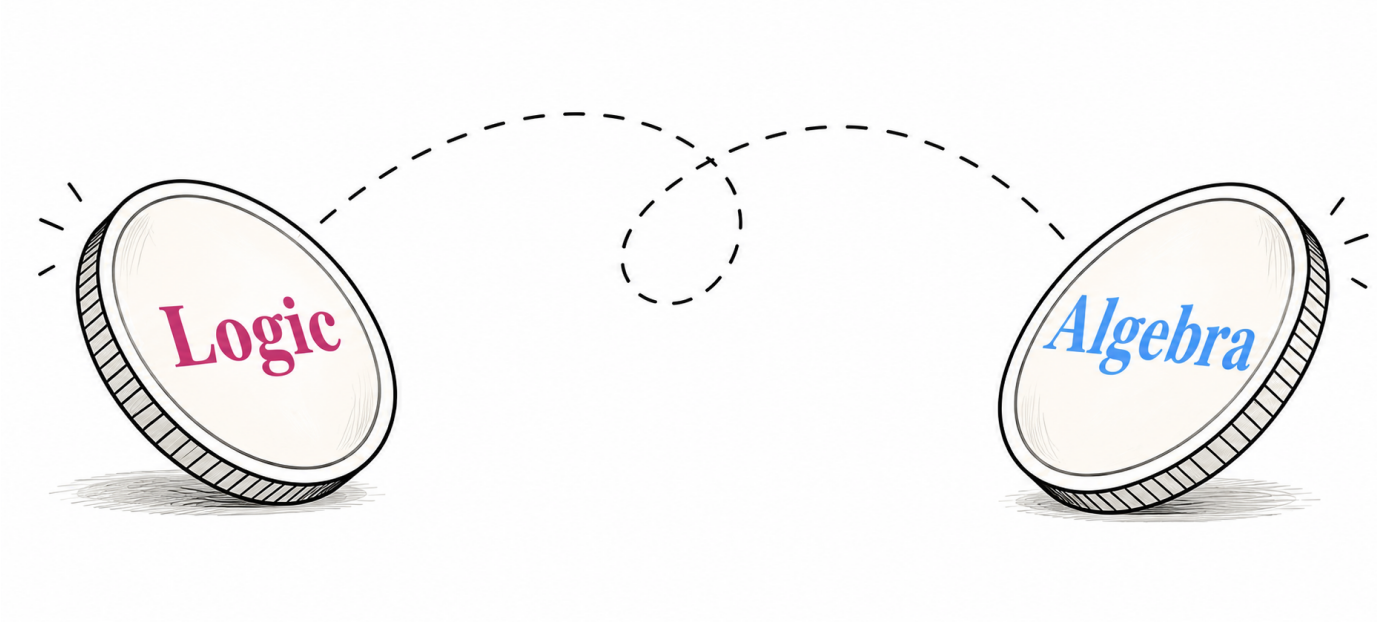
Logic	**Algebra**
Classical logic	Boolean Algebras
Intuitionistic Logic	Heyting Algebras
Basic Logic	BL-algebras
Łukasiewicz Logic	MV-algebras
Gödel Logic	Gödel algebras
MTL Logic	MTL-algebras

Bridge theorems



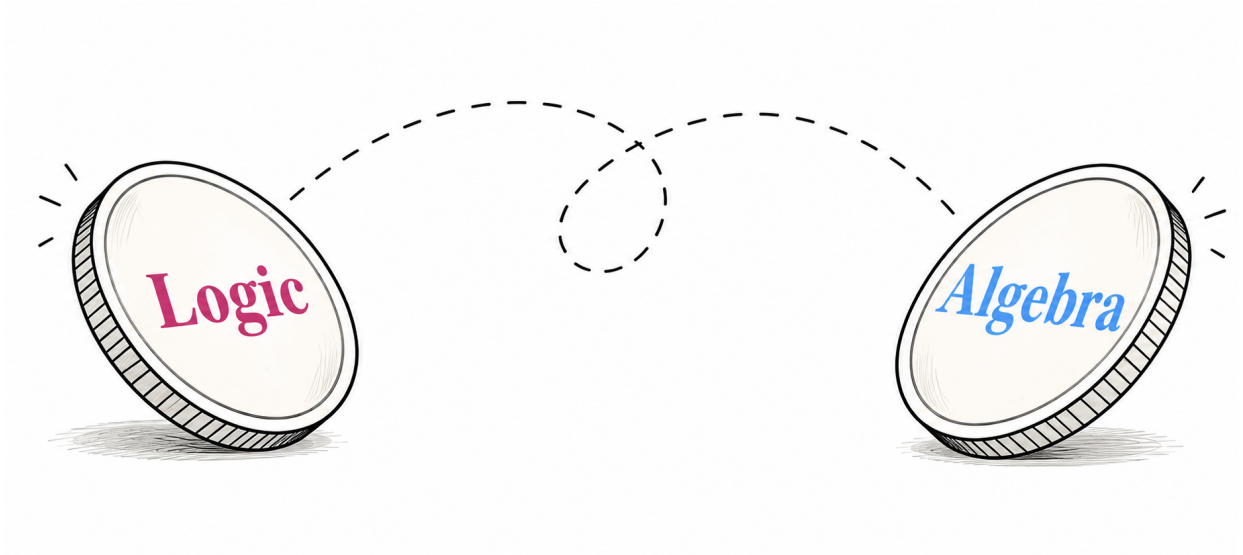
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Bridge theorems



Theorems	Valid equations
Deductions	Valid quasi-equations
Lattice of axiomatic extensions	Lattice of subvarieties
finite model property	generation by finite algebras
deduction theorem	property of congruences (EDPC)
interpolation	**amalgamation**
...	...

Interpolation property

Suppose that a set of formulas Γ derives another formula ψ, then it is possible to explain this derivation using only their shared variables.

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A logic $\mathcal{L}$ has the **deductive interpolation property** if for any set of formulas $\Gamma \cup \{\psi\}$, if $\Gamma \vdash \psi$, there exists a formula θ such that

$$\Gamma \vdash \theta, \qquad \theta \vdash \psi, \qquad \mathsf{Var}(\theta) \subseteq \mathsf{Var}(\Gamma) \cap \mathsf{Var}(\psi).$$

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The interpolant θ passes only information written in the common language.

Example in Classical Logic

Consider

$$\Gamma = \{p \to q,\ q \to r,\ p,\ r \to s\}, \qquad \varphi = s \vee t.$$

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Choose the interpolant $\theta = s$.

$$\underbrace{\Gamma}_{\text{premises}} \vdash_{\mathrm{CPC}} \underbrace{s}_{\text{interpolant}} \vdash_{\mathrm{CPC}} \underbrace{s \vee t}_{\text{conclusion}} .$$

Some application of deductive interpolation

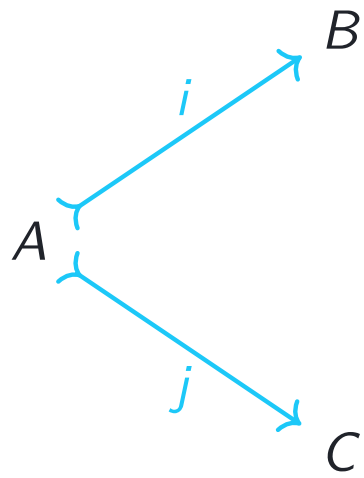
- **Formal Verification and Automated Theorem Proving**
  - Invariants: Interpolants help define properties that must hold during program execution.
  - Model refinement: It can be used to improve abstract models by removing spurious counterexamples.
  - Debugging: Provide intermediate formulas to explain and localize errors.
- **Modular Reasoning and Proof Decomposition**
  - Intermediate formulas: Given $A \models B$, interpolation provides a formula I such that $A \models I$ and $I \models B$.
  - Modularity: Enables proof reuse and verification of components in isolation.

The algebraic counterpart: amalgamation

Thanks to bridge theorems, the study of interpolation for an algebrizable logic is equivalent to the study of the **amalgamation property** for the corresponding class of algebras, when such class has the additional property of having the congruence extension property.

The algebraic counterpart: amalgamation

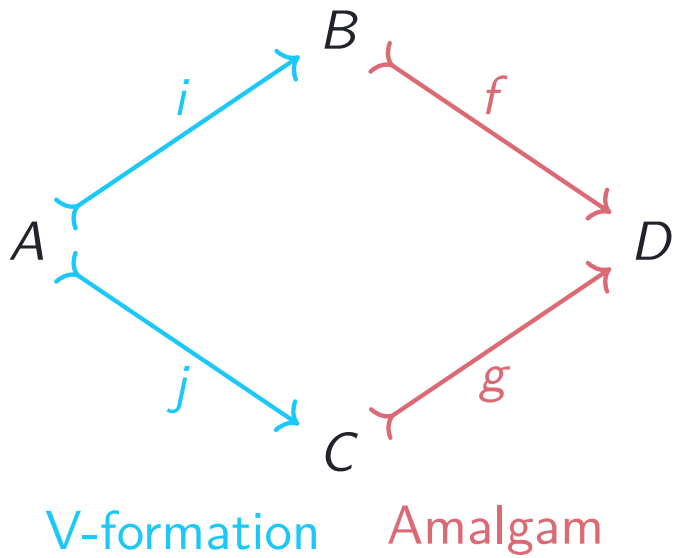
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V-formation

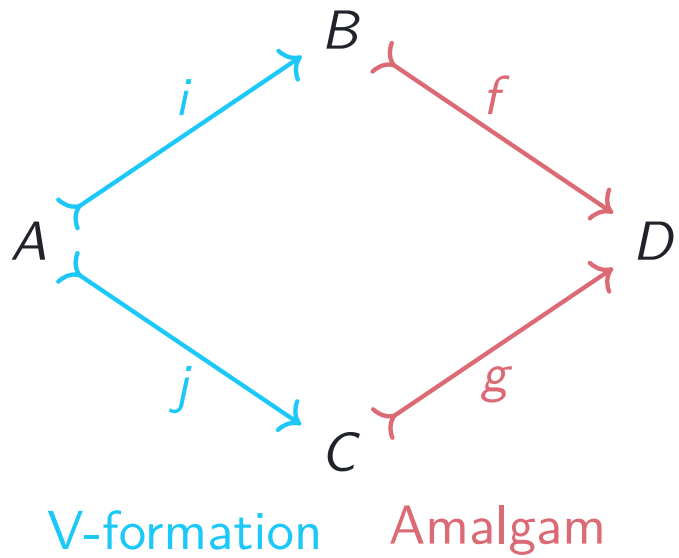
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A class K of algebras has the **amalgamation property (AP)** if any V-formation on K admits an amalgam in K.

The algebraic counterpart: amalgamation

An embedding $\phi : \mathbf{A} \to \mathbf{B}$ is called essential if for each homomorphism $\psi : \mathbf{B} \to \mathbf{C}$, if $\psi \circ \phi$ is an embedding, then ψ is an embedding.

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In case ϕ is just the inclusion map, then ϕ is essential if for each $\theta \in \mathrm{Con}(\mathbf{B})$, with $\theta \neq \Delta_{\mathbf{B}}$, also $\theta \cap A^2 \neq \Delta_A$.

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A V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ is called essential span if j is an essential embedding and a class of algebras K has the **essential amalgamation property (EAP)** if any essential span in K has an amalgam in K.

The algebraic counterpart: amalgamation

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Theorem (Fussner, Santschi)

Let V *be a variety. If* V_{SI} *does not have the essential amalgamation property, then* V *does not have the amalgamation property.*

Algebraic framework

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A **residuated lattice**, or RL, is a structure $\mathbf{A} = (A, \cdot, \backslash, /, \wedge, \vee, 1)$ where:

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, 1)$ is a monoid,
- residuation law holds, i.e.

$$xy \leq z \text{ iff } y \leq x \backslash z \text{ iff } x \leq z/y.$$

It can be seen as an "algebraic deduction theorem" where the divisions are implications or equivalently as a way of solving inequalities.

Algebraic framework

We call a residuated lattice:

- **integral** if 1 is the top element of the lattice (IRL).
- **bounded** if integral and there is an extra constant 0 in the signature that is the least element of the lattice (BIRL).
- **commutative** if the monoidal operation is commutative (CRL).
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In particular **MTL-algebras** are SBCIRL and **BL-algebras** are MTL-algebras that are divisible, i.e. $x \wedge y = x \cdot (x \to y)$.

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Residuated lattices are rich and complex structures. Although they have been extensively studied over the past century, many significant subclasses still lack a structural understanding.

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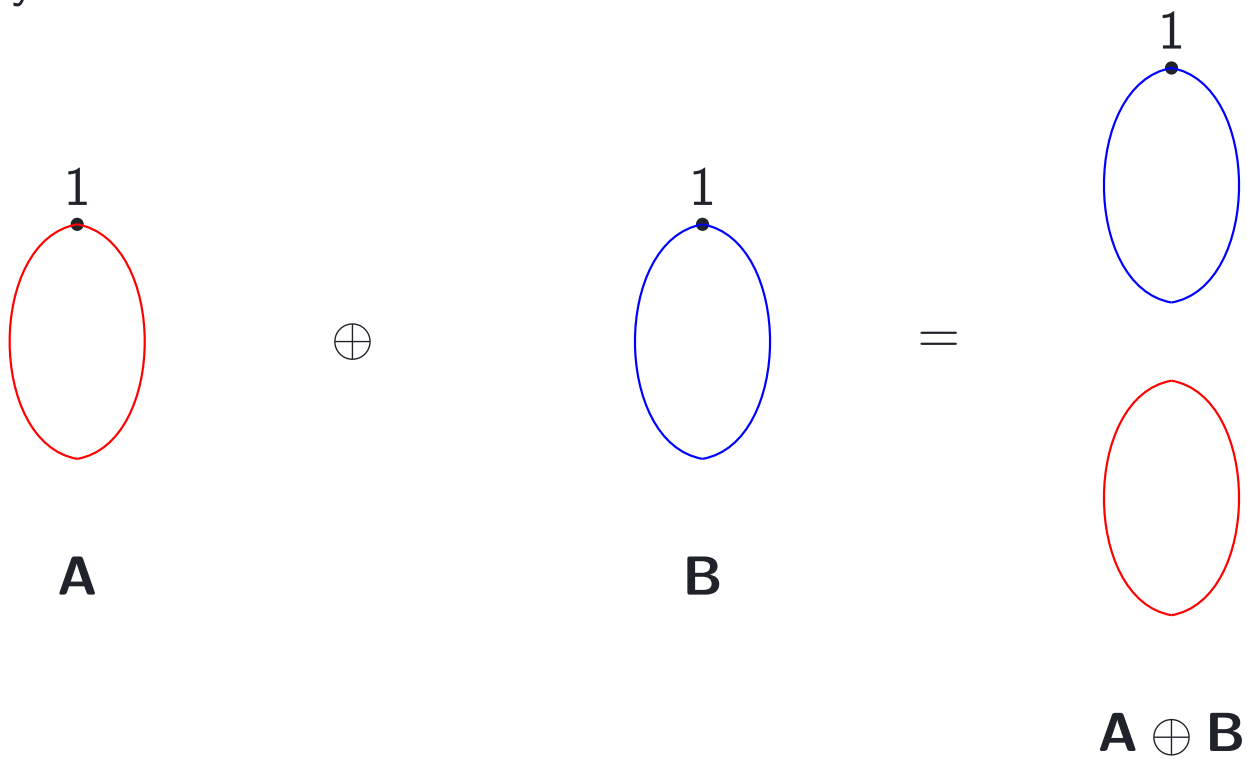
Also, algebraic constructions may help in the study of logical properties connected to algebraic ones. In particular, we will see that they are of crucial importance for the study of the interpolation property.

Ordinal sum construction

The ordinal sum construction was introduced fist by Clifford of semigrous and the by Ferreirim for residuated structures.

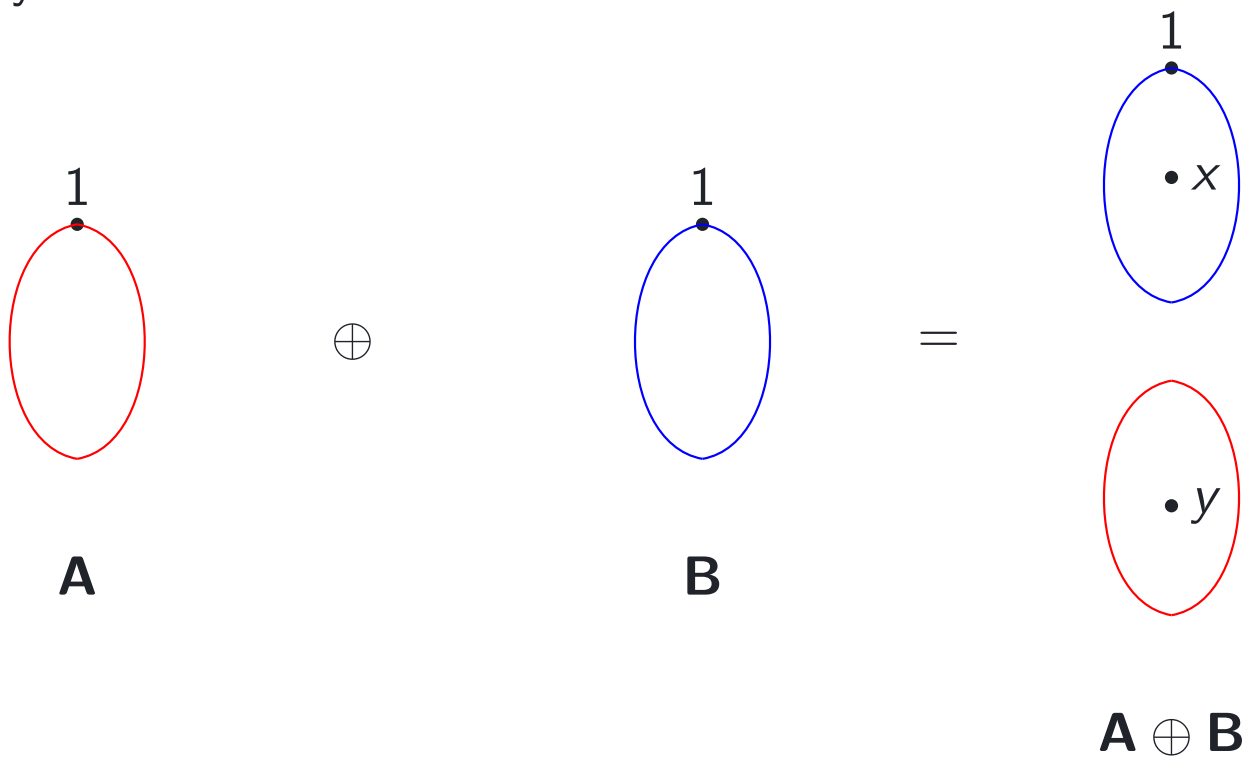
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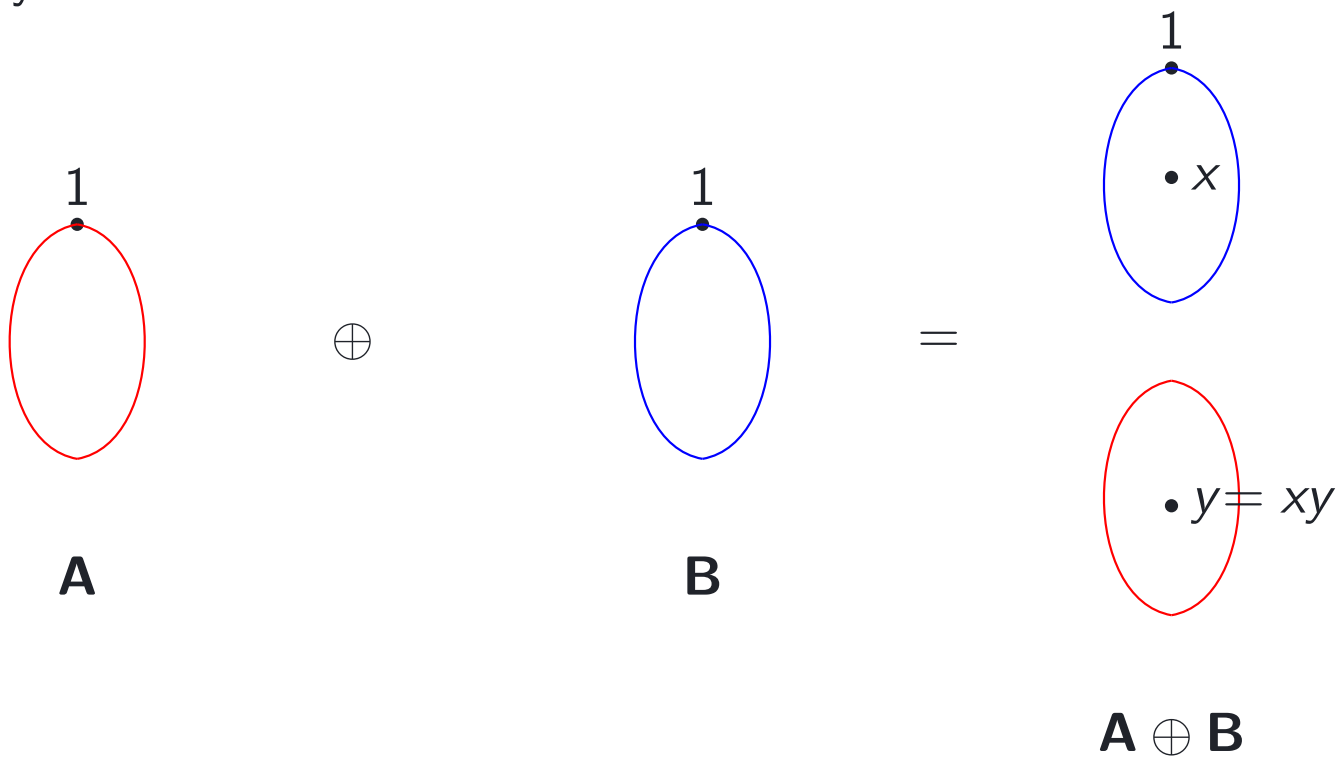
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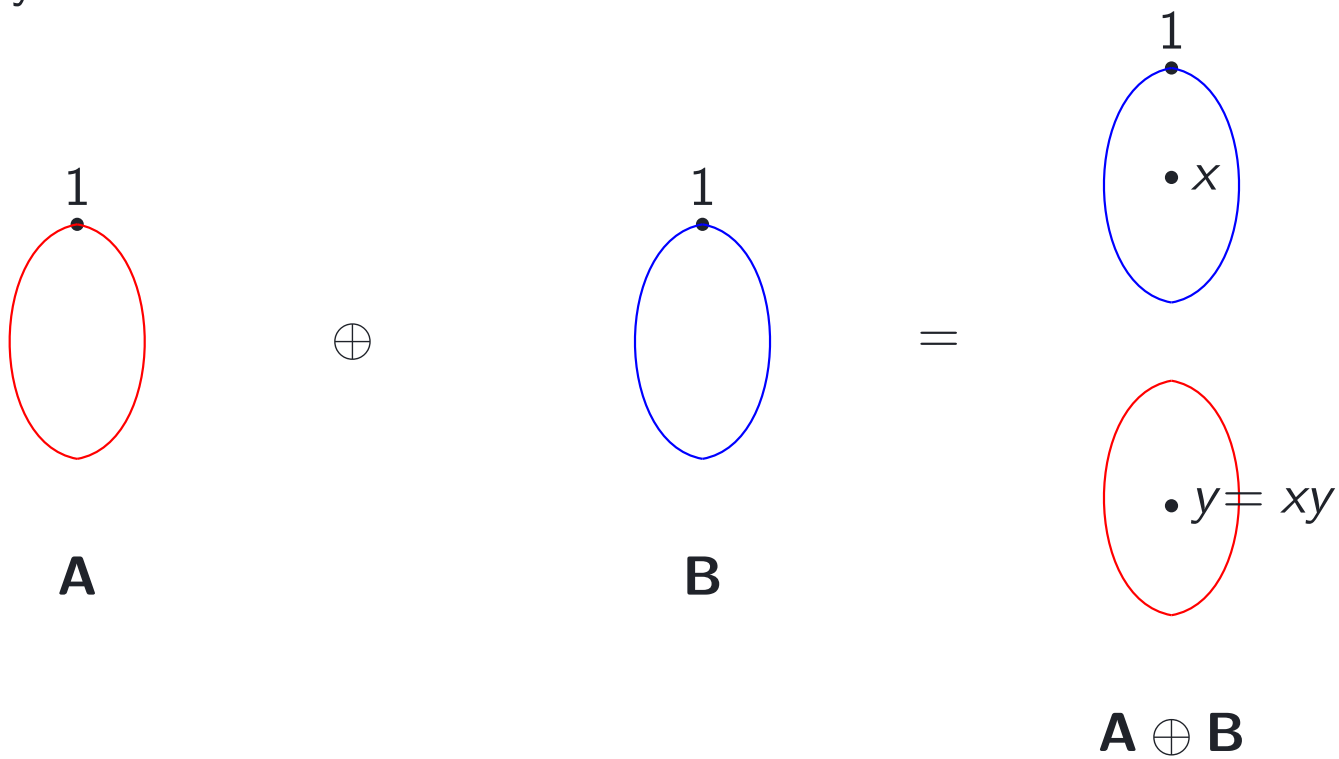
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The ordinal sum can be iterated and it is associative, so one can write

$$\mathbf{A}_0 \oplus \mathbf{A}_1 \oplus \ldots$$

Power of the ordinal sum

Fussner and Santchi used the ordinal sum construction together with their transfer theorem in order to give a complete characterization for the subvariety of BL-algebras having the amalgamation property.

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This result strongly relies on the fact that every BL-chain can be decomposed as an ordinal sum of simpler BL-chains.

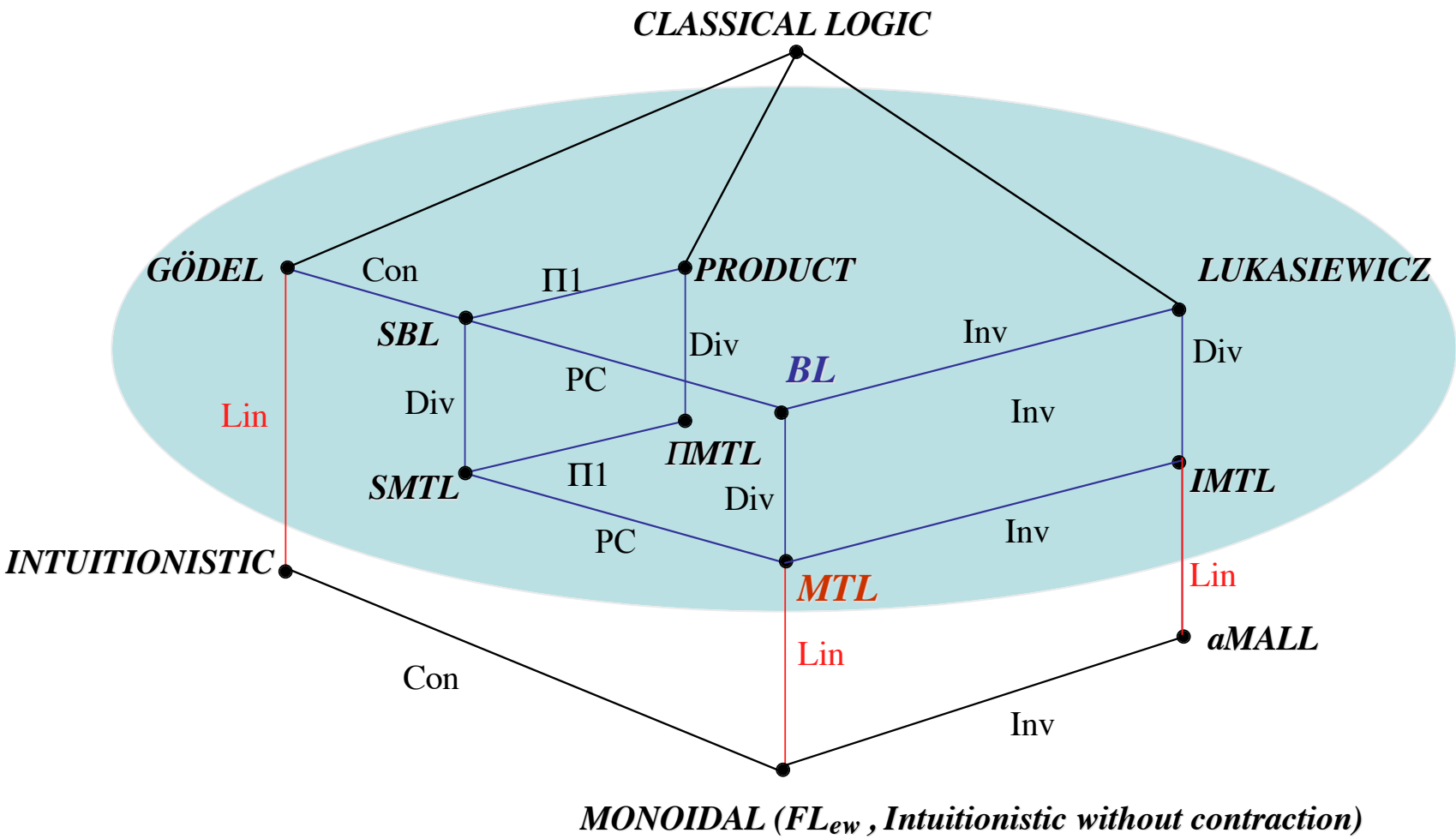
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On the logical side, since the variety of BL-algebras has the CEP, they manage to characterize all the axiomatic extensions of the Basic Logic having the deductive interpolation property.

Limits


CLASSICAL LOGIC
GÖDEL
Con
Π1
PRODUCT
LUKASIEWICZ
SBL
Div
Inv
Div
BL
PC
Lin
Div
Inv
ΠMTL
SMTL
Π1
IMTL
Div
Inv
PC
INTUITIONISTIC
MTL
Lin
Lin
aMALL
Con
Inv
MONOIDAL (FLew , Intuitionistic without contraction)

The missing pieces

Variety	**Label**	**AP**
Residuated lattices	*RL*	no
Commutative *RL*	*CRL*	yes
Semilinear *RL*	*SemRL*	no
Commutative *SemRL*	*CSemRL*	?
MTL-algebras	*MTL*	?
IMTL-algebras	*IMTL*	?
SMTL-algebras	*SMTL*	?
GBL-algebras	*GBL*	no
Commutative *GBL*	*CGBL*	?
Semilinear *GBL*	*SemGBL*	no
Commutative *SemGBL*	*CSemGBL*	yes
GMV-algebras	*GMV*	no
Commutative *GMV*	*CGMV*	yes
ℓ-groups	*LG*	no
Abelian ℓ-groups	*AbLG*	yes
Integral *RL*	*IRL*	?
Commutative *IRL*	*CIRL*	yes
Semilinear *IRL*	*SemIRL*	?
Commutative *SemIRL*	*CSemIRL*	?
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Commutative *IGBL*	*CIGBL*	?
Semilinear *IGBL*	*SemIGBL*	no
Commutative *SemIGBL*	*CSemIGBL*	yes
Integral *GMV*	*IGMV*	?
Commutative *IGMV*	*CIGMV*	yes
Semilinear FL_e	$SemFL_e$	?
Negative cones of ℓ-groups	LG^-	no
Abelian ℓ-groups (cones)	$AbLG^-$	yes
Brouwerian algebras	*Brw*	yes
Relative Stone algebras	*RSA*	yes

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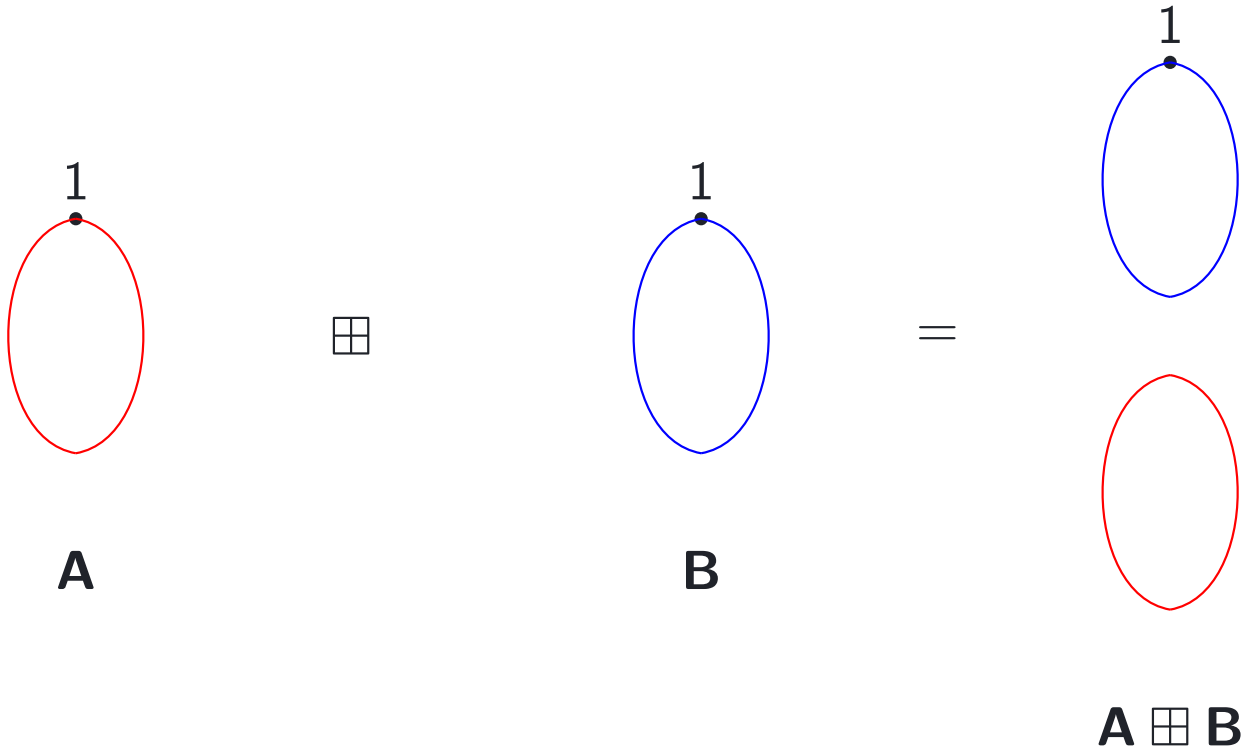
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Relative Stone algebras	*RSA*	yes

Partial gluing construction

The **partial gluing construction**, introduced by Galatos and Ugolini, generalizes the ordinal sum construction allowing products between two different components to be different form the meet.

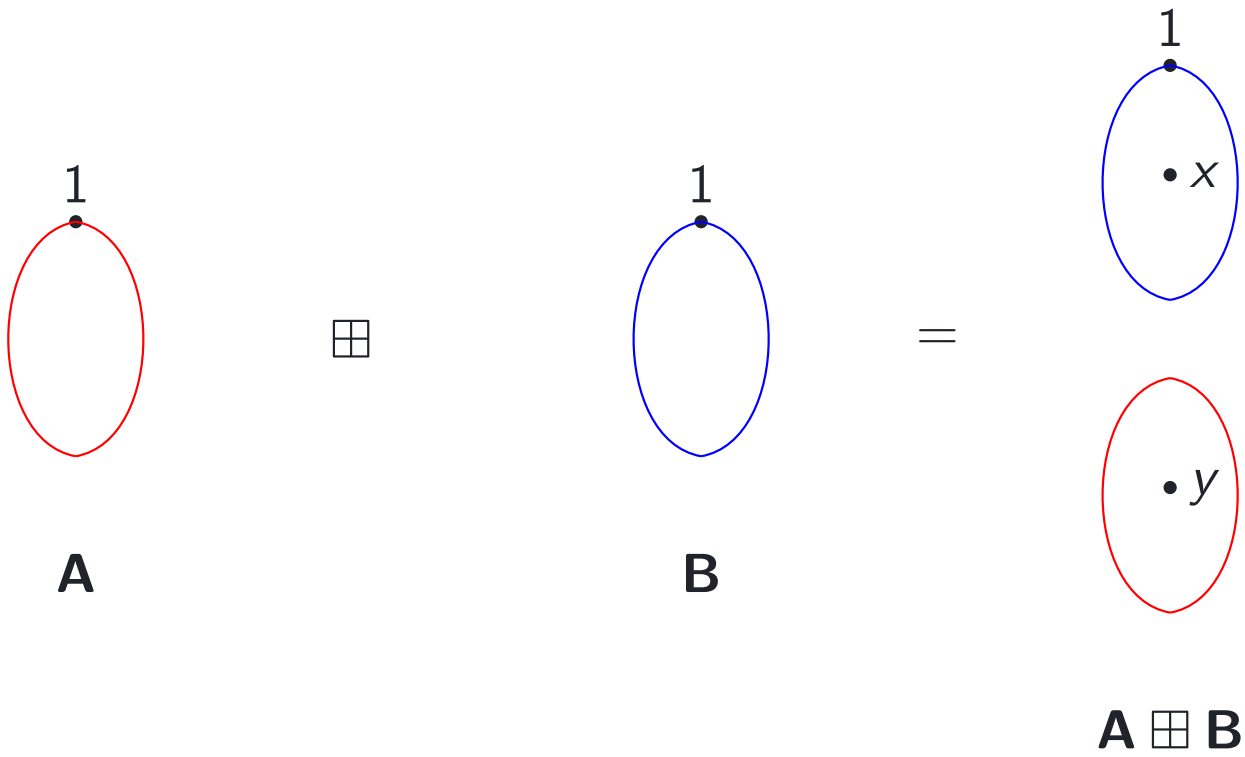
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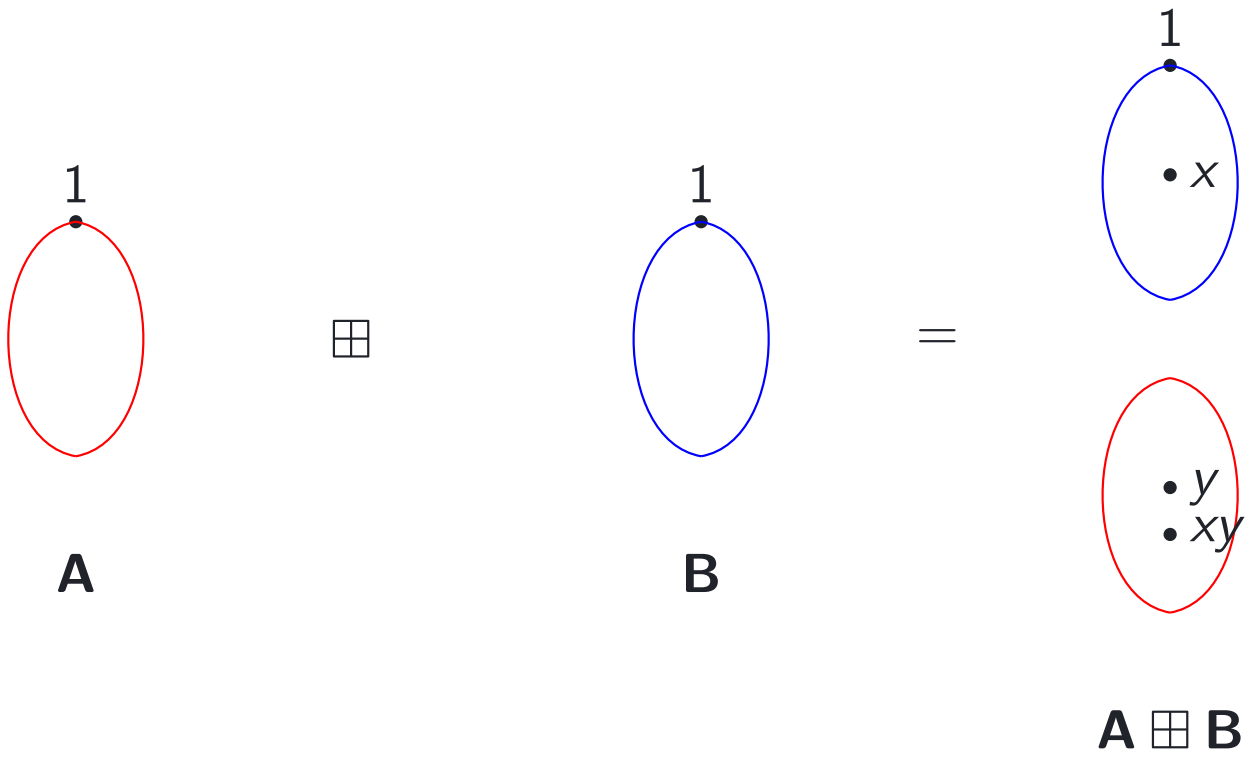
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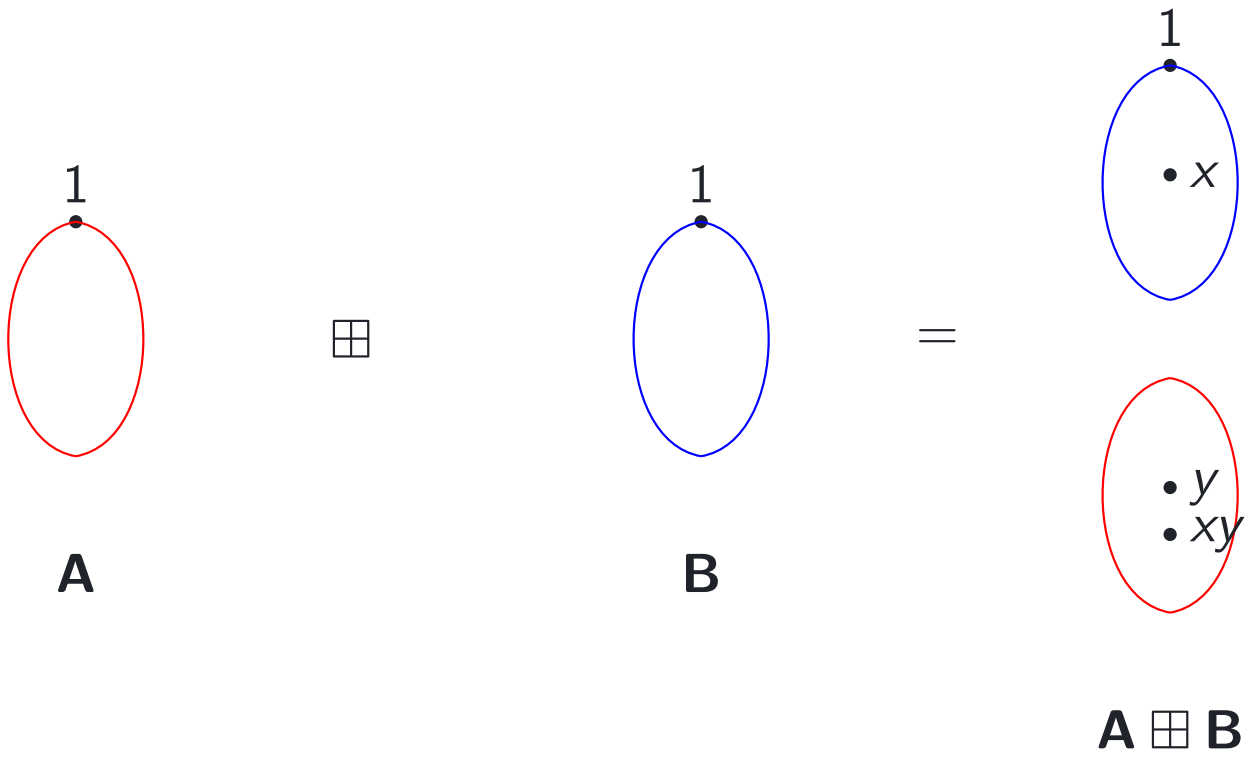
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The **blockwise gluing construction** is the non-commutative version of the partial gluing and it is iterable.

Application: amalgamation failure

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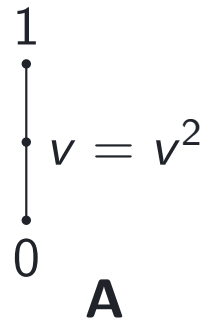
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Hence, we will see now the power of these construction and Fussner and Santschi's transfer theorem when we consider semilinear varieties.

The goal is to find a V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$, which we will call $\mathcal{VS}$**-formation**, that does not have an amalgam in chains. As usual, one can consider i, j to be the identity, i.e. $\mathbf{A}$ is a subalgebra of both $\mathbf{B}$ and $\mathbf{C}$.

The algebras in the V-formation

Let us take as **A** the Gödel chain of three elements, i.e.

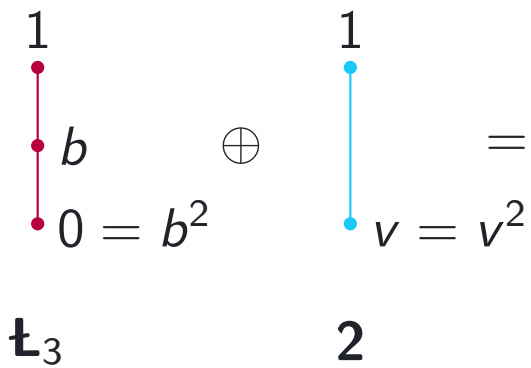


A

We would need now to construct the algebras **B** and **C**.

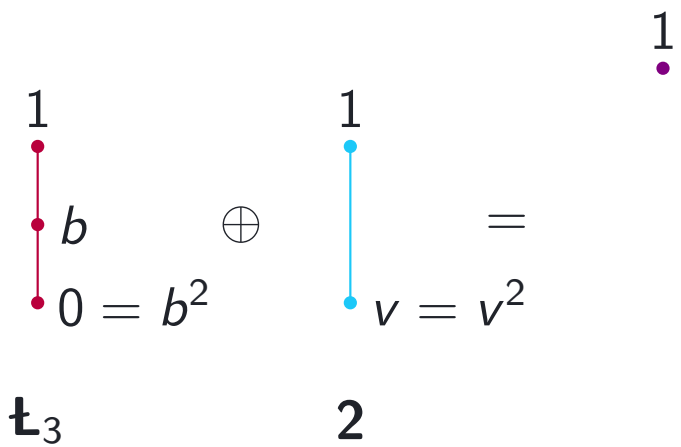
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To construct the algebra **B**, we would like to use the ordinal sum of the two element Boolean algebra **2** and the there elements MV-chain.



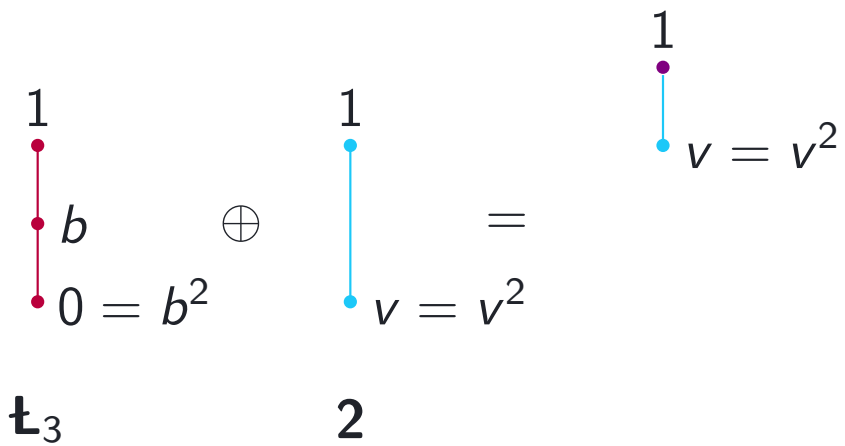
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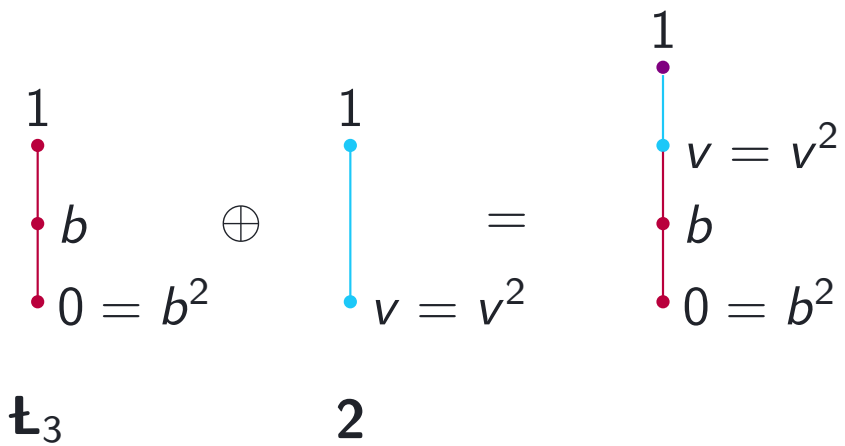
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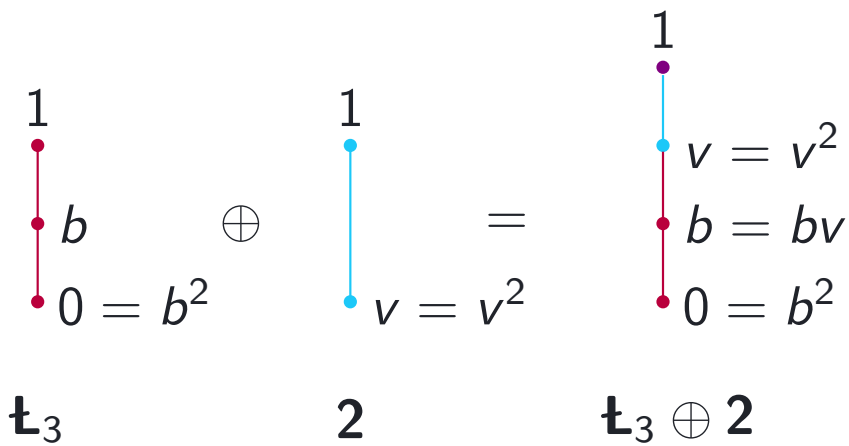
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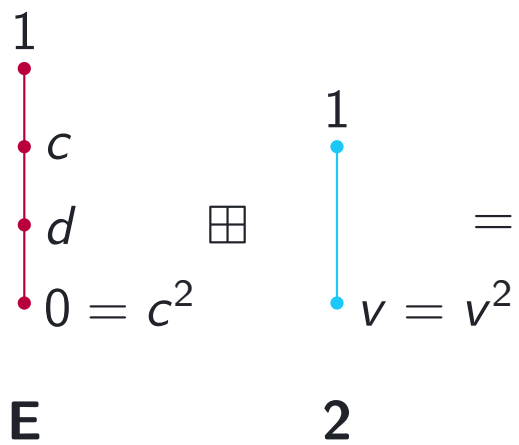
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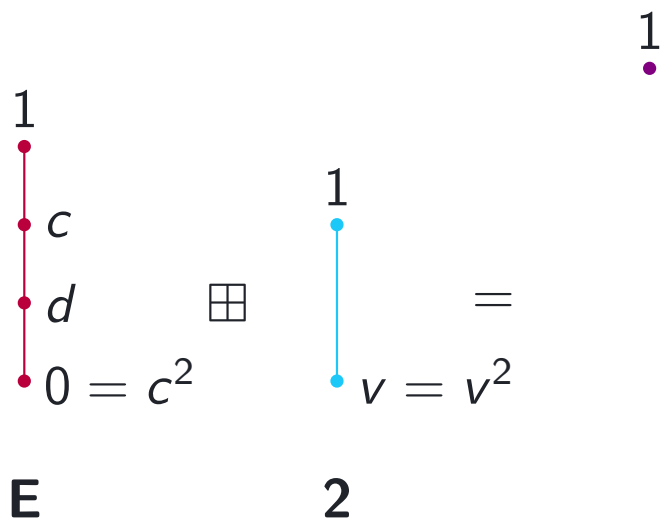
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It is left to construct the algebra **C**. To do so we would like to use the blockwise gluing construction. In particular, **C** would be given as the blockwise gluing of the two element Boolean algebra and the algebra, **E** having universe $\{1 > c > d > 0\}$ such that $c^2 = 0$.



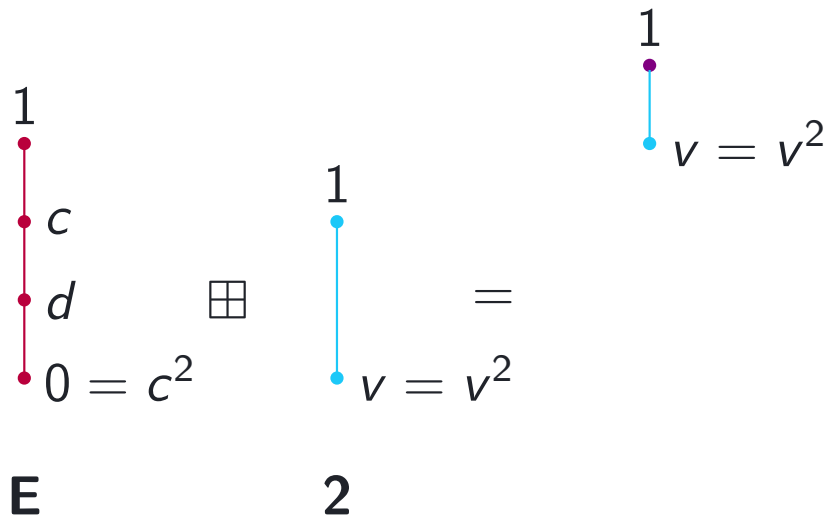
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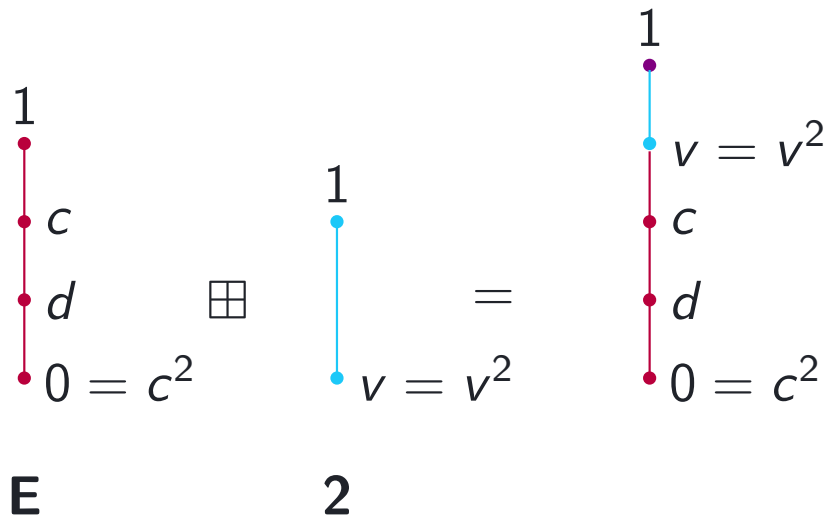
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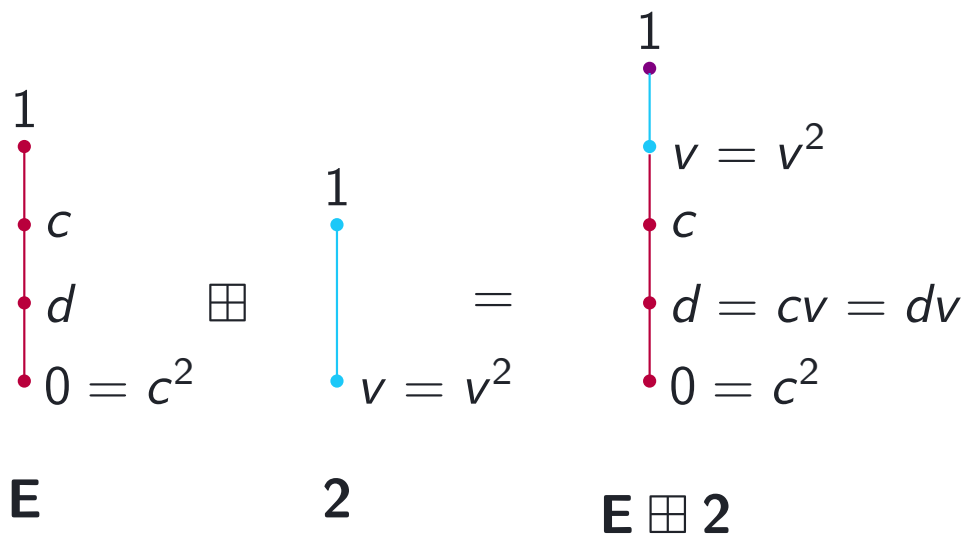
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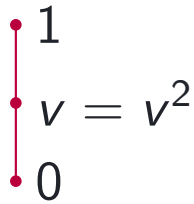


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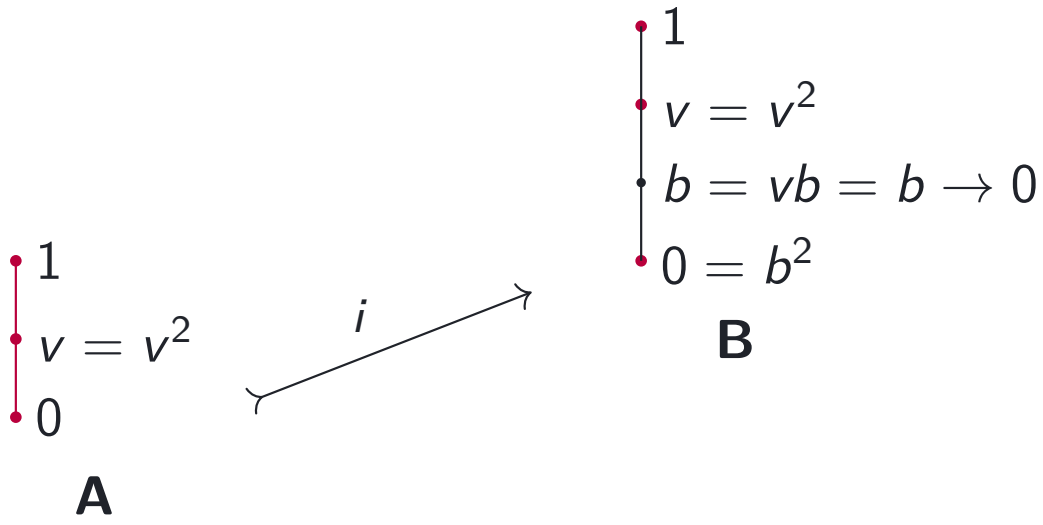


Amalgamation failure

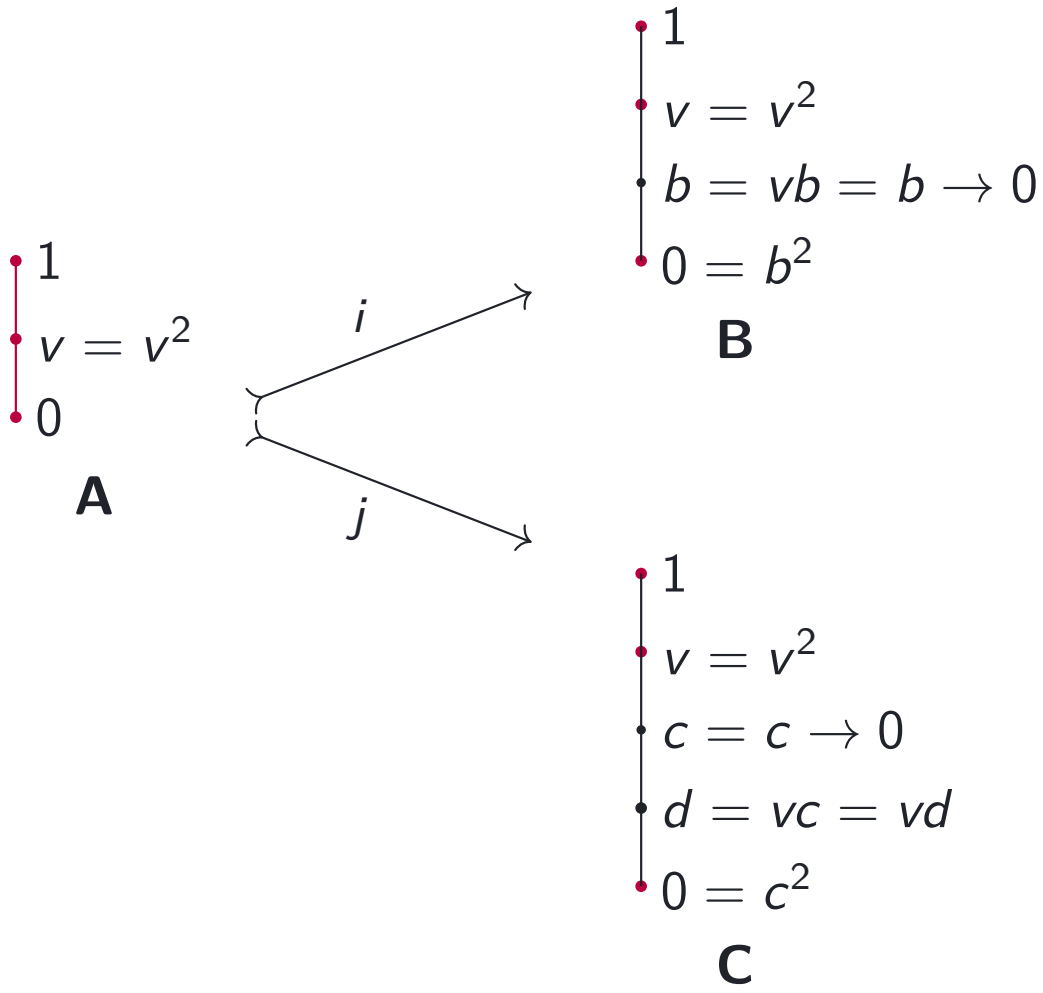


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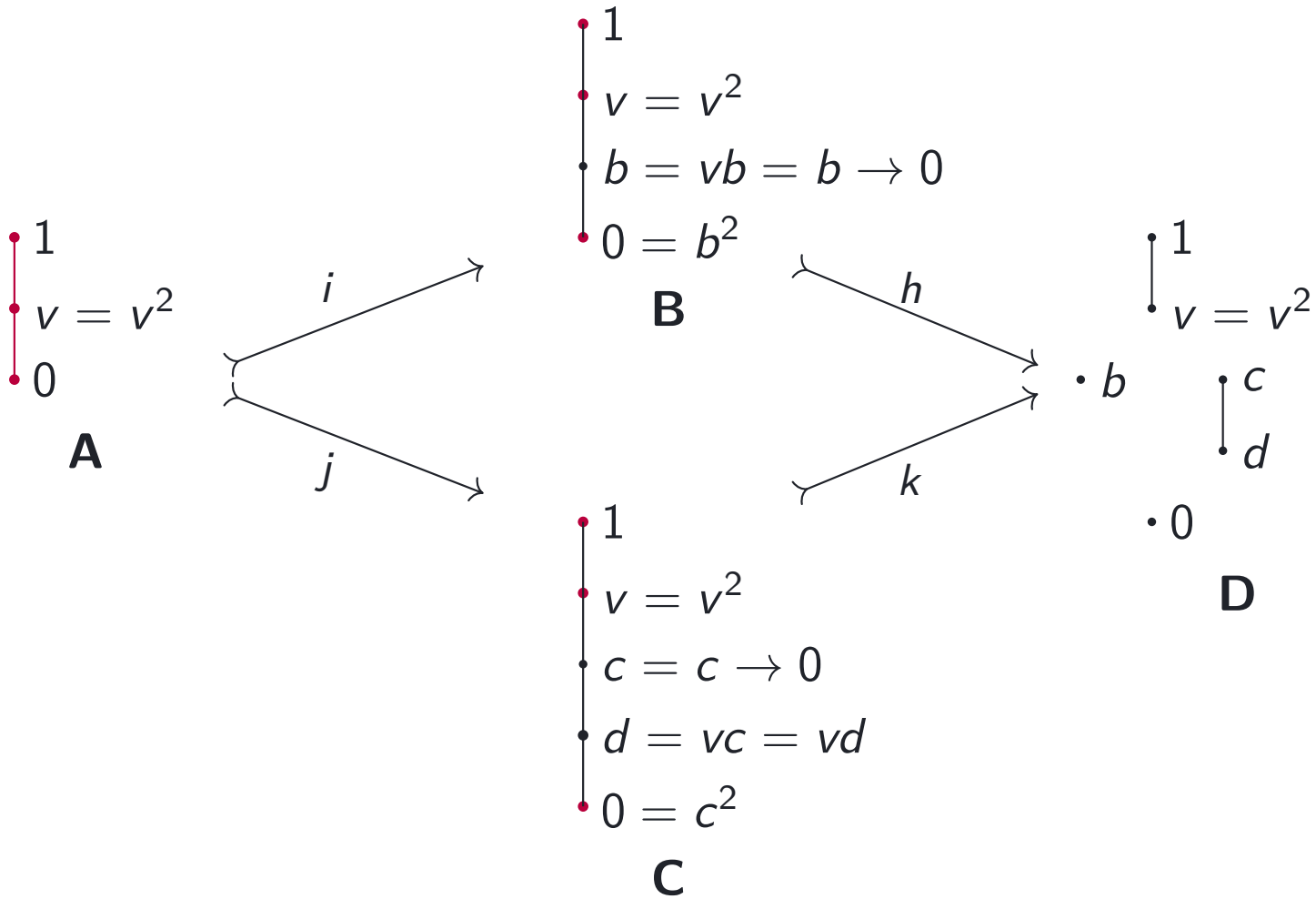
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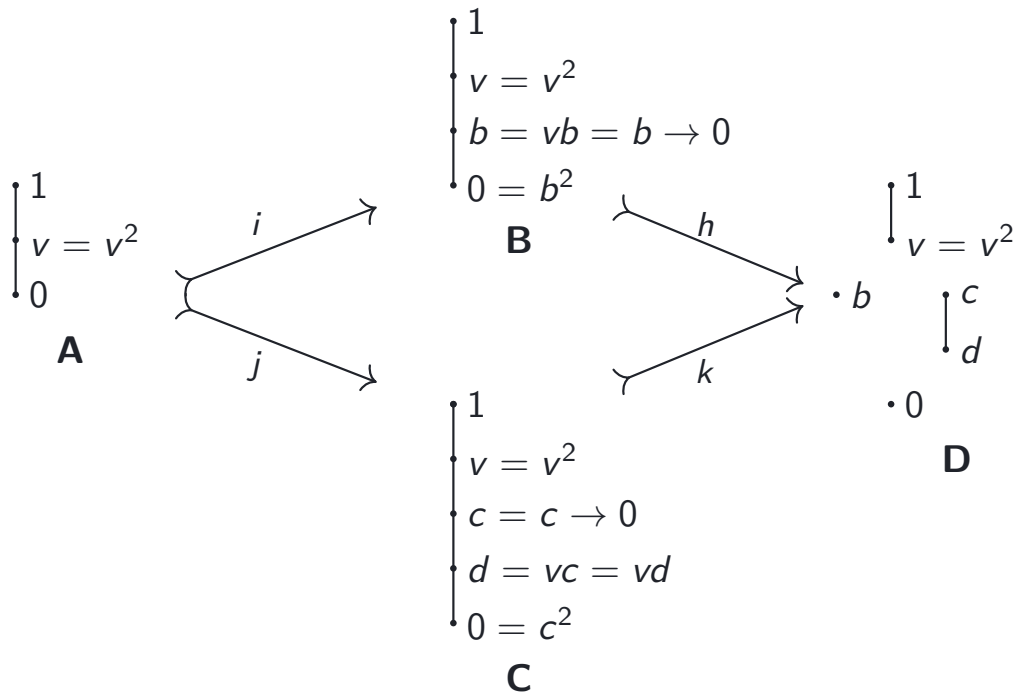
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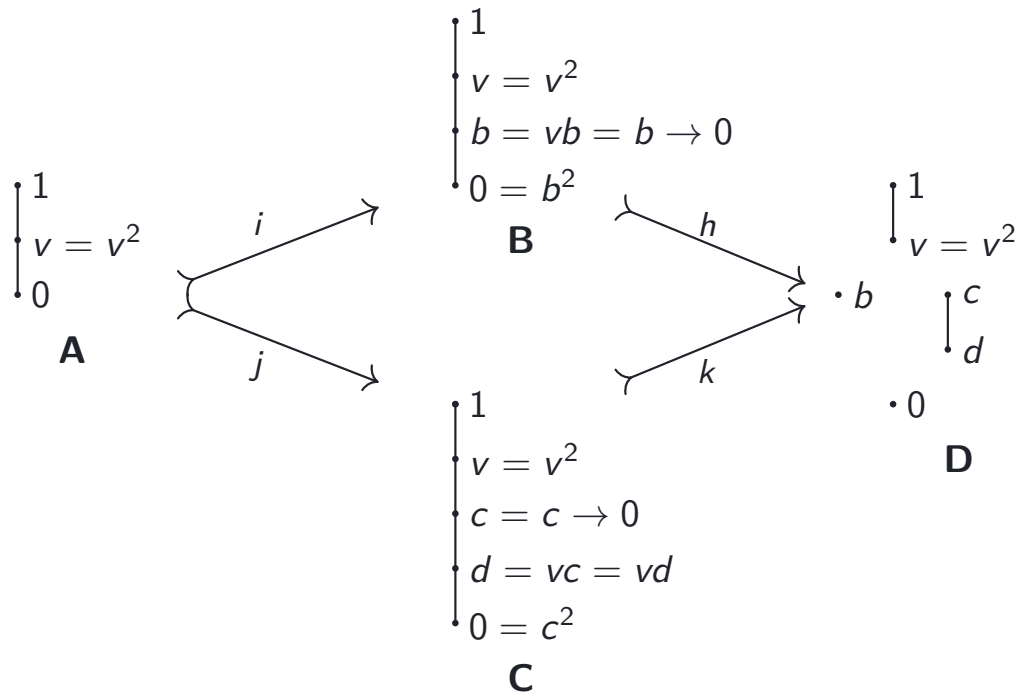


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If $b < c$, then, by order preservation, $bc \leq c^2 = 0$ and so $c \leq b \to 0 = b$, **CONTRADICTION!**

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Amalgamation failure

Theorem

The $\mathcal{VS}$-formation is an essential span and there is no essential amalgam $(\mathbf{D}, h, k)$ to $\mathcal{VS}$ in the class of RL-chains.

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Hence,

Theorem

Let V be a variety of semilinear RLs (or FL-algebras) such that the algebras $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathcal{VS}$ belong to V. Then V does not have the AP.

Amalgamation failure

Theorem

The following varieties do not have the AP:

1. MTL
2. IMTL
3. SMTL
4. SemCIRL
5. SemIRL
6. SemFL
7. $\mathsf{SemFL_w}$
8. SemRL *(Gil-Ferez, Ledda and Tsinakis)*
9. SemCRL *(Fussner and Santschi)*
10. $\mathsf{SemFL_e}$ *(Fussner and Santschi)*
11. *all n-potent subvarieties of the above ones for* $n \geq 2$

Thanks for your attention