Tight complexity bounds for substructural logics

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Outline

1. Substructural logics and residuated lattices.
2. Complexity results in the literature (for provability and deducibility).
3. Finiteness conditions and the role of wqo's.
4. Proof-theoretic analysis: Upper bounds.
5. Encoding machines: Lower bounds.
6. Extensions to: weak commutativity, hypersequents.

Based on:

G.-Greati-Ramanayake-St.John. Fast-growing complexities of substructural logics. manuscript, 184 pages, 2025.

Figure: https://tinyurl.com/3fsdty5t

Gentzen's system **LJ** for intuitionistic logic

A ***sequent*** is an expression $a_1, \ldots, a_n \Rightarrow a_0$, where a's are formulas. For $a, b, c \in Fm$, $x, y, z, x_1, x_2 \in Fm^*$, we have the inference rules:

$$\frac{x \Rightarrow a \quad y, a, z \Rightarrow c}{y, x, z \Rightarrow c}\ (\mathsf{cut}) \qquad \frac{}{a \Rightarrow a}\ (\mathsf{Id})$$

$$\frac{y, x_1, x_2, z \Rightarrow c}{y, x_2, x_1, z \Rightarrow c}\ (\mathsf{e}) \qquad \frac{y, z \Rightarrow c}{y, x, z \Rightarrow c}\ (\mathsf{w}) \qquad \frac{y, x, x, z \Rightarrow c}{y, x, z \Rightarrow c}\ (\mathsf{c})$$

$$\frac{y, a, z \Rightarrow c}{y, a \wedge b, z \Rightarrow c}\ (\wedge \mathsf{L}\ell) \quad \frac{y, b, z \Rightarrow c}{y, a \wedge b, z \Rightarrow c}\ (\wedge \mathsf{L}r) \quad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge \mathsf{R})$$

$$\frac{y, a, z \Rightarrow c \quad y, b, z \Rightarrow c}{y, a \vee b, z \Rightarrow c}\ (\vee \mathsf{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee \mathsf{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee \mathsf{R}r)$$

$$\frac{x \Rightarrow a \quad y, b, z \Rightarrow c}{y, x, a \to b, z \Rightarrow c}\ (\to \mathsf{L}) \qquad \frac{a, x \Rightarrow b}{x \Rightarrow a \to b}\ (\to \mathsf{R})$$

$$\frac{y, z \Rightarrow c}{y, 1, z \Rightarrow c}\ (1\mathsf{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\mathsf{R})$$

Basic substructural logics

In $\mathbf{LJ}$, the sequent $a_1, \ldots, a_n \Rightarrow a_0$ is provable iff the sequent $a_1 \wedge \ldots \wedge a_n \Rightarrow a_0$ is, so comma corresponds to $\wedge$. The proof system $\mathbf{FL}$ of Full Lambek calculus is obtained from Gentzen's proof system $\mathbf{LJ}$ for intuitionistic logic by removing the three basic structural rules:

$$\frac{u[x,y] \Rightarrow c}{u[y,x] \Rightarrow c}\ (e) \qquad \text{(exchange)} \qquad [x \to (y \to z)] \to [y \to (x \to z)] \qquad xy \leqslant yx$$

$$\frac{u[x,x] \Rightarrow c}{u[x] \Rightarrow c}\ (c) \qquad \text{(contraction)} \qquad [x \to (x \to y)] \to (x \to y) \qquad x \leqslant x^2$$

$$\frac{u[\varepsilon] \Rightarrow c}{u[x] \Rightarrow c}\ (i) \qquad \text{(integrality)} \qquad y \to (x \to y) \qquad x \leqslant 1$$

In $\mathbf{FL}$, comma and $\wedge$ do not correspond any more. But we can conservatively add a new connective $\cdot$ (*fusion* or *multiplication*) that does correspond to comma and rules:

$$\frac{y, a, b, z \Rightarrow c}{y, a \cdot b, z \Rightarrow c}\ (\cdot\mathrm{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x, y \Rightarrow a \cdot b}\ (\cdot\mathrm{R})$$

Also, $a \to b$ splits into $a \backslash b$ and b / a.

FL

$$\frac{x \Rightarrow a \quad y, a, z \Rightarrow c}{y, x, z \Rightarrow c}\ (\mathsf{cut}) \qquad \frac{}{a \Rightarrow a}\ (\mathsf{Id})$$

$$\frac{y, a, z \Rightarrow c}{y, a \wedge b, z \Rightarrow c}\ (\wedge \mathsf{L}\ell) \quad \frac{y, b, z \Rightarrow c}{y, a \wedge b, z \Rightarrow c}\ (\wedge \mathsf{L}r) \quad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge \mathsf{R})$$

$$\frac{y, a, z \Rightarrow c \quad y, b, z \Rightarrow c}{y, a \vee b, z \Rightarrow c}\ (\vee \mathsf{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee \mathsf{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee \mathsf{R}r)$$

$$\frac{x \Rightarrow a \quad y, b, z \Rightarrow c}{y, x, (a \backslash b), z \Rightarrow c}\ (\backslash \mathsf{L}) \qquad \frac{a, x \Rightarrow b}{x \Rightarrow a \backslash b}\ (\backslash \mathsf{R})$$

$$\frac{x \Rightarrow a \quad y, b, z \Rightarrow c}{y, (b/a), x, z \Rightarrow c}\ (/\mathsf{L}) \qquad \frac{x, a \Rightarrow b}{x \Rightarrow b/a}\ (/\mathsf{R})$$

$$\frac{y, a, b, z \Rightarrow c}{y, a \cdot b, z \Rightarrow c}\ (\cdot \mathsf{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x, y \Rightarrow a \cdot b}\ (\cdot \mathsf{R})$$

$$\frac{y, z \Rightarrow c}{y, 1, z \Rightarrow c}\ (1\mathsf{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\mathsf{R})$$

where $a, b, c \in Fm$, $x, y, z \in Fm^*$. Extensions of $\mathbf{FL}$ are known as *substructural logics*.

Substructural logics

Classical logic studies truth.

Intuitionistic logic (Brouwer, Heyting) deals with provability or constructibility. The algebraic models are Heyting algebras.

Many-valued logic (Łukasiewicz) allows different degrees of truth. [Ulam's game] $[x \to (x \to y)] \to (x \to y)$ is not a theorem. The algebraic models fail $x \leqslant x \cdot x$.

Relevance logic (Anderson, Belnap) deals with relevance.

$p \to (q \to q)$ is not a theorem. The algebraic models do not satisfy integrality $x \leqslant 1$.

$p \to (\neg p \to q)$ [or $(p \cdot \neg p) \to q$] is not a theorem, where $\neg p = p \to 0$. The algebraic models do not satisfy $0 \leqslant x$.

Linear logic (Girard) studies preservation of resourses.

$p \to (p \to p)$ [or $(p \cdot p) \to p$] and $p \to (p \cdot p)$ are not theorems.

The algebraic models do not satisfy mingle $x^2 \leqslant x$ nor contraction $x \leqslant x^2$.

FL and its variations are used in:

- Mathematical linguistics: Context-free grammars, pregroups. (Lambek, Buzskowski)
- CS: Memory allocation, pointer management, concurrent programming. (Separation logic, bunched implication logic).

Residuated lattices

A *residuated lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, 1)$ is a monoid and
- for all $a, b, c \in A$, $\quad ab \leqslant c \;\Leftrightarrow\; b \leqslant a\backslash c \;\Leftrightarrow\; a \leqslant c/b$.

Examples:

1. Boolean and Heyting algebras, where $x \cdot y = x \wedge y$ and $x \to y = x\backslash y = y/x$. We also add a constant 0 and define $\neg x = x \to 0$.
2. Also, MV-algebras and other algebras of substructural logics: Linear, relevance, MV, BL, MTL, where multiplication is strong conjunction.
3. Lattice-ordered groups: $x\backslash y = x^{-1}y$ and $y/x = yx^{-1}$. (and ℓ-pregroups)
4. Quantales (relating to quantal-valued model theory, C*-algebras)
5. Relation algebras: $R\backslash S = (R^{\cup} \circ S^{c})^{c}$, $\quad S/R = (S^{c} \circ R^{\cup})^{c}$.
6. Lattices of ideals of rings, under the usual multiplication and division of ideals. (Ward and Dilworth 1930's)
7. Computer Science: Action algebras, Kleene algebras with tests. (Pratt, Kozen)

Varieties of residuated lattices form *equivalent algebraic semantics* (a la Lindenbaum-Blok-Pigozzi) for various substructural logics.

Known complexities

We write $\mathbf{FL_{ec}}$ for $\mathbf{FL} + (e) + (c)$, etc.
Provability/theoremhood of intuitionistic logic $\mathbf{FL_{ecw}}$ (the equational theory of the variety of Heyting algebras) is PSPACE-complete. (Statman, TCS, 1979)
The same holds for $\mathbf{FL}$, $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$. (Horčik-Terui, TCS, 2011)

However, provability of $\mathbf{FL_{ec}}$ is Ackermanian-complete (Urquhart, JSL, 1999).

Provability of $\mathbf{FL_c}$ is undecidable (Chvalovský-Horčik, JSL, 2016).

Deducibility (quasiequational theory) for $\mathbf{FL_{ew}}$ is TOWER-complete (Tanaka, CSL, 2023), for $\mathbf{FL_{ec}}$ is Ackermanian complete (Urquhart, JSL, 1999), for $\mathbf{FL_e}$ is undecidable (Lincoln et all, APAL, 1992).

Actually, deducubility for most structural extensions of $\mathbf{FL_e}$ (varieties of commutative residuated lattices) are undecidable (G.-St.John, JSL, 2022).

Note that if the logic (e.g., $\mathbf{FL_{ec}}$) has a deduction theorem $\Gamma, \phi \vdash \psi \Leftrightarrow \Gamma \vdash \phi \to \psi$, then provabilty and deducibility coincide.

We generalize these special cases to a uniform class of logics/varieties. The generalization occurs in multiple directions.

Knotted rules

$$\frac{u[x,x] \Rightarrow c}{u[x] \Rightarrow c}\ (c) \qquad \frac{u[x] \Rightarrow c}{u[x,x] \Rightarrow c}\ (m) \qquad \frac{u[x,x,x] \Rightarrow c}{u[x,x] \Rightarrow c}\ (k(2,3))$$

The above three rules correspond to the algebraic axioms $x \leqslant x^2$, $x^2 \leqslant x$ and $x^2 \leqslant x^3$. In general *knotted rules* $k(n,m)$ allow for controlled duplication of resources and correspond to $x^n \leqslant x^m$. For $m > n$: *knotted contraction rule*; for $m < n$: *knotted weakening*.

To use proof-theoretic methods as in (Urquhart, JSL, 1999) for $\mathbf{FL_{ec}}$, we want to have an analytic calculus (cut elimination). Unfortunately, adding (m) or (k(3,2)) breaks cut-elimination for the calculus.

By results of (G.-Jipsen, TAMS, 2013) we can get a structural rule that has cut elimination, but it has multiple premisses and it corresponds to the (equivalent) linearization of $x^2 \leqslant x^3$: $xy \leqslant x^3 \vee x^2y \vee xy^2 \vee y^3$.

$$\frac{u[x,x,x] \Rightarrow c \quad u[x,x,y] \Rightarrow c \quad u[x,y,y] \Rightarrow c \quad u[y,y,y] \Rightarrow c}{u[x,y] \Rightarrow c}\ (k(2,3)')$$

We end up working with both rules in suitable parts of the proof (for example for height-preserving admisibility/Curry's lemma and for cut elimination).

Simple rules

Fact. Every equation over $\{\vee, \cdot, 1\}$ is equivalent to a conjunction of simple equations.

An equation is called *simple* if it is of the form $t_0 \leqslant t_1 \vee \cdots \vee t_n$, where t_i are $\{\cdot, 1\}$-terms and t_0 is linear.

Proof For an equation ε over $\{\vee, \cdot, 1\}$ we distribute products over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: monoid terms.

$s_1 \vee \cdots \vee s_m \leqslant t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leqslant s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leqslant t_1 \vee \cdots \vee t_n)$.

We proceed by example: $x^2 y \leqslant xy \vee yx \Rightarrow (x_1 \vee x_2)^2 y \leqslant (x_1 \vee x_2) y \vee y(x_1 \vee x_2)$

$\Rightarrow x_1^2 y \vee x_1 x_2 y \vee x_2 x_1 y \vee x_2^2 y \leqslant x_1 y \vee x_2 y \vee y x_1 \vee y x_2$

$\Rightarrow x_1 x_2 y \leqslant x_1 y \vee x_2 y \vee y x_1 \vee y x_2 \qquad \Rightarrow \qquad x^2 y \leqslant xy \vee yx$

(G.-Jipsen, TAMS, 2013): The structural rule corresponding to the simple equation $\varepsilon : t_0 \leqslant t_1 \vee \cdots \vee t_n$ preserves cut elimination

$$\frac{u[t_1] \Rightarrow a \quad \cdots \quad u[t_n] \Rightarrow a}{u[t_0] \Rightarrow a} \ (R(\varepsilon))$$

FEP

So, extensions with (linearized variants) of knotted rules enjoy cut elimination. Unfortunately, extensions with almost all (linearized variants of) knotted rules, such as contraction $x \leqslant x^2$, are not tame by (Chvalovský-Horčik, JSL, 2016): Provability of $\mathbf{FL_c}$ is undecidable and the same holds for $x^n \leqslant x^m$ with $m > 2$.

In the presence of exchange/commutativity, things are more tame by (Kripke, JSL, 1959): Provability of $\mathbf{FL_{ec}}$ is decidable.

Unfortunately, deducubility for most structural extensions of $\mathbf{FL_e}$ (varieties of commutative residuated lattices) are undecidable (G.-St.John, JSL, 2022). Fortunately, the few exceptions include the knotted rules.

The FEP holds for $\mathbf{FL_e}$ plus a knotted rule (van Alten, JSL, 2005) plus any set of simple rules (G.-Cardona, IJAC, 2015).

We say that a class $\mathcal{K}$ of algebras over the same language has the *finite embeddability property* (FEP) if for every $\mathbf{A} \in \mathcal{K}$ and finite $B \subseteq A$, there is a finite algebra $\mathbf{D} \in \mathcal{K}$ such that B embeds in $\mathbf{D}$ as a partial algebra.
Then, if a universal formula fails in $\mathcal{K}$, then it fails in a finite algebra of $\mathcal{K}$.

So, if a universal class is finitely axiomatized and has the FEP then its universal theory is decidable. This holds for $\mathbf{FL_e}$ plus a knotted rule.

FEP with simple rules

The proof for Boolean algebras is easy: Take $\mathbf{D}$ to be the subalgebra of $\mathbf{A}$ generated by B; it is a finite subalgebra of $\mathbf{A}$ containing B.

The proof for Heyting algebras is a bit more interesting, as we do not have *local finiteness*: Take $\mathbf{C}$ to be the $\{\vee, \wedge, 1\}$-subalgebra of $\mathbf{A}$ generated by B; it is a finite $\{\vee, \wedge, 1\}$-subalgebra of $\mathbf{A}$ containing B (distributive lattices are locally finite). Also, since $\mathbf{C}$ is a finite distributive lattice it supports a (unique) Heyting algebra expansion $\mathbf{D}$.

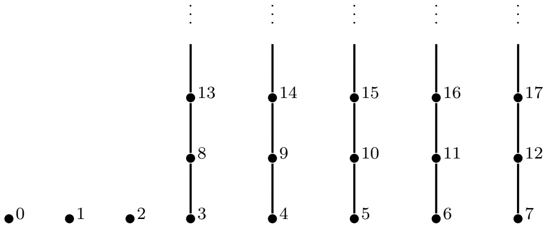
The proof for knotted commutative residuated lattices is much more complex. Take $\mathbf{C}$ to be the $\{\vee, \cdot, 1\}$-subalgebra of $\mathbf{A}$ generated by $B = \{b_1, \ldots, b_k\}$; if we have a knotted equality $x^n = x^{n+1}$, then the elements of $\mathbf{C}$ are joins of products $b_1^{n_1} \cdots b_k^{n_k}$, where $0 \leqslant n_i \leqslant n$ for all i. So, $\mathbf{C}$ is finite. Also, because $\mathbf{C}$ supports a (unique) residuated lattice expansion $\mathbf{D}$.

...but if we simply have a knotted inequality $x^n \leqslant x^m$, then the monoid $\mathbf{C}$ generated by B is infinite: there is no bound on the exponents n_i. Nevertheless, the set of products is isomorphic to the poset $\mathbb{N}^k$, which is a well-ordered set.

Well-ordered sets

In a poset antichains and infinite descending chains are special cases of bad sequences. In a quasi-ordered set, a sequence $a_1, a_2, \ldots$ is called *bad* if for all $i < j$ we have $a_i \nleqslant a_j$. A qosets that lack bad sequencs is called a well-quasi-ordered set (a wqo).
A bad sequence in $(\mathbb{N}, \leqslant)$: $9, 8, 5, 2, 1$. $(\mathbb{N}, \leqslant)$ is a wqo.
A bad sequence in $(\mathbb{N}^2, \leqslant)$: $(2,3), (2,2), (100,1), (99,1), (50,1), (2,1)$.
Some key features of wqos:

- Products and disjoint unions of wqo's are wqo's.
- Every finitely-generated downset is finite. (Mathematical induction.)
- Every upset is finitely generated.



So, the monoid $\mathbf{C}$ generated by B is infinite but also a wqo $\mathbb{N}^k_{n,m}$ (in the figure is $\mathbb{N}_{3,8}$ for the case $x^3 \leqslant x^8$). We proceed by constructing a *residuated frame*.

Residuated frames

A *residuated frame* is a structure $\mathbf{W} = (W, W', N, \circ, \varepsilon)$ where W and W' are sets $N \subseteq W \times W'$, $(W, \circ, \varepsilon)$ is a monoid and for all $x, y \in W$ and $w \in W'$ there exist subsets $x \mathbin{\backslash\!\backslash} w, w \mathbin{/\!/} y \subseteq W'$ such that

$$(x \circ y)\ N\ w \ \Leftrightarrow\ y\ N\ (x \mathbin{\backslash\!\backslash} w) \ \Leftrightarrow\ x\ N\ (w \mathbin{/\!/} y)$$

Notation $X^{\rhd} := \{w' \in W' : X\ N\ w'\}$, $Y^{\lhd} := \{w \in W : w\ N\ Y\}$, $\gamma_N(X) := X^{\rhd\lhd}$.

(G.-Jipsen, TAMS, 2013) The *Galois algebra* $\mathbf{W}^+ = \mathcal{P}(W, \circ, \varepsilon)_{\gamma_N}$ is a residuated lattice.

Example 1: If $\mathbf{L}$ is a RL, $\mathbf{W_L} = (L, L, \leqslant, \cdot, \{1\})$ is a residuated frame. Moreover, for $\mathbf{W_L}$, $x \mapsto \{x\}^{\lhd}$ is an embedding. The underlying poset of $\mathbf{W}^+_{\mathbf{L}}$ is the *Dedekind-MacNeille completion* of $\mathbf{L}$.

Example 2: We define the frame $\mathbf{W_{FL}}$, where

- $(W, \circ, \varepsilon)$ is the free monoid over the set Fm of all formulas
- $W' = S_W \times Fm$, where S_W is the set of all *contexts* $u[x] = y{\circ}x{\circ}z$ of W,
- $x\ N\ (u, a)$ iff $\vdash_{\mathbf{FL}} u[x] \ \Rightarrow\ a$.
- $(u, a) \mathbin{/\!/} x = \{(u[\_ \circ x], a)\}$ and $x \mathbin{\backslash\!\backslash} (u, a) = \{(u[x \circ \_], a)\}$

$$\begin{array}{lll} x \circ y N(u,a) & \text{iff } \vdash_{\mathbf{FL}} u[x \circ y] \ \Rightarrow\ a & \\ & \text{iff } \vdash_{\mathbf{FL}} u[x{\circ}y] \ \Rightarrow\ a & \\ & \text{iff } xN(u[\_ \circ y], a) & \text{iff } yN(u[x \circ \_], a). \end{array}$$

Gentzen frames

A residuated frame $\mathbf{W} = (W, W', N, \circ, \varepsilon)$ together with a common subset S of W and W' that is a partial algebra $\mathbf{S}$, is called a *Gentzen frame* if it satisfies versions of the rules of $\mathbf{FL}$ such as:

$$\frac{aNz \quad bNz}{a \vee bNz}\ (\vee\mathsf{L})$$

for $a, b \in S$, $z \in W'$ with $a \wedge b$ defined in S.

(G.-Jipsen, TAMS, 2013) The map $s \mapsto s^{\triangleleft}$ is a partial embedding of $\mathbf{S}$ into the Galois algebra $\mathbf{W}^+$. Also, any $\{\vee, \cdot, 1\}$ equations that hold in $\mathbf{W}$ are preserved in $\mathbf{W}^+$.

For the FEP application we consider the frame $\mathbf{W_{A,B}}$, where

- $\mathbf{C} = (W, \cdot, 1)$ to be the submonoid of $\mathbf{A}$ generated by B,
- $W' = S_B \times B$, where S_W is the set of all *unary linear polynomials* $u[x] = y \circ x \circ z$ of $(W, \cdot, 1)$,
- $x\ N\ (u, b)$ iff $u[x] \leqslant_{\mathbf{A}} b$,
- $(u, a) \mathbin{/\!\!/} x = \{(u[\_ \cdot x], a)\}$ and $x \mathbin{\backslash\!\!\backslash} (u, a) = \{(u[x \cdot \_], a)\}$.

Then the Galois algebra $\mathbf{D} := \mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ is a residuated lattice that satisfies all $\{\vee, \cdot, 1\}$ equations that hold in $\mathbf{A}$.

(G.-Cardona, IJAC, 2015) Using the fact that $\mathbf{C}$ is a wqo, we get that $\mathbf{D}$ is finite.

Proof-theoretic analysis for upper bounds

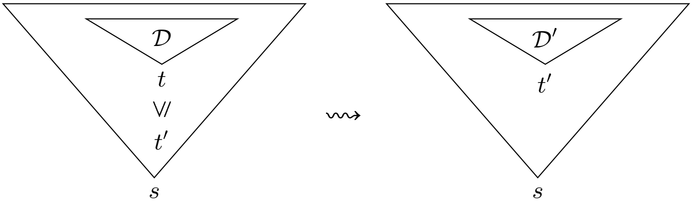
We will start by following (Kripke, JSL, 1959): provability of $\mathbf{FL_{ec}}$ is decidable.
Given a sequent s we want to check if it is provable in $\mathbf{FL_{ekR}}$, where k is a knotted contraction rule and R is any finite set of simple rules.

We call Ω the set of all formulas in s and consider only Ω-sequents in our analysis. We fix a listing of Ω so, modulo commutativity, every Ω-sequents is an element of $\mathbb{N}^\Omega$.

We design a new cut-free calculus $\mathbf{FL^*_{ekR}}$ that is equivalent to $\mathbf{FL_{ekR}}$ and

- the logical rules include a fixed number, $g(k,R)$, of applications of k below them.
- does not contain k but does contain its linearization k'. (This differs from Kripke.)

We prove 'Curry's Lemma' for $\mathbf{FL^*_{ekR}}$: If t has a derivation $\mathcal{D}$ of height at most h and $t' \leqslant t$ (in the wqo $\mathbb{N}^\Omega_k$), then t is also has a derivation $\mathcal{D}'$ of height at most h.



Proof-theoretic analysis for upper bounds

So if a proof has a branch where t' appears lower than t and $t' \leqslant t$, we can delete that segment of the branch. In the end we end up with proofs where all branches (read upward) form bad sequences.

We can define a 'proof search tree' where we we start with the end sequent s and recursively, for existing leaves, we add as children all the premisses of rules that have the leaf as conclusion (if the new children are not already on the branch). The proof search tree is finitely-branching and all branches are finite so by Kruskal's Lemma it is finite.

Theorem. Provability is decidable for $\mathbf{FL_{ekR}}$, where k is any knotted contraction rule and R is any finite set of simple rules.

To get complexity upper bounds we follow (Urquhart, JSL, 1999): provability of $\mathbf{FL_{ec}}$ is Ackermanian-complete.

Since there are no absolute bounds for the length of bad sequence in a wqo, we resort to relative length theorems: the sequences have jumps that are controlled. A *normed* wqo is a wqo $\mathbf{Q}$ endowed with a norm function $a \mapsto [a]$ into the naturals, where the preimage of every number is finite.

The branches of the proof search are bad sequences where $[a_{i+1}] \leqslant M[a_i]$, where M depends only on k and R. So, $[a_i] \leqslant M^i[s]$, for all i.

Fast-growing hierarchy and lenght functions

We consider the fast-growing hierarchy $\mathbf{F}_\alpha$ of complexity classes, where for α and ordinal less than Cantor's ε_0 (smallest ordinal such that $\varepsilon = \omega^\varepsilon$). $\mathbf{F}_\alpha$ is defined as the class of problems that have run time bounded by function in $\mathbf{F}^*_\alpha$: functions that are the composition of a single application of s function at complexity α with a function that is in a lower level than α in the Grzegorczyk hierarchy.

$\mathbf{F}_\omega$ is the class of Ackermann complexity. All n-EXP are in $\mathbf{F}_3$ (elementary), which is contained in TOWER= $\mathbf{F}_4$ which is contained the union of the $\mathbf{F}_n$'s (primitive recursive), which is contained in ACK= $\mathbf{F}_\omega$.

Theorem. The class of nwqo's of the form $r\mathbb{N}^k$, with $r, k \in \mathbb{N}$ have lengths of bad sequences (as functions of the norm of their first entry) that are in $\mathbf{F}^*_\omega$.

We are also able to control the size of sequent in a branch, in terms of the height of the node and the size of the end sequent.

Theorem. Provability of $\mathbf{FL_{ekR}}$ is at most Ackermaniann, where k is any knotted contraction rule and R is any finite set of simple rules.

Theorem. The deduction theorem holds for $\mathbf{FL_{ekR}}$, where k is any knotted contraction.

Theorem. Deducibility of $\mathbf{FL_{ekR}}$ is at most Ackermaniann, where k is any knotted contraction rule and R is any finite set of simple rules.

Knotted weakening

For extensions with knotted weakening we employ a forward search.

- We start from axioms and consider larger and larger sets D_i of provable sequents, checking whether the end sequent is contained at every stage; these sets form a sequence.
- If the end sequent is not contained in any of them and there is no stabilization $D_{i+1} = D_i$, then we form a sequence $s_i \in D_{i+1} \backslash D_i$.
- We show that $s_1, s_2, \ldots$ is a bad sequence.
- We employ the length theorems to get a complexity bound.

Theorem. Provability of $\mathbf{FL_{ekR}}$ is at most Ackermaniann, where k is any knotted weakening rule and R is any finite set of simple rules.

The deduction theorem fails for knotted weakening extensions, so we cannot transfer the result to deducibility.

- Given a set S of assumption sequents (for the deduction $S \vdash s$) we design an auxiliary equivalent calculus $\mathbf{FL}_S$ in which S gets replaces by suitable inference rules.
- All of the proof-theoretic results (e.g., Curry's lemma) hold for the new calculus.
- All of the complexity results are proved to be uniform in the complexity size of S.

Theorem. Deducibility of $\mathbf{FL_{ekR}}$ is at most Ackermaniann, where k is any knotted weakening rule and R is any finite set of simple rules.

Generalizing exchange/commutativity

We move from $xy = yx$ to $xyx = xxy$ and more generally to equations:

$$xy_1xy_2 \cdots y_rx = x^{a_0}y_1x^{a_1}y_2 \cdots y_rx^{a_r}.$$

where $a_0 + a_1 + \cdots + a_r = r + 1$ and at least one a_i is 0. If $a = (a_0, a_1, \ldots, a_r)$, then the equation is called a-commutativity, or *weak commutativity*, if a can vary.

Recall that under commutativity, in the finitely-generated case we get a single form $x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}$. So, the infinite behavior is moved to the 1-generated case, where it is tamed by using well quasi orders. If we lack commutativity potentilly infinitely many form can appear, such as $x_1^{n_1}x_2^{n_2}x_3^{n_3}$, $x_2^{n_2}x_1^{n_1}x_3^{n_3}$, $x_2^{n_2}x_1^{n_1}x_2^{n_2'}x_3^{n_3}x_1^{n_1'}$ can appear.

In (G.-Cardona, IJAC, 2015) we study the dynamics of the Zimin-like words and prove that in the finitely-generated cases we get finitely-many forms (and the number is elementarily computable).

We undertake a complex proof-theoretic analysis based on this fact and define suitable new nwqo's for weak commutativity.

Theorem. Deducibility of $\mathbf{FL_{akR}}$ is at most Ackermaniann, where k is any knotted rule, a is any weak commutativity, and R is any finite set of simple rules.

Counter machines: hardware

Counter machines store numbers and can *increment*, *decrement* or *test* if the number is zero. The hardware of a *counter machine* consists of

- a finite set $R = \{r_1, \ldots, r_k\}$ of *registers* (or counters), which can be thought of as empty boxes labeled by the name of the register, and *tokens* each of which can be in some register,
- a final set Q of internal *states* in which the machine can be in, with designated initial state q_I and final state q_F.

A *configuration* consists of a state and a natural number for each register. The configuration of a machine can be represented by the (commutative) monoid term

$$qr_1^{n_1} r_2^{n_2} \cdots r_k^{n_k}.$$

The machine will be able to move to other configurations during the computation:

$$qr_1^{n_1} r_2^{n_2} \cdots r_k^{n_k} \leqslant q' r_1^{m_1} r_2^{m_2} \cdots r_k^{m_k} \leqslant \ldots \leqslant q_F$$

via applications of instructions, and will be *accepted* if there is some way to reach q_F.

The set of instructions always contains $qr \leqslant rq$ and $rq \leqslant qr$, for all $q \in Q$, and $r \in R$.

Counter machines: software

The software consists of a finite set P of instructions taken from three different types.

- Increment instructions $q \leqslant q' r_i$: when in state q, increment register r_i by one token and change the internal state to q'.

  Intended application: $q r_1^{n_1} r_2^{n_2} \cdots r_i^{n_i} \cdots r_k^{n_k} \leqslant q' r_1^{n_1} r_2^{n_2} \cdots r_i^{n_i+1} \cdots r_k^{n_k}$.
- Decrement instructions $q r_i \leqslant q'$: when in state q, decrement register r_i (if possible) by one token and change the internal state to q'.

  Intended application: $q r_1^{n_1} r_2^{n_2} \cdots r_i^{n_i+1} \cdots r_k^{n_k} \leqslant q' r_1^{n_1} r_2^{n_2} \cdots r_i^{n_i} \cdots r_k^{n_k}$.
- (For CM) Zero-test instructions $q \leqslant q'$: when in state q, check the contents of register r_i and if they are empty then move to state q'.

  Intended application: $q r_1^{n_1} r_2^{n_2} \cdots r_i^{0} \cdots r_k^{n_k} \leqslant q' r_1^{n_1} r_2^{n_2} \cdots r_i^{0} \cdots r_k^{n_k}$.
- (For ACM) Copy instructions $q \leqslant q' \vee q''$: when in state q, duplicate the data and move to states q' and q''.

  Intended application: $q r_1^{n_1} \cdots r_k^{n_k} \leqslant q' r_1^{n_1} \cdots r_k^{n_k} \vee q'' r_1^{n_1} \cdots r_k^{n_k}$.

This works well as in RLs: $qR \leqslant (q' \vee q'')R = q'R \vee q''R$.

The *computation relation* $\leqslant$ of a machine is defined as the reflexive-transitive closure of the smallest compatible (with multiplication and join) relation containing the instructions.

Different encoding

So we update our hardware of an *And-branching Counter Machine* to support joins of configurations, which we call *instantaneous descriptions*, IDs. and we represent by

$$C_1 \vee \cdots \vee C_m,$$

where the C_i's are configurations; so ID's of the machine are elements of the free join-semilattice over the set QR^*. We assume that this sits inside the commutative idempotent semiring generated by $Q \cup R$.

Recall that in lattices

$$C_1 \vee \cdots \vee C_m \leqslant q_F \Leftrightarrow (C_1 \leqslant q_F \;\&\; \ldots \;\&\; C_m \leqslant q_F)$$

so $\vee$ behaves conjunctively: all parallel computations/branches must be accepted.

Fact. ACMs can simulate CMs.

In (G.-St.John, JSL, 2022) we encode undecidable acceptance problems for machines to quasiequations $\&P \implies u \leqslant q_F$, where P is the set of instructions of a single machine. (This corresponds to the word problem, with input u, of the finitely-presented algebra with presentation P.)

Here, we fix the conclusion of the quasi-equations $\&P \implies q_I \leqslant q_F$ and vary the antecedent P that ranges over the instructions of machines in a given class (of Ackermanian complexity).

The residuated frame

All computations in the machine are interpreted as valid in the residuated lattice presented by relations corresponding to the instructions. So RL satisfies the quasiequation

$$\&P \ \Rightarrow\ q_I \leqslant q_F$$

where P is the set of instructions of an undecidable machine and u is an accepted configuration of the machine.

Conversely, if some configuration is not accepted then we can construct a residuated lattice that falsifies the quasiequation; it will be the Galois algebra of a residuated frame.

Let M be a machine and $W := (Q \cup R_k \cup S)^*$ be the free monoid generated by $Q \cup R_k \cup S$ and $W' = W \times W$. We define the relation $N \subseteq W \times W'$ via

$$x \ N \ (u, v) \quad \text{iff} \quad uxv \leqslant q_F,$$

for all $x, z \in W$. Observe that, for any $x, y, u, v \in W$,

$$xy \ N \ (u, v) \iff uxyv \leqslant q_F \iff x \ N \ (u, yv) \iff y \ N \ (ux, v).$$

Theorem. $\mathbf{W} := (W, W', N, \circ, \varepsilon)$ is a residuated frame, $\mathbf{W}^+ \in$ RL, and there exists a valuation $\nu : \mathbf{Fm} \to \mathbf{W}^+$ that falsifies the quasiequation of the machine.

Lower bounds

But our models are RLs that satisfy a knotted rule $x^n \leqslant x^m$. So in the computations of the machine n-copies of a register can become m-copies *spontaneously*; this is a *glitch* that can compute/accept unintended instantaneous descriptions.

Theorem. Our machines are resilient/impervious to these knotted glitches. (And to many more different axioms.)

Main Theorem. Deducibility of $\mathbf{FL}+$ a knotted rule $+$ a (weak) commutativity is Ackermann-complete. In the case of a knotted contraction rule and commutativity, the same holds for the equational theory.

Beyond sequent rules

We mentioned that, by results of (G.-Jipsen, TAMS, 2013), $\{\vee, \cdot, 1\}$-equations give rise to analytic structural *sequent* rules (cut elimination holds).

By results of (G.-Ciabbatoni-Terui, LICS, 2008) and (G.-Ciabbatoni-Terui, APAL, 2012) strongly analytic sequent rules are essentially defined only by $\{\vee, \cdot, 1\}$-equations.

A hypersequent is a multiset $s_1 \mid \cdots \mid s_m$ of sequents s_i. Hypersequent structural rules:

$$\frac{H \mid s'_1 \quad H \mid s'_2 \quad \ldots \quad H \mid s'_n}{H \mid s_1 \mid \cdots \mid s_m}$$

Hypersequent calculi allow for the proof-theoretic study of many more extensions, such as the Gödel-Dummet logic modeled by $(x \to y) \vee (y \to x)$, as $\mid$ is a form of disjunction.

We make heavy use of results in a series of papers on *Algebraic Proof Theory* by G.-Ciabbatoni-Terui: (LICS, 2008), (AU, 2011), (APAL, 2012), (APAL, 2017).

(i) Hypersequents allow access to *finitely subdirectly irreducible* algebras in the variety and to $\mathsf{HSP_U}$-classes (positive universal classes).

(ii) Full description of analytic (hyper)sequent rules and a transformation proceedure.

(iii) The *substructural hierarchy* (similar to the arithmetical hierarchy) is defined by alternations of *positive* and *negative* connectives.

Bi-modules

Let's assume that $P = N$ is the underlying set of a residuated lattice.

- $x \cdot 1 = x = 1 \cdot x$, $(xy)z = x(yz)$
- $x(y \vee z) = xy \vee xz$ and $(y \vee z)x = yx \vee zx$

So, $(P, \vee, \cdot, 1)$ is a semiring. [In the complete case, a quantale.]

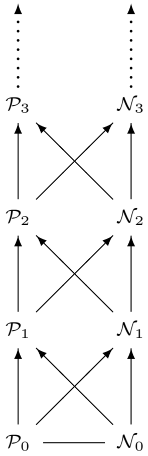
- $x\backslash(y \wedge z) = (x\backslash y) \wedge (x\backslash z)$ and $(y \wedge z)/x = (y/x) \wedge (z/x)$
- $(y \vee z)\backslash x = (y\backslash x) \wedge (z\backslash x)$ and $x/(y \vee z) = (x/y) \wedge (x/z)$
- $x\backslash(y/z) = (x\backslash y)/z$
- $1\backslash x = x = x/1$
- $(yz)\backslash x = z\backslash(y\backslash x)$ and $x/(zy) = (x/y)/z$

So, $(P, \vee, \cdot, 1)$ acts on both sides on $(N, \wedge)$ by $p \star n = n/p$ and $n \star p = p\backslash n$. Thus, $((N, \wedge), \star)$ becomes a $(P, \vee, \cdot, 1)$-bimodule. This split matches the notion of *polarity*. It also extends to $\bigvee$, $\bigwedge$.

The bimodule can be viewed as a two-sorted algebra $(P, \vee, \cdot, 1, N, \wedge, \backslash, /)$.

The absolutely free algebra for $P = N$ generated by $P_0 = N_0 = Var$ (the set of propositional variables) gives the set of all formulas. The steps of the generation process yield the *substructural hierarchy*.

Substructural hierarchy



- The sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas are defined by:
  - (0) $\mathcal{P}_0 = \mathcal{N}_0 =$ the set of variables
  - (P1) $\mathcal{N}_n \subseteq \mathcal{P}_{n+1}$
  - (P2) $a, b \in \mathcal{P}_{n+1} \;\Rightarrow\; a \vee b, a \cdot b, 1 \in \mathcal{P}_{n+1}$
  - (N1) $\mathcal{P}_n \subseteq \mathcal{N}_{n+1}$
  - (N2) $a, b \in \mathcal{N}_{n+1} \;\Rightarrow\; a \wedge b \in \mathcal{N}_{n+1}$
  - (N3) $a \in \mathcal{P}_{n+1}, b \in \mathcal{N}_{n+1} \;\Rightarrow\; a\backslash b, b/a, 0 \in \mathcal{N}_{n+1}$
- $\mathcal{P}_{n+1} = \langle \mathcal{N}_n \rangle_{\vee, \prod}$; $\mathcal{N}_{n+1} = \langle \mathcal{P}_n \rangle_{\wedge, \mathcal{P}_{n+1}\backslash, /\mathcal{P}_{n+1}}$
- $\mathcal{P}_n \subseteq \mathcal{P}_{n+1}, \mathcal{N}_n \subseteq \mathcal{N}_{n+1}, \bigcup \mathcal{P}_n = \bigcup \mathcal{N}_n = Fm$
- $\mathcal{P}_1$-reduced: $\bigvee \prod p_i$
- $\mathcal{N}_1$-reduced: $\bigwedge (p_1 p_2 \cdots p_n \backslash r / q_1 q_2 \cdots q_m)$
  $p_1 p_2 \cdots p_n q_1 q_2 \cdots q_m \leqslant r$
- Sequent: $a_1, a_2, \ldots, a_n \;\Rightarrow\; a_0 \; (a_i \in Fm)$

Theorem: Deducibility of $\mathbf{FL}+$ a (weak) commutativity $+$ a knotted $+$ any finite set of $\mathcal{P}_3$ formulas is in hyper-ACK $\mathbf{F}_{\omega^\omega}$.

	Logic(s)	Provability			Deducibility		
		Decidability	LB	UB	Decidability	LB	UB
Base logics	$\mathbf{FL_e}$	FMP[113] PS[114]	PSPACE [54]	PSPACE [54]	N[93]	–	–
	$\mathbf{FL_{ew}}$	FMP[113] PS[114]	PSPACE [54]	PSPACE [54]	FEP[52] PS(6.5)[b]	TOWER [96]	TOWER [96]
	$\mathbf{FL_{ec}}$	FMP[113] PS[115]	$\mathbf{F}_\omega$[58]	$\mathbf{F}_\omega$[58]	FEP[52] PS[115][a]	$\mathbf{F}_\omega$[58]	$\mathbf{F}_\omega$[58][a]
	$\mathbf{FL_{ec(\mathit{m},1)}}, m > 2$	FEP[52] PS[53]	$\mathbf{F}_\omega$(10.4)[a]	$\mathbf{F}_\omega$(5.20)	FEP[52] PS[53]	$\mathbf{F}_\omega$(10.4)	$\mathbf{F}_\omega$(5.20)
	$\mathbf{FL_{ec(\mathit{m},\mathit{n})}}, n \geq 2$	FEP[52] PS(5.19)	$\mathbf{F}_\omega$(10.4)[a]	$\mathbf{F}_\omega$(5.20)	FEP[52] PS(5.19)	$\mathbf{F}_\omega$(10.4)	$\mathbf{F}_\omega$(5.20)
	$\mathbf{FL_{e(\vec{\mathit{a}})c(\mathit{m},\mathit{n})}}$	FEP[56] PS(7.30)	PSPACE [54]	$\mathbf{F}_\omega$(7.31)	FEP[56] PS(7.30)	$\mathbf{F}_\omega$(10.4)	$\mathbf{F}_\omega$(7.31)
	$\mathbf{FL_{ew(1,\mathit{n})}}, n \geq 2$	FEP[52] PS[53]	PSPACE [54]	PSPACE [54]	FEP[52] PS(6.5)	$\mathbf{F}_\omega$(11.15)	$\mathbf{F}_\omega$(6.14)
	$\mathbf{FL_{ew(\mathit{m},\mathit{n})}}, m \geq 2$	FEP[52] PS(6.5)	PSPACE [54]	$\mathbf{F}_\omega$(6.14)	FEP[52] PS(6.5)	$\mathbf{F}_\omega$(11.15)	$\mathbf{F}_\omega$(6.14)
	$\mathbf{FL_{e(\vec{\mathit{a}})w(\mathit{m},\mathit{n})}}$	FEP[56] PS(7.33)	PSPACE [54]	$\mathbf{F}_\omega$(7.35)	FEP[56] PS(7.33)	$\mathbf{F}_\omega$(11.15)	$\mathbf{F}_\omega$(7.35)
	$\mathbf{FL_i}$	FMP[113] PS[114]	PSPACE [54]	PSPACE [54]	FEP[105] PS[59]	$\mathbf{F}_{\omega^\omega}$[59]	$\mathbf{F}_{\omega^\omega}$[59]
	$\mathbf{FL_{c(\mathit{m},\mathit{n})}}$	N[60]	–	–	N[94]	–	–
	$\mathbf{FL_{w(1,2)}}$	FMP[67] PS[54]	PSPACE [54]	PSPACE [54]	FEP[107]	open	open
	$\mathbf{FL_{w(1,\mathit{n})}}$	FMP[67] PS[54]	PSPACE [54]	PSPACE [54]	open	open	open
	$\mathbf{FL_{w(\mathit{m},\mathit{n})}}, m > 1$	open	PSPACE [54]	open	N[94]	–	–
$\mathcal{A} \subseteq \mathcal{N}_2$	$\mathbf{FL_{ec}}(\mathcal{A})$	FEP[56] PS[66]	$\mathbf{F}_\omega$(10.12)[c]	$\mathbf{F}_\omega$[65]	FEP[56] PS[65][a]	$\mathbf{F}_\omega$(10.4)[c]	$\mathbf{F}_\omega$[65][a]
	$\mathbf{FL_{ec(\mathit{m},\mathit{n})}}(\mathcal{A})$	FEP[56] PS(5.19)	$\mathbf{F}_\omega$(10.12)[c]	$\mathbf{F}_\omega$(5.20)	FEP[56] PS(5.19)	$\mathbf{F}_\omega$(10.4)[c]	$\mathbf{F}_\omega$(5.20)
	$\mathbf{FL_{e(\vec{\mathit{a}})c(\mathit{m},\mathit{n})}}(\mathcal{A})$	FEP[56] PS(7.30)	$\mathbf{F}_\omega$(10.12)[c]	$\mathbf{F}_\omega$(7.31)	FEP[56] PS(7.30)	$\mathbf{F}_\omega$(10.4)[c]	$\mathbf{F}_\omega$(7.31)
	$\mathbf{FL_{ew}}(\mathcal{A})$	FEP[56] PS[65]	PSPACE [54]	$\mathbf{F}_\omega$[65]	FEP[56] PS(6.5)	$\mathbf{F}_\omega$(11.15)[c]	$\mathbf{F}_\omega$[65][a]
	$\mathbf{FL_{ew(\mathit{m},\mathit{n})}}(\mathcal{A})$	FEP[56] PS(6.5)	PSPACE [54]	$\mathbf{F}_\omega$(6.14)	FEP[56] PS(6.5)	$\mathbf{F}_\omega$(11.15)[c]	$\mathbf{F}_\omega$(6.14)
	$\mathbf{FL_{e(\vec{\mathit{a}})w(\mathit{m},\mathit{n})}}(\mathcal{A})$	FEP[56] PS(7.33)	PSPACE [54]	$\mathbf{F}_\omega$(7.35)	FEP[56] PS(7.33)	$\mathbf{F}_\omega$(11.15)[c]	$\mathbf{F}_\omega$(7.35)
	$\mathbf{FL_i}(\mathcal{A})$	FEP[67] PS[59]	PSPACE [54]	$\mathbf{F}_{\omega^\omega}$[59]	FEP[67] PS[59]	PSPACE [54]	$\mathbf{F}_{\omega^\omega}$[59]
$\mathcal{A} \subseteq \mathcal{P}_3^\flat$	$\mathbf{FL_{ec}}(\mathcal{A})$	FEP(3.3) PS[65]	open	$\mathbf{F}_{\omega^\omega}$[65]	FEP(3.3) PS[65][a]	open	$\mathbf{F}_{\omega^\omega}$[65][a]
	$\mathbf{FL_{ec(\mathit{m},\mathit{n})}}(\mathcal{A})$	FEP(3.3) PS(5.19)	open	$\mathbf{F}_{\omega^\omega}$(5.20)	FEP(3.3) PS(5.19)	open	$\mathbf{F}_{\omega^\omega}$(5.20)
	$\mathbf{FL_{e(\vec{\mathit{a}})c(\mathit{m},\mathit{n})}}(\mathcal{A})$	FEP(3.3) PS(7.30)	open	$\mathbf{F}_{\omega^\omega}$(7.31)	FEP(3.3) PS(7.30)	open	$\mathbf{F}_{\omega^\omega}$(7.31)
	$\mathbf{FL_{ew}}(\mathcal{A})$	FEP(3.3) PS[65]	open	$\mathbf{F}_{\omega^\omega}$[65]	FEP(3.3) PS(6.5)	open	$\mathbf{F}_{\omega^\omega}$[65][a]
	$\mathbf{FL_{ew(\mathit{m},\mathit{n})}}(\mathcal{A})$	FEP(3.3) PS(6.5)	open	$\mathbf{F}_{\omega^\omega}$(6.14)	FEP(3.3) PS(6.5)	open	$\mathbf{F}_{\omega^\omega}$(6.14)
	$\mathbf{FL_{e(\vec{\mathit{a}})w(\mathit{m},\mathit{n})}}(\mathcal{A})$	FEP(3.3) PS(7.33)	open	$\mathbf{F}_{\omega^\omega}$(7.35)	FEP(3.3) PS(7.33)	open	$\mathbf{F}_{\omega^\omega}$(7.35)
	$\mathbf{FL_i}(\mathcal{A})$	FEP(3.3) PS(8.13)	open	$\mathbf{F}_{\omega^{\omega^{\omega^\omega}}}$(8.20)	FEP(3.3) PS(8.13)	open	$\mathbf{F}_{\omega^{\omega^{\omega^\omega}}}$(8.20)