Well-ordering principles and the reverse mathematics zoo

Anton Freund

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Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Emmy
Noether-
Programm
DFG Deutsche
Forschungsgemeinschaft

A well-ordering principle

For each linear order α, consider

$$2^\alpha = \left\{2^{\alpha_{n-1}} + \ldots + 2^{\alpha_0} \,\middle|\, \alpha_{n-1} > \ldots > \alpha_0 \text{ in } \alpha\right\}$$
$$= \left\{\langle \alpha_{n-1}, \ldots, \alpha_0 \rangle \,\middle|\, \alpha_0 < \ldots < \alpha_{n-1} \text{ in } \alpha\right\}$$

with the lexicographic order.

Theorem (Girard; Hirst).

The following are equivalent over RCA_0:

- ACA_0,
- $\alpha \mapsto 2^\alpha$ preserves well-foundedness.

More examples from the literature

- ACA_0^+ corresponds to $\alpha \mapsto \varepsilon_\alpha$ (Marcone&Montalbán; Afshari&Rathjen),
- ATR_0 corresponds to $\alpha \mapsto \varphi_\alpha(0)$ (Veblen hierarchy) (Friedman; Rathjen&Weiermann; Marcone&Montalbán),
- ω-models of ATR_0 correspond to $\alpha \mapsto \Gamma_\alpha$ (Rathjen),
- Π^1_1-transfinite induction corresponds to $D \mapsto D'$ (derivatives of normal functions; F.&Rathjen),
- Π^1_1-comprehension corresponds to $D \mapsto \vartheta D$ (Bachmann-Howard construction; F.).

A proof strategy

- Classical ordinal analysis proves, e.g., that if ε_0 is well-founded, then ACA_0 is consistent (over RCA_0). By completeness, we get a model.
- By a modified argument, if $\alpha \mapsto \varepsilon_\alpha$ preserves well-foundedness, we get ω-models of ACA_0 (using ω-completeness due to Shoenfield/Schütte). Having these models yields ACA_0^+ (i.e. ω-jumps).

Application I (classical)

Kruskal's theorem says that for finite trees $T_1, T_2, \ldots$, we always find $i < j$ such that T_i embeds into T_j.

Theorem (H. Friedman): Kruskal theorem is not provable in ATR_0.

Proof idea: Notations for Γ_0 can be seen as finite trees.
In fact, this extends to notations somewhat above Γ_0,
involving a collapsing function ϑ (Rathjen&Weiermann).

Application II (new)

Finite trees form a 'recursive data type': One can construct them by adding a root below a collection of previously constructed trees.

Theorem (F.&Rathjen&Weiermann):

The following are equivalent (over RCA_0 due to Uftring):

(1) a uniform Kruskal theorem for all recursive data types,

(2) Π^1_1-comprehension,

(3) the 'minimal bad-sequence lemma' of Nash-Williams.

Note that (2)⇔(3) is due to Marcone.

Application III (new)

Being a better-quasi-order (**bqo**) is equivalent to well-foundedness for linear orders but much stronger without linearity.

That the antichain **3** with three elements is Δ^0_2-bqo plays an important role in Montalbán's analysis of Fraïssé's conjecture. It would be great to know that this is provable in ATR_0.

Theorem (F.): If **3** is Δ^0_2-bqo, we get ATR_0 (over RCA_0).

The previous considerations took place above ACA_0 (Turing jump).

Question: Are there well-ordering principles below ACA_0?

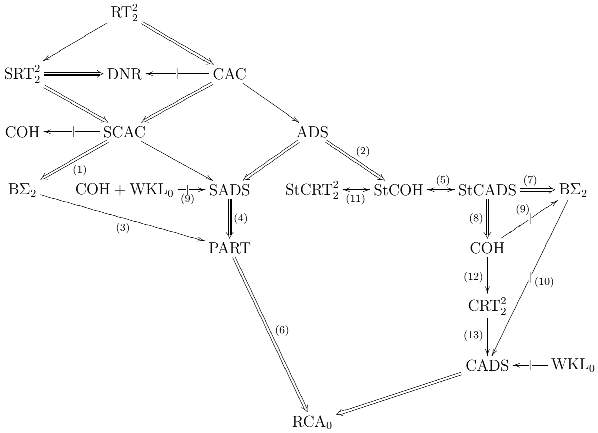
Theorem (Uftring; known to experts).

The following are equivalent over RCA_0:

- $\alpha \mapsto \alpha^\omega$ preserves well-foundedness,
- Σ^0_2-induction.

Are there proper second-order examples?

Are there examples from the zoo?



(diagram by Hirschfeldt&Shore)

What counts as an example?

Definition (Girard). A dilator is a functor $\alpha \mapsto D(\alpha)$ on well-orders that preserves direct limits and pullbacks; equivalently, elements of $D(\alpha)$ are terms with constants from α.

Fact. The notion of dilator is Π^1_2-complete over ACA_0.

Theorem (Uftring). There is no countable dilator D such that

$$\mathsf{WKL}_0 \quad \Leftrightarrow \quad D \text{ preserves well-foundedness}$$

holds in all ω-models of RCA_0.

For each linear order $L = (\mathbb{N}, \leq_L)$, define permutations π^L_n of $\{0, \ldots, n-1\}$ by

$$\pi^L_n(i) \leq_{\mathbb{N}} \pi^L_n(j) \quad \Leftrightarrow \quad i \leq_L j.$$

Definition. Let D_L be the dilator given by

$$D_L(\alpha) = \left\{ \langle \alpha_{\pi^L_n(0)}, \ldots, \alpha_{\pi^L_n(n-1)} \rangle \,\middle|\, \alpha_0 < \ldots < \alpha_{n-1} \text{ in } \alpha \right\}$$

with lexicographic comparisons.

Example. For $L = -\omega = (\mathbb{N}, \geq)$, we have

$$D_L(\alpha) = 2^\alpha = \left\{ \langle \alpha_{n-1}, \ldots, \alpha_0 \rangle \,\middle|\, \alpha_0 < \ldots < \alpha_{n-1} \text{ in } \alpha \right\}.$$

Fact (ACA_0). D_L preserves well-foundedness iff L is ill-founded.

Recall. If $\alpha \mapsto D_{-\omega}(\alpha) = 2^\alpha$ preserves well-foundedness, we have ACA_0 or equivalently (Hirst) arithmetical transfinite induction.

Theorem (F.; RCA_0). If D_L preserves well-foundedness for some L, we have **slow transfinite Π^0_2-induction (Π^0_2-$\mathsf{TI}^\star$)**: whenever 2^α is well-founded, Π^0_2-induction along α holds.

Fun Fact. The proof uses $2^{-\omega} \cong 1 + \mathbb{Q}$.

When L has finite Hausdorff rank, the conclusion of the theorem can be strengthened from Π^0_2-$\mathsf{TI}^\star$ to ACA_0.

Question. How strong is the statement that $D_{\mathbb{Q}}$ preserves well-foundedness?

How strong is slow transfinite Π^0_2-induction?

Theorem.

(1) Π^0_2-$\mathsf{TI}^\star$ does not hold in all ω-models of WKL_0 (with help from Aguilera&Pakhomov).

(2) Π^0_2-$\mathsf{TI}^\star$ is not provable in $\mathsf{WKL}_0 + \mathsf{RT}^2_2 + \mathsf{I}\Sigma^1_0$ (with help from Beklemishev).

The proof of (2) uses a result of Le Houérou, Patey and Yokoyama.

Conjecture. Π^0_2-$\mathsf{TI}^\star$ is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$.

We write $\mathsf{RCA}_0 \vDash_\omega \varphi$ if there is an $X \subseteq \mathbb{N}$ such that $\mathcal{M} \vDash \varphi$ holds for all ω-models $\mathcal{M} \ni X$ of RCA_0 (i.e., on a cone).

Theorem (F.).
For each countable dilator D, precisely one of the following holds:

- $\mathsf{RCA}_0 \vDash_\omega$ 'D preserves well-foundedness';
- $\mathsf{RCA}_0 \vDash_\omega$ 'D preserves well-foundedness' $\rightarrow \Pi^0_2\text{-}\mathsf{TI}^\star$.

The proof exploits the fine structure of dilators: It asks whether one of the dilators D_L can be embedded into D.

Corollary (F.). If we have $\mathsf{RCA}_0 \nvDash_\omega \psi$ but $\mathsf{WKL}_0 + \mathsf{RT}^2_2 + \mathsf{I}\Sigma^1_0 \vdash \psi$, there is no computable dilator D with

$$\mathsf{RCA}_0 + \mathsf{I}\Sigma^1_0 \vdash \psi \leftrightarrow \text{`}D \text{ preserves well-foundedness'}.$$

In particular, we can take $\psi = \mathsf{RT}^2_2$.

This negative result should not be over-interpreted: in fact, well-orders play a central role in the analysis of RT^2_2 (cf. the result that $\omega^{300n} \to (\omega^n)^2_2$ due to Kołodziejczyk and Yokoyama).

Thank you very much!

Do you have questions or comments?

Details and references can be found in:

> Anton Freund, *Dilators and the reverse mathematics zoo*, Journal of Mathematical Logic, to appear, `https://doi.org/10.1142/S0219061325500102`.