

Multiplicity of minimal types in a semiminimal analysis.

Christine Eagles

University of Illinois at Chicago

October, 2026

Overview

- 1 Motivate in the DCF_0 setting
- 2 Internality and analyzability to $\mathcal{C}$
- 3 Minimal types and semiminimal analyses
- 4 Understanding minimal types associated to p , invariant on choice of minimal analysis

Differential fields

- A **differential field** is a (characteristic zero) field F with $\delta : F \rightarrow F$, an additive function satisfying the Leibniz rule $\delta(xy) = \delta(x)y + x\delta(y)$.

Differential fields

- A **differential field** is a (characteristic zero) field F with $\delta : F \rightarrow F$, an additive function satisfying the Leibniz rule $\delta(xy) = \delta(x)y + x\delta(y)$.
- Let $\bar{a} = (a_1, \dots, a_n)$. We set $\delta(\bar{a}) = (\delta(a_1), \dots, \delta(a_n))$.

Differential fields

- A **differential field** is a (characteristic zero) field F with $\delta : F \rightarrow F$, an additive function satisfying the Leibniz rule $\delta(xy) = \delta(x)y + x\delta(y)$.
- Let $\bar{a} = (a_1, \dots, a_n)$. We set $\delta(\bar{a}) = (\delta(a_1), \dots, \delta(a_n))$.
- $\delta^n(a) = \delta(\dots(\delta(a))\dots)$.

Differential fields

- A **differential field** is a (characteristic zero) field F with $\delta : F \rightarrow F$, an additive function satisfying the Leibniz rule $\delta(xy) = \delta(x)y + x\delta(y)$.
- Let $\bar{a} = (a_1, \dots, a_n)$. We set $\delta(\bar{a}) = (\delta(a_1), \dots, \delta(a_n))$.
- $\delta^n(a) = \delta(\dots(\delta(a))\dots)$.
- Denote the differential field generated by F and $\bar{a}$ by $F\langle\bar{a}\rangle$.

Differential fields

- A **differential field** is a (characteristic zero) field F with $\delta : F \rightarrow F$, an additive function satisfying the Leibniz rule $\delta(xy) = \delta(x)y + x\delta(y)$.
- Let $\bar{a} = (a_1, \dots, a_n)$. We set $\delta(\bar{a}) = (\delta(a_1), \dots, \delta(a_n))$.
- $\delta^n(a) = \delta(\dots(\delta(a))\dots)$.
- Denote the differential field generated by F and $\bar{a}$ by $F\langle\bar{a}\rangle$.
- (L, δ') is **differential field extension** of (F, δ) if $L \supseteq F$ and $\delta'|_F = \delta$.

Differential fields

- A **differential field** is a (characteristic zero) field F with $\delta : F \rightarrow F$, an additive function satisfying the Leibniz rule $\delta(xy) = \delta(x)y + x\delta(y)$.
- Let $\bar{a} = (a_1, \dots, a_n)$. We set $\delta(\bar{a}) = (\delta(a_1), \dots, \delta(a_n))$.
- $\delta^n(a) = \delta(\dots(\delta(a))\dots)$.
- Denote the differential field generated by F and $\bar{a}$ by $F\langle\bar{a}\rangle$.
- (L, δ') is **differential field extension** of (F, δ) if $L \supseteq F$ and $\delta'|_F = \delta$.

Example

- $(\mathbb{C}(t), \delta)$ where $\delta = \frac{d}{dt}$.

Differential fields

- A **differential field** is a (characteristic zero) field F with $\delta : F \rightarrow F$, an additive function satisfying the Leibniz rule $\delta(xy) = \delta(x)y + x\delta(y)$.
- Let $\bar{a} = (a_1, \dots, a_n)$. We set $\delta(\bar{a}) = (\delta(a_1), \dots, \delta(a_n))$.
- $\delta^n(a) = \delta(\cdots (\delta(a)) \cdots)$.
- Denote the differential field generated by F and $\bar{a}$ by $F\langle\bar{a}\rangle$.
- (L, δ') is **differential field extension** of (F, δ) if $L \supseteq F$ and $\delta'|_F = \delta$.

Example

- $(\mathbb{C}(t), \delta)$ where $\delta = \frac{d}{dt}$.
- $(\mathbb{C}, \delta)$ where $\delta(x) = 0$ is a differential sub-field.

Differential fields

- A **differential field** is a (characteristic zero) field F with $\delta : F \rightarrow F$, an additive function satisfying the Leibniz rule $\delta(xy) = \delta(x)y + x\delta(y)$.
- Let $\bar{a} = (a_1, \dots, a_n)$. We set $\delta(\bar{a}) = (\delta(a_1), \dots, \delta(a_n))$.
- $\delta^n(a) = \delta(\cdots(\delta(a))\cdots)$.
- Denote the differential field generated by F and $\bar{a}$ by $F\langle\bar{a}\rangle$.
- (L, δ') is **differential field extension** of (F, δ) if $L \supseteq F$ and $\delta'|_F = \delta$.

Example

- $(\mathbb{C}(t), \delta)$ where $\delta = \frac{d}{dt}$.
- $(\mathbb{C}, \delta)$ where $\delta(x) = 0$ is a differential sub-field.
- Fix $(\mathcal{U}, \delta)$ a **differentially closed** differential field: every consistent system of finitely many differential polynomial equations and in-equations has a solution in $\mathcal{U}$.

Differential fields

- A **differential field** is a (characteristic zero) field F with $\delta : F \rightarrow F$, an additive function satisfying the Leibniz rule $\delta(xy) = \delta(x)y + x\delta(y)$.
- Let $\bar{a} = (a_1, \dots, a_n)$. We set $\delta(\bar{a}) = (\delta(a_1), \dots, \delta(a_n))$.
- $\delta^n(a) = \delta(\dots(\delta(a))\dots)$.
- Denote the differential field generated by F and $\bar{a}$ by $F\langle\bar{a}\rangle$.
- (L, δ') is **differential field extension** of (F, δ) if $L \supseteq F$ and $\delta'|_F = \delta$.

Example

- $(\mathbb{C}(t), \delta)$ where $\delta = \frac{d}{dt}$.
- $(\mathbb{C}, \delta)$ where $\delta(x) = 0$ is a differential sub-field.
- Fix $(\mathcal{U}, \delta)$ a **differentially closed** differential field: every consistent system of finitely many differential polynomial equations and in-equations has a solution in $\mathcal{U}$.
- Let $\mathcal{C} = \{a \in \mathcal{U} \mid \delta(a) = 0\}$, the field of constants for $\mathcal{U}$.

System of algebraic differential equations

Let (F, δ) be a differential field and $f_1, \dots, f_n \in F[x_1, \dots, x_n]$.

We look at systems of of order one ordinary differential equations of the form

$$(E) = \begin{cases} \delta(x_1) = f_1(x_1, \dots, x_n) \\ \vdots \\ \delta(x_n) = f_n(x_1, \dots, x_n). \end{cases}$$

System of algebraic differential equations

Let (F, δ) be a differential field and $f_1, \dots, f_n \in F[x_1, \dots, x_n]$.

We look at systems of of order one ordinary differential equations of the form

$$(E) = \begin{cases} \delta(x_1) = f_1(x_1, \dots, x_n) \\ \vdots \\ \delta(x_n) = f_n(x_1, \dots, x_n). \end{cases}$$

When can (E) be “built” from the constants?

The basic model theory

ACF₀

- Language of rings

DCF₀

- $L_\delta = \{0, 1, +, -, \times, \delta\}$

The basic model theory

ACF₀

- Language of rings
- Models are algebraically closed fields

DCF₀

- $L_\delta = \{0, 1, +, -, \times, \delta\}$
- Models are differentially closed fields

The basic model theory

ACF₀

- Language of rings
- Models are algebraically closed fields
- Strongly minimal theory, eliminates imaginaries, admits quantifier elimination

DCF₀

- $L_\delta = \{0, 1, +, -, \times, \delta\}$
- Models are differentially closed fields
- ω -stable theory, eliminates imaginaries, admits quantifier elimination

The basic model theory

ACF₀

- Language of rings
- Models are algebraically closed fields
- Strongly minimal theory, eliminates imaginaries, admits quantifier elimination
- $\text{dcl}(A)$ is field-theoretic closure, $\text{acl}(A)$ is field-theoretic algebraic closure

DCF₀

- $L_\delta = \{0, 1, +, -, \times, \delta\}$
- Models are differentially closed fields
- ω -stable theory, eliminates imaginaries, admits quantifier elimination
- $\text{dcl}(A) = \langle A \rangle$ and $\text{acl}(A) = \langle A \rangle^{\text{alg}}$

The basic model theory

ACF₀

- Language of rings
- Models are algebraically closed fields
- Strongly minimal theory, eliminates imaginaries, admits quantifier elimination
- $\text{dcl}(A)$ is field-theoretic closure, $\text{acl}(A)$ is field-theoretic algebraic closure
- **definable sets** are finite Boolean combinations of Zariski closed sets

DCF₀

- $L_\delta = \{0, 1, +, -, \times, \delta\}$
- Models are differentially closed fields
- ω -stable theory, eliminates imaginaries, admits quantifier elimination
- $\text{dcl}(A) = \langle A \rangle$ and $\text{acl}(A) = \langle A \rangle^{\text{alg}}$
- definable sets are finite Boolean combinations of **Kolchin** closed sets (zero sets of differential polynomials)

DCF₀

- Fix $(\mathcal{U}, \delta)$, a sufficiently saturated model of DCF₀.

DCF₀

- Fix $(\mathcal{U}, \delta)$, a sufficiently saturated model of DCF₀.
 - ▶ Any consistent system of differential polynomial equations and in-equations with coefficients in a small differential subfield of $\mathcal{U}$ has a solution in $(\mathcal{U}, \delta)$.

DCF₀

- Fix $(\mathcal{U}, \delta)$, a sufficiently saturated model of DCF₀.
 - ▶ Any consistent system of differential polynomial equations and in-equations with coefficients in a small differential subfield of $\mathcal{U}$ has a solution in $(\mathcal{U}, \delta)$.
 - ▶ Any isomorphism between small differential subfields of $(\mathcal{U}, \delta)$ extends to an automorphism of $(\mathcal{U}, \delta)$.

DCF₀

- Fix $(\mathcal{U}, \delta)$, a sufficiently saturated model of DCF₀.
 - ▶ Any consistent system of differential polynomial equations and in-equations with coefficients in a small differential subfield of $\mathcal{U}$ has a solution in $(\mathcal{U}, \delta)$.
 - ▶ Any isomorphism between small differential subfields of $(\mathcal{U}, \delta)$ extends to an automorphism of $(\mathcal{U}, \delta)$.
- The field of constants $\mathcal{C}$ is defined by $\delta(x) = 0$.

DCF₀

- Fix $(\mathcal{U}, \delta)$, a sufficiently saturated model of DCF₀.
 - ▶ Any consistent system of differential polynomial equations and in-equations with coefficients in a small differential subfield of $\mathcal{U}$ has a solution in $(\mathcal{U}, \delta)$.
 - ▶ Any isomorphism between small differential subfields of $(\mathcal{U}, \delta)$ extends to an automorphism of $(\mathcal{U}, \delta)$.
- The field of constants $\mathcal{C}$ is defined by $\delta(x) = 0$.
- The **generic type** $p \in S(F)$ of a system of differential equations (E) is the collection $\{\bar{x} \in E\} \cup \{\bar{x} \notin Z \mid Z \text{ proper Kolchin closed subset over } F\}$.

DCF₀

- Fix $(\mathcal{U}, \delta)$, a sufficiently saturated model of DCF₀.
 - ▶ Any consistent system of differential polynomial equations and in-equations with coefficients in a small differential subfield of $\mathcal{U}$ has a solution in $(\mathcal{U}, \delta)$.
 - ▶ Any isomorphism between small differential subfields of $(\mathcal{U}, \delta)$ extends to an automorphism of $(\mathcal{U}, \delta)$.
- The field of constants $\mathcal{C}$ is defined by $\delta(x) = 0$.
- The **generic type** $p \in S(F)$ of a system of differential equations (E) is the collection $\{\bar{x} \in E\} \cup \{\bar{x} \notin Z \mid Z \text{ proper Kolchin closed subset over } F\}$.
- We only consider **finite dimensional** types meaning that for some $\bar{a} \models p$ and $k \in \mathbb{N}$, $\delta^{k+1}(\bar{a}) \in F(\bar{a}, \delta(\bar{a}), \dots, \delta^k(\bar{a}))$.

DCF₀

- Fix $(\mathcal{U}, \delta)$, a sufficiently saturated model of DCF₀.
 - ▶ Any consistent system of differential polynomial equations and in-equations with coefficients in a small differential subfield of $\mathcal{U}$ has a solution in $(\mathcal{U}, \delta)$.
 - ▶ Any isomorphism between small differential subfields of $(\mathcal{U}, \delta)$ extends to an automorphism of $(\mathcal{U}, \delta)$.
- The field of constants $\mathcal{C}$ is defined by $\delta(x) = 0$.
- The **generic type** $p \in S(F)$ of a system of differential equations (E) is the collection $\{\bar{x} \in E\} \cup \{\bar{x} \notin Z \mid Z \text{ proper Kolchin closed subset over } F\}$.
- We only consider **finite dimensional** types meaning that for some $\bar{a} \models p$ and $k \in \mathbb{N}$, $\delta^{k+1}(\bar{a}) \in F(\bar{a}, \delta(\bar{a}), \dots, \delta^k(\bar{a}))$.

$$\dim(p) = \text{trdeg}(F\langle\bar{a}\rangle/F) = \text{trdeg}(F(\bar{a}, \delta(\bar{a}), \dots)/F).$$

DCF₀

- Fix $(\mathcal{U}, \delta)$, a sufficiently saturated model of DCF₀.
 - ▶ Any consistent system of differential polynomial equations and in-equations with coefficients in a small differential subfield of $\mathcal{U}$ has a solution in $(\mathcal{U}, \delta)$.
 - ▶ Any isomorphism between small differential subfields of $(\mathcal{U}, \delta)$ extends to an automorphism of $(\mathcal{U}, \delta)$.
- The field of constants $\mathcal{C}$ is defined by $\delta(x) = 0$.
- The **generic type** $p \in S(F)$ of a system of differential equations (E) is the collection $\{\bar{x} \in E\} \cup \{\bar{x} \notin Z \mid Z \text{ proper Kolchin closed subset over } F\}$.
- We only consider **finite dimensional** types meaning that for some $\bar{a} \models p$ and $k \in \mathbb{N}$, $\delta^{k+1}(\bar{a}) \in F(\bar{a}, \delta(\bar{a}), \dots, \delta^k(\bar{a}))$.

$$\dim(p) = \text{trdeg}(F\langle\bar{a}\rangle/F) = \text{trdeg}(F(\bar{a}, \delta(\bar{a}), \dots)/F).$$

- Let $p \in S(F)$, $F = F^{\text{alg}}$. The **non-forking extension** of p to $(L, \delta) \supseteq (F, \delta)$ is $\text{tp}(\bar{a}/L)$ for some $\bar{a} \models p$ with $\dim(\bar{a}/L) = \dim(\bar{a}/F)$.

Internality and analyzability

Let (F, δ) be an algebraically closed differential field and $p \in S(F)$.

p is almost $\mathcal{C}$ -**internal** if there is $(L, \delta) \supseteq (F, \delta)$ such that for every $\bar{a} \models p$, $\bar{a} \in (L(c_1, \dots, c_m))^{\text{alg}}$ for some $c_1, \dots, c_m \in \mathcal{C}$.

Internality and analyzability

Let (F, δ) be an algebraically closed differential field and $p \in S(F)$.

p is almost $\mathcal{C}$ -**internal** if there is $(L, \delta) \supseteq (F, \delta)$ such that for every $\bar{a} \models p$, $\bar{a} \in (L(c_1, \dots, c_m))^{\text{alg}}$ for some $c_1, \dots, c_m \in \mathcal{C}$.

$$(LV_{1,b,1,b}) = \begin{cases} \delta(x) = x(y + b) \\ \delta(y) = y(x + b) \end{cases} \quad \text{is not almost } \mathcal{C}\text{-internal,}$$

Internality and analyzability

Let (F, δ) be an algebraically closed differential field and $p \in S(F)$.

p is almost $\mathcal{C}$ -**internal** if there is $(L, \delta) \supseteq (F, \delta)$ such that for every $\bar{a} \models p$, $\bar{a} \in (L(c_1, \dots, c_m))^{\text{alg}}$ for some $c_1, \dots, c_m \in \mathcal{C}$.

$(LV_{1,b,1,b}) = \begin{cases} \delta(x) = x(y + b) \\ \delta(y) = y(x + b) \end{cases}$ is not almost $\mathcal{C}$ -internal, but there is a

$$f : \text{tp}(u, v/F) \rightarrow \text{tp}(u - v/F)$$

Internality and analyzability

Let (F, δ) be an algebraically closed differential field and $p \in S(F)$.

p is almost $\mathcal{C}$ -**internal** if there is $(L, \delta) \supseteq (F, \delta)$ such that for every $\bar{a} \models p$, $\bar{a} \in (L(c_1, \dots, c_m))^{\text{alg}}$ for some $c_1, \dots, c_m \in \mathcal{C}$.

$(LV_{1,b,1,b}) = \begin{cases} \delta(x) = x(y + b) \\ \delta(y) = y(x + b) \end{cases}$ is not almost $\mathcal{C}$ -internal, but there is a

$$f : \text{tp}(u, v/F) \rightarrow \text{tp}(u - v/F)$$

where $\text{tp}(u - v/F)$ and the fiber $\text{tp}(u, v/F(u - v))$ are almost $\mathcal{C}$ -internal [4].

Internality and analyzability

Let (F, δ) be an algebraically closed differential field and $p \in S(F)$.

- A **fibration** is a definable map $f : p \rightarrow q$ such that every fiber is stationary.

Internality and analyzability

Let (F, δ) be an algebraically closed differential field and $p \in S(F)$.

- A **fibration** is a definable map $f : p \rightarrow q$ such that every fiber is stationary.
- A type is stationary if there is a unique non-forking extension to any $L \supseteq F$.

Internality and analyzability

Let (F, δ) be an algebraically closed differential field and $p \in S(F)$.

- A **fibration** is a definable map $f : p \rightarrow q$ such that every fiber is stationary.
- A type is stationary if there is a unique non-forking extension to any $L \supseteq F$.
- Every type over an algebraically closed field is stationary.

Internality and analyzability

Let (F, δ) be an algebraically closed differential field and $p \in S(F)$.

- A **fibration** is a definable map $f : p \rightarrow q$ such that every fiber is stationary.
- A type is stationary if there is a unique non-forking extension to any $L \supseteq F$.
- Every type over an algebraically closed field is stationary.

$p \in S(F)$ is **$\mathcal{C}$ -analyzable** if there is a sequence of fibrations

$p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ such that

- $\dim(p_n) > \dim(p_{n-1}) > \cdots > \dim(p_1) > \dim(p_0) = 0$,

Internality and analyzability

Let (F, δ) be an algebraically closed differential field and $p \in S(F)$.

- A **fibration** is a definable map $f : p \rightarrow q$ such that every fiber is stationary.
- A type is stationary if there is a unique non-forking extension to any $L \supseteq F$.
- Every type over an algebraically closed field is stationary.

$p \in S(F)$ is **$\mathcal{C}$ -analyzable** if there is a sequence of fibrations

$p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ such that

- $\dim(p_n) > \dim(p_{n-1}) > \cdots > \dim(p_1) > \dim(p_0) = 0$,
- p_n is *interalgebraic* with p , and

Internality and analyzability

Let (F, δ) be an algebraically closed differential field and $p \in S(F)$.

- A **fibration** is a definable map $f : p \rightarrow q$ such that every fiber is stationary.
- A type is stationary if there is a unique non-forking extension to any $L \supseteq F$.
- Every type over an algebraically closed field is stationary.

$p \in S(F)$ is **$\mathcal{C}$ -analyzable** if there is a sequence of fibrations

$p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ such that

- $\dim(p_n) > \dim(p_{n-1}) > \cdots > \dim(p_1) > \dim(p_0) = 0$,
- p_n is *interalgebraic* with p , and
- for all $i = 1, \dots, n$, every fiber of f_i is almost internal to the constants.

Internality and analyzability

A **fibration** is a definable map $f : p \rightarrow q$ such that every fiber is stationary.

p is **$\mathcal{C}$ -analyzable** if there is a sequence of fibrations

$p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ such that

- $\dim(p_n) > \dim(p_{n-1}) > \cdots > \dim(p_1) > \dim(p_0) = 0$,
- p_n is *interalgebraic* with p , and
- for all $i = 1, \dots, n$, every fiber of f_i is almost internal to the constants.

① Let p be the generic type of $(E) = \begin{cases} \delta(x) = 0 \\ \delta(y) = xy \end{cases}$ over $F < \mathcal{C}$.

Internality and analyzability

A **fibration** is a definable map $f : p \rightarrow q$ such that every fiber is stationary.

p is **$\mathcal{C}$ -analyzable** if there is a sequence of fibrations

$p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ such that

- $\dim(p_n) > \dim(p_{n-1}) > \cdots > \dim(p_1) > \dim(p_0) = 0$,
- p_n is *interalgebraic* with p , and
- for all $i = 1, \dots, n$, every fiber of f_i is almost internal to the constants.

- ① Let p be the generic type of $(E) = \begin{cases} \delta(x) = 0 \\ \delta(y) = xy \end{cases}$ over $F < \mathcal{C}$. This is not almost $\mathcal{C}$ -internal, but is $\mathcal{C}$ -analyzable [2]:

Internality and analyzability

A **fibration** is a definable map $f : p \rightarrow q$ such that every fiber is stationary.

p is **$\mathcal{C}$ -analyzable** if there is a sequence of fibrations

$p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ such that

- $\dim(p_n) > \dim(p_{n-1}) > \cdots > \dim(p_1) > \dim(p_0) = 0$,
- p_n is *interalgebraic* with p , and
- for all $i = 1, \dots, n$, every fiber of f_i is almost internal to the constants.

① Let p be the generic type of $(E) = \begin{cases} \delta(x) = 0 \\ \delta(y) = xy \end{cases}$ over $F < \mathcal{C}$. This

is not almost $\mathcal{C}$ -internal, but is $\mathcal{C}$ -analyzable [2]:

$\text{tp}(u, v/F) \xrightarrow{\text{al. } \mathcal{C}\text{-int.}} \text{tp}(u/F) := \text{tp}(\frac{\delta(v)}{v}/F) \xrightarrow{\text{al. } \mathcal{C}\text{-int.}} \text{tp}(1/F)$ is a $\mathcal{C}$ -analysis.

Internality and analyzability

A **fibration** is a definable map $f : p \rightarrow q$ such that every fiber is stationary.

p is **$\mathcal{C}$ -analyzable** if there is a sequence of fibrations

$p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ such that

- $\dim(p_n) > \dim(p_{n-1}) > \cdots > \dim(p_1) > \dim(p_0) = 0$,
- p_n is *interalgebraic* with p , and
- for all $i = 1, \dots, n$, every fiber of f_i is almost internal to the constants.

- ① Let p be the generic type of $(E) = \begin{cases} \delta(x) = 0 \\ \delta(y) = xy \end{cases}$ over $F < \mathcal{C}$. This

is not almost $\mathcal{C}$ -internal, but is $\mathcal{C}$ -analyzable [2]:

$\text{tp}(u, v/F) \xrightarrow{\text{al. } \mathcal{C}\text{-int.}} \text{tp}(u/F) := \text{tp}(\frac{\delta(v)}{v}/F) \xrightarrow{\text{al. } \mathcal{C}\text{-int.}} \text{tp}(1/F)$ is a $\mathcal{C}$ -analysis.

- ② The generic type of $\delta(x) = x^3(x - 1)$ is **not** $\mathcal{C}$ -analyzable.

Internality and analyzability

A **fibration** is a definable map $f : p \rightarrow q$ such that every fiber is stationary.

p is **$\mathcal{C}$ -analyzable** if there is a sequence of fibrations

$p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ such that

- $\dim(p_n) > \dim(p_{n-1}) > \cdots > \dim(p_1) > \dim(p_0) = 0$,
- p_n is *interalgebraic* with p , and
- for all $i = 1, \dots, n$, every fiber of f_i is almost internal to the constants.

- ① Let p be the generic type of $(E) = \begin{cases} \delta(x) = 0 \\ \delta(y) = xy \end{cases}$ over $F < \mathcal{C}$. This

is not almost $\mathcal{C}$ -internal, but is $\mathcal{C}$ -analyzable [2]:

$\text{tp}(u, v/F) \xrightarrow{\text{al. } \mathcal{C}\text{-int.}} \text{tp}(u/F) := \text{tp}(\frac{\delta(v)}{v}/F) \xrightarrow{\text{al. } \mathcal{C}\text{-int.}} \text{tp}(1/F)$ is a $\mathcal{C}$ -analysis.

- ② The generic type of $\delta(x) = x^3(x - 1)$ is **not** $\mathcal{C}$ -analyzable.
- ▶ however this type is “simple”

Minimal types

Let (F, δ) be algebraically closed. A type is **minimal** if it has a unique non-algebraic extension to any differential field extension.

Minimal types

Let (F, δ) be algebraically closed. A type is **minimal** if it has a unique non-algebraic extension to any differential field extension.

- Every 1-dimensional type is minimal.

Minimal types

Let (F, δ) be algebraically closed. A type is **minimal** if it has a unique non-algebraic extension to any differential field extension.

- Every 1-dimensional type is minimal.
- The generic type of the constants is minimal.

Minimal types

Let (F, δ) be algebraically closed. A type is **minimal** if it has a unique non-algebraic extension to any differential field extension.

- Every 1-dimensional type is minimal.
- The generic type of the constants is minimal.
- Not every minimal type is 1-dimensional.

Minimal types

Let (F, δ) be algebraically closed. A type is **minimal** if it has a unique non-algebraic extension to any differential field extension.

- Every 1-dimensional type is minimal.
 - The generic type of the constants is minimal.
 - Not every minimal type is 1-dimensional.
-
- p is **almost internal** to a minimal type r if there is $(L, \delta) \supset (F, \delta)$ such that for all $\bar{a} \models p$, $\bar{a} \in (L\langle c_1, \dots, c_m \rangle)^{\text{alg}}$ for some $c_1, \dots, c_m \models r$.

Minimal types

Let (F, δ) be algebraically closed. A type is **minimal** if it has a unique non-algebraic extension to any differential field extension.

- Every 1-dimensional type is minimal.
 - The generic type of the constants is minimal.
 - Not every minimal type is 1-dimensional.
-
- p is **almost internal** to a minimal type r if there is $(L, \delta) \supset (F, \delta)$ such that for all $\bar{a} \models p$, $\bar{a} \in (L\langle c_1, \dots, c_m \rangle)^{\text{alg}}$ for some $c_1, \dots, c_m \models r$.
 - A **semiminimal analysis** for p is a sequence of fibrations $p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_1} p_0$ such that
 - ▶ $\dim(p_n) > \dim(p_{n-1}) > \dots > \dim(p_1) > \dim(p_0) = 0$,
 - ▶ p_n is *interalgebraic* with p , and
 - ▶ for all $i = 1, \dots, n$, every fiber of f_i is almost internal to a minimal type.

Semiminimal analyses

A **semiminimal analysis** for p is a sequence of definable fibrations $p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ where every fiber of f_i is almost internal to a minimal type.

While not every type is almost internal to a minimal type, every finite rank type has a semiminimal analysis.

Semiminimal analyses

A **semiminimal analysis** for p is a sequence of definable fibrations $p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ where every fiber of f_i is almost internal to a minimal type.

While not every type is almost internal to a minimal type, every finite rank type has a semiminimal analysis.

Semiminimal analyses

A **semiminimal analysis** for p is a sequence of definable fibrations $p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ where every fiber of f_i is almost internal to a minimal type.

While not every type is almost internal to a minimal type, every finite rank type has a semiminimal analysis.

The generic type over $F < \mathcal{C}$, p , of $\begin{cases} \delta(x) = x^3(x-1) \\ \delta(y) = xy \end{cases}$ is not $\mathcal{C}$ -analyzable [7].

Semiminimal analyses

A **semiminimal analysis** for p is a sequence of definable fibrations $p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ where every fiber of f_i is almost internal to a minimal type.

While not every type is almost internal to a minimal type, every finite rank type has a semiminimal analysis.

The generic type over $F < \mathcal{C}$, p , of $\begin{cases} \delta(x) = x^3(x-1) \\ \delta(y) = xy \end{cases}$ is not $\mathcal{C}$ -analyzable [7].

- Let q be the generic type of $\delta(x) = x^3(x-1)$. This type is minimal and not almost $\mathcal{C}$ -internal.

Semiminimal analyses

A **semiminimal analysis** for p is a sequence of definable fibrations $p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ where every fiber of f_i is almost internal to a minimal type.

While not every type is almost internal to a minimal type, every finite rank type has a semiminimal analysis.

The generic type over $F < \mathcal{C}$, p , of $\begin{cases} \delta(x) = x^3(x-1) \\ \delta(y) = xy \end{cases}$ is not $\mathcal{C}$ -analyzable [7].

- Let q be the generic type of $\delta(x) = x^3(x-1)$. This type is minimal and not almost $\mathcal{C}$ -internal.
- $p \xrightarrow[\text{al. } \mathcal{C}\text{-int.}]{\pi_x} q \xrightarrow[\text{al. } q\text{-int.}]{} \text{tp}(1/F)$ is a semiminimal analysis for p .

Distinct semiminimal analyses

- Let $r, s \in S(F)$ be minimal types which are not interalgebraic over any $L \supseteq F$ (i.e. orthogonal).

Set $r \otimes s = \text{tp}(ab/F)$ for some $a \models r$ and $b \models s$.

- We have two distinct semiminimal analyses of $r \otimes s$.

$$r \otimes s \xrightarrow[\text{al } s\text{-int}]{(a,b) \mapsto a} r \xrightarrow[\text{al } r\text{-int}]{} \text{tp}(1/F)$$

and

$$r \otimes s \xrightarrow[\text{al } r\text{-int}]{(a,b) \mapsto b} s \xrightarrow[\text{al } s\text{-int}]{} \text{tp}(1/F).$$

This can also happen when p is not a product.

Semiminimal analyses

A **semiminimal analysis** for p is a sequence of definable fibrations $p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ where every fiber of f_i is almost internal to a minimal type.

In the general totally transcendental setting we explore these semiminimal analyses.

Understanding what invariants of a finite rank type are captured by a semiminimal analysis.

Semiminimal analyses

A **semiminimal analysis** for p is a sequence of definable fibrations $p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ where every fiber of f_i is almost internal to a minimal type.

In the general totally transcendental setting we explore these semiminimal analyses.

Understanding what invariants of a finite rank type are captured by a semiminimal analysis.

- The multiplicity with which a minimal type appears in a semiminimal analysis p , does not depend on the choice of analysis.

Working now in the general setting, I will replace $\dim$ with U , the Lascar rank.

Semiminimal analyses

A **semiminimal analysis** for p is a sequence of definable fibrations $p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ where every fiber of f_i is almost internal to a minimal type.

In the general totally transcendental setting we explore these semiminimal analyses.

Understanding what invariants of a finite rank type are captured by a semiminimal analysis.

- The multiplicity with which a minimal type appears in a semiminimal analysis p , does not depend on the choice of analysis.

Working now in the general setting, I will replace $\dim$ with U , the Lascar rank. In DCF_0 , for p, q finite dimensional types,

- $U(p) \leq \dim(q)$,
- $\dim(p) \leq \dim(q)$ if and only if $U(p) \leq U(q)$.

Semiminimal analyses

A **semiminimal analysis** for p is a sequence of definable fibrations $p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ where every fiber of f_i is almost internal to a minimal type.

In the general totally transcendental setting we explore these semiminimal analyses.

Working now in the general setting, I will replace $\dim$ with U , the Lascar rank.

Semiminimal analyses

A **semiminimal analysis** for p is a sequence of definable fibrations $p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ where every fiber of f_i is almost internal to a minimal type.

In the general totally transcendental setting we explore these semiminimal analyses.

Working now in the general setting, I will replace $\dim$ with U , the Lascar rank. In DCF_0 , for p, q finite dimensional types,

- $U(p) \leq \dim(q)$,
- $\dim(p) \leq \dim(q)$ if and only if $U(p) \leq U(q)$.

While p being minimal did not imply $\dim(p) = 1$, p is minimal if and only if $U(p) = 1$ and p is stationary.

Invariants of the type p

We will say a minimal type r **appears** in a semiminimal analysis

$p_n \xrightarrow{f_n} p_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_0$ if for some $i = 1, \dots, n$, some fiber of $f_i : p_i \rightarrow p_{i-1}$ is almost r -internal.

Fact

Let p be stationary and finite rank. Let r be a minimal type. Then r appears in some semiminimal analysis of p if and only if it appears in every semiminimal analysis.

That is, the minimal type appearing in a semiminimal analysis is an invariant of the type p .

Theorem

Let p be stationary and finite rank. Let r be a minimal type. The “multiplicity” with which r appears in a given semiminimal analysis is an invariant of p and r , independent of the choice of semiminimal analysis.

Multiplicity of r in p

Theorem (Technical version)

Let p be a stationary type of finite U -rank and $\mathbf{f} = (f_i : p_i \rightarrow p_{i-1})_{i=1,\dots,n}$ be a semiminimal analysis of p . For any minimal type r , let

Multiplicity of r in p

Theorem (Technical version)

Let p be a stationary type of finite U -rank and $\mathbf{f} = (f_i : p_i \rightarrow p_{i-1})_{i=1,\dots,n}$ be a semiminimal analysis of p . For any minimal type r , let

$$I_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$$

and let $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$.

Multiplicity of r in p

Theorem (Technical version)

Let p be a stationary type of finite U -rank and $\mathbf{f} = (f_i : p_i \rightarrow p_{i-1})_{i=1,\dots,n}$ be a semiminimal analysis of p . For any minimal type r , let

$$I_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$$

and let $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$. $\leftarrow$ size of the fibers where r appears

Multiplicity of r in p

Theorem (Technical version)

Let p be a stationary type of finite U -rank and $\mathbf{f} = (f_i : p_i \rightarrow p_{i-1})_{i=1,\dots,n}$ be a semiminimal analysis of p . For any minimal type r , let

$$I_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$$

and let $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$. $\leftarrow$ size of the fibers where r appears

Then m_r is an invariant of p and r , independent of the semiminimal analysis $\mathbf{f}$.

Multiplicity of r in p

Theorem (Technical version)

Let p be a stationary type of finite U -rank and $\mathbf{f} = (f_i : p_i \rightarrow p_{i-1})_{i=1,\dots,n}$ be a semiminimal analysis of p . For any minimal type r , let

$$I_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$$

and let $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$. $\leftarrow$ size of the fibers where r appears

Then m_r is an invariant of p and r , independent of the semiminimal analysis $\mathbf{f}$.

We call m_r the **multiplicity of r in $\mathbf{f}$** .

Multiplicity of r in p

Theorem (Technical version)

Let p be a stationary type of finite U -rank and $\mathbf{f} = (f_i : p_i \rightarrow p_{i-1})_{i=1,\dots,n}$ be a semiminimal analysis of p . For any minimal type r , let

$$I_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$$

and let $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$. $\leftarrow$ size of the fibers where r appears

Then m_r is an invariant of p and r , independent of the semiminimal analysis $\mathbf{f}$.

We call m_r the **multiplicity of r in $\mathbf{f}$** .

In fact, we can give an intrinsic characterization of multiplicity (which gets even more technical).

Multiplicity of r in p

- $I_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$
- $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$.

① If p is $\mathcal{C}$ -analyzable, then $p_n \xrightarrow[\text{al. } \mathcal{C}\text{-int}]{f_n} p_{n-1} \xrightarrow[\text{al. } \mathcal{C}\text{-int}]{} \cdots \xrightarrow[\text{al. } \mathcal{C}\text{-int}]{f_1} p_0$,

Multiplicity of r in p

- $I_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$
- $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$.

- ① If p is $\mathcal{C}$ -analyzable, then $p_n \xrightarrow[\text{al. } \mathcal{C}\text{-int}]{f_n} p_{n-1} \xrightarrow[\text{al. } \mathcal{C}\text{-int}]{} \cdots \xrightarrow[\text{al. } \mathcal{C}\text{-int}]{f_1} p_0$,
 $I_{\mathcal{C}} = \{1, \dots, n\}$

Multiplicity of r in p

- $I_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$
- $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$.

- ① If p is $\mathcal{C}$ -analyzable, then $p_n \xrightarrow[\text{al. } \mathcal{C}\text{-int}]{f_n} p_{n-1} \xrightarrow[\text{al. } \mathcal{C}\text{-int}]{} \cdots \xrightarrow[\text{al. } \mathcal{C}\text{-int}]{f_1} p_0$,
 $I_{\mathcal{C}} = \{1, \dots, n\}$ and
$$m_{\mathcal{C}} = \sum_{i=1}^n U(p_i) - U(p_{i-1}) = U(p_n) - U(p_0) = U(p).$$

Multiplicity of r in p

- $I_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$
- $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$.

- ① If p is $\mathcal{C}$ -analyzable, then $p_n \xrightarrow[\text{al. } \mathcal{C}\text{-int}]{f_n} p_{n-1} \xrightarrow[\text{al. } \mathcal{C}\text{-int}]{} \cdots \xrightarrow[\text{al. } \mathcal{C}\text{-int}]{f_1} p_0$,

$I_{\mathcal{C}} = \{1, \dots, n\}$ and

$$m_{\mathcal{C}} = \sum_{i=1}^n U(p_i) - U(p_{i-1}) = U(p_n) - U(p_0) = U(p).$$

- ② Let q be the generic type of $\delta(x) = x^3(x-1)$ and

p the generic type of $\begin{cases} \delta(x) = x^3(x-1) \\ \delta(y) = yx \end{cases}$ over $F < \mathcal{C}$. We have the

semiminimal analysis

Multiplicity of r in p

- $I_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$
- $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$.

① If p is $\mathcal{C}$ -analyzable, then $p_n \xrightarrow[\text{al. } \mathcal{C}\text{-int.}]{f_n} p_{n-1} \xrightarrow[\text{al. } \mathcal{C}\text{-int.}]{} \cdots \xrightarrow[\text{al. } \mathcal{C}\text{-int.}]{f_1} p_0$,

$I_{\mathcal{C}} = \{1, \dots, n\}$ and

$$m_{\mathcal{C}} = \sum_{i=1}^n U(p_i) - U(p_{i-1}) = U(p_n) - U(p_0) = U(p).$$

② Let q be the generic type of $\delta(x) = x^3(x-1)$ and

p the generic type of $\begin{cases} \delta(x) = x^3(x-1) \\ \delta(y) = yx \end{cases}$ over $F < \mathcal{C}$. We have the

semiminimal analysis $p \xrightarrow[\text{al. } \mathcal{C}\text{-int.}]{\pi_x} q \xrightarrow[\text{al. } q\text{-int.}]{} \text{tp}(1/F)$.

Multiplicity of r in p

- $I_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$
- $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$.

① If p is $\mathcal{C}$ -analyzable, then $p_n \xrightarrow[\text{al. } \mathcal{C}\text{-int.}]{f_n} p_{n-1} \xrightarrow[\text{al. } \mathcal{C}\text{-int.}]{} \cdots \xrightarrow[\text{al. } \mathcal{C}\text{-int.}]{f_1} p_0$,

$I_{\mathcal{C}} = \{1, \dots, n\}$ and

$$m_{\mathcal{C}} = \sum_{i=1}^n U(p_i) - U(p_{i-1}) = U(p_n) - U(p_0) = U(p).$$

② Let q be the generic type of $\delta(x) = x^3(x-1)$ and

p the generic type of $\begin{cases} \delta(x) = x^3(x-1) \\ \delta(y) = yx \end{cases}$ over $F < \mathcal{C}$. We have the

semiminimal analysis $p \xrightarrow[\text{al. } \mathcal{C}\text{-int.}]{\pi_x} q \xrightarrow[\text{al. } q\text{-int.}]{} \text{tp}(1/F)$.

Then $I_{\mathcal{C}} = \{2\}$ and $I_q = \{1\}$.

Multiplicity of r in p

- $I_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$
- $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$.

① If p is $\mathcal{C}$ -analyzable, then $p_n \xrightarrow[\text{al. } \mathcal{C}\text{-int.}]{f_n} p_{n-1} \xrightarrow[\text{al. } \mathcal{C}\text{-int.}]{} \cdots \xrightarrow[\text{al. } \mathcal{C}\text{-int.}]{f_1} p_0$,

$I_{\mathcal{C}} = \{1, \dots, n\}$ and

$$m_{\mathcal{C}} = \sum_{i=1}^n U(p_i) - U(p_{i-1}) = U(p_n) - U(p_0) = U(p).$$

② Let q be the generic type of $\delta(x) = x^3(x-1)$ and

p the generic type of $\begin{cases} \delta(x) = x^3(x-1) \\ \delta(y) = yx \end{cases}$ over $F < \mathcal{C}$. We have the

semiminimal analysis $p \xrightarrow[\text{al. } \mathcal{C}\text{-int.}]{\pi_x} q \xrightarrow[\text{al. } q\text{-int.}]{} \text{tp}(1/F)$.

Then $I_{\mathcal{C}} = \{2\}$ and $I_q = \{1\}$. So $m_{\mathcal{C}} = U(p) - U(q) = 1$ and $m_q = U(q) - U(1/F) = U(q) = 1$.

Multiplicity of r in p

- $I_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$
- $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$.

① If p is $\mathcal{C}$ -analyzable, then $p_n \xrightarrow[\text{al. } \mathcal{C}\text{-int.}]{f_n} p_{n-1} \xrightarrow[\text{al. } \mathcal{C}\text{-int.}]{} \cdots \xrightarrow[\text{al. } \mathcal{C}\text{-int.}]{f_1} p_0$,

$I_{\mathcal{C}} = \{1, \dots, n\}$ and

$$m_{\mathcal{C}} = \sum_{i=1}^n U(p_i) - U(p_{i-1}) = U(p_n) - U(p_0) = U(p).$$

② Let q be the generic type of $\delta(x) = x^3(x-1)$ and

p the generic type of $\begin{cases} \delta(x) = x^3(x-1) \\ \delta(y) = yx \end{cases}$ over $F < \mathcal{C}$. We have the

semiminimal analysis $p \xrightarrow[\text{al. } \mathcal{C}\text{-int.}]{\pi_x} q \xrightarrow[\text{al. } q\text{-int.}]{} \text{tp}(1/F)$.

Then $I_{\mathcal{C}} = \{2\}$ and $I_q = \{1\}$. So $m_{\mathcal{C}} = U(p) - U(q) = 1$ and $m_q = U(q) - U(1/F) = U(q) = 1$.

③ $m_r > 0$ if and only if r appears in some (any) semiminimal analysis.

Multiplicity of r in p

- $I_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$
- $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$.

Let $r, s \in S(F)$ be orthogonal minimal types.

- 1 For the analysis $r \otimes s \xrightarrow[\text{al } s\text{-int}]{(a,b) \mapsto a} r \xrightarrow[\text{al } r\text{-int}]{} \text{tp}(1/F)$

Multiplicity of r in p

- $I_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$
- $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$.

Let $r, s \in S(F)$ be orthogonal minimal types.

- 1 For the analysis $r \otimes s \xrightarrow[\text{al } s\text{-int}]{(a,b) \mapsto a} r \xrightarrow[\text{al } r\text{-int}]{} \text{tp}(1/F)$
 - $I_r = \{1\}$ and $I_s = \{2\}$.

Multiplicity of r in p

- $I_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$
- $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$.

Let $r, s \in S(F)$ be orthogonal minimal types.

- 1 For the analysis $r \otimes s \xrightarrow[\text{al } s\text{-int}]{(a,b) \mapsto a} r \xrightarrow[\text{al } r\text{-int}]{} \text{tp}(1/F)$
 - ▶ $I_r = \{1\}$ and $I_s = \{2\}$.
 - ▶ $m_r = U(r) - U(1/F) = 1$ and $m_s = U(r \otimes s) - U(r) = 1$.

Multiplicity of r in p

- $I_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$
- $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$.

Let $r, s \in S(F)$ be orthogonal minimal types.

- 1 For the analysis $r \otimes s \xrightarrow[\text{al } s\text{-int}]{(a,b) \mapsto a} r \xrightarrow[\text{al } r\text{-int}]{} \text{tp}(1/F)$
 - ▶ $I_r = \{1\}$ and $I_s = \{2\}$.
 - ▶ $m_r = U(r) - U(1/F) = 1$ and $m_s = U(r \otimes s) - U(r) = 1$.
- 2 For the analysis $r \otimes s \xrightarrow[\text{al } r\text{-int}]{(a,b) \mapsto b} s \xrightarrow[\text{al } s\text{-int}]{} \text{tp}(1/F)$

Multiplicity of r in p

- $I_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$
- $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$.

Let $r, s \in S(F)$ be orthogonal minimal types.

- 1 For the analysis $r \otimes s \xrightarrow[\text{al } s\text{-int}]{(a,b) \mapsto a} r \xrightarrow[\text{al } r\text{-int}]{} \text{tp}(1/F)$
 - ▶ $I_r = \{1\}$ and $I_s = \{2\}$.
 - ▶ $m_r = U(r) - U(1/F) = 1$ and $m_s = U(r \otimes s) - U(r) = 1$.
- 2 For the analysis $r \otimes s \xrightarrow[\text{al } r\text{-int}]{(a,b) \mapsto b} s \xrightarrow[\text{al } s\text{-int}]{} \text{tp}(1/F)$
 - ▶ $I_r = \{2\}$ and $I_s = \{1\}$.

Multiplicity of r in p

- $l_r := \{1 \leq i \leq n \mid \text{some fiber of } f_i : p_i \rightarrow p_{i-1} \text{ is almost } r\text{-internal}\}$
- $m_r := \sum_{i \in l_r} U(p_i) - U(p_{i-1})$.

Let $r, s \in S(F)$ be orthogonal minimal types.

- 1 For the analysis $r \otimes s \xrightarrow[\text{al } s\text{-int}]{(a,b) \mapsto a} r \xrightarrow[\text{al } r\text{-int}]{} \text{tp}(1/F)$
 - ▶ $l_r = \{1\}$ and $l_s = \{2\}$.
 - ▶ $m_r = U(r) - U(1/F) = 1$ and $m_s = U(r \otimes s) - U(r) = 1$.
- 2 For the analysis $r \otimes s \xrightarrow[\text{al } r\text{-int}]{(a,b) \mapsto b} s \xrightarrow[\text{al } s\text{-int}]{} \text{tp}(1/F)$
 - ▶ $l_r = \{2\}$ and $l_s = \{1\}$.
 - ▶ $m_r = U(r \otimes s) - U(s) = 1$ and $m_s = U(s) - U(1/F) = 1$.

Intrinsic characterization of multiplicity

- The idea is to take specific extensions of p and sum the rank of the parts of these extensions which are almost r -internal.
- First, what is the part which is almost r -internal of a type?

Intrinsic characterization of multiplicity

- The idea is to take specific extensions of p and sum the rank of the parts of these extensions which are almost r -internal.
- First, what is the part which is almost r -internal of a type? Image of maximal rank which is almost r -internal.

Intrinsic characterization of multiplicity

- The idea is to take specific extensions of p and sum the rank of the parts of these extensions which are almost r -internal.
- First, what is the part which is almost r -internal of a type? Image of maximal rank which is almost r -internal.

Definition (r -reductions)

Let $p = \text{tp}(a/A)$ be stationary. Let r be a minimal type.

- An **r -reduction** of a over A is a tuple $b \in \text{dcl}(Aa)$ such that:
 - 1 $\text{tp}(b/A)$ is almost r -internal, and
 - 2 if $c \in \text{dcl}(Aa)$ with $\text{tp}(c/A)$ almost r -internal, then $c \in \text{acl}(Ab)$.

Intrinsic characterization of multiplicity

- The idea is to take specific extensions of p and sum the rank of the parts of these extensions which are almost r -internal.
- First, what is the part which is almost r -internal of a type? Image of maximal rank which is almost r -internal.

Definition (r -reductions)

Let $p = \text{tp}(a/A)$ be stationary. Let r be a minimal type.

- An **r -reduction** of a over A is a tuple $b \in \text{dcl}(Aa)$ such that:
 - 1 $\text{tp}(b/A)$ is almost r -internal, and
 - 2 if $c \in \text{dcl}(Aa)$ with $\text{tp}(c/A)$ almost r -internal, then $c \in \text{acl}(Ab)$.
- An r -reduction of p is $\text{tp}(b/A)$ where b is an r -reduction of a over A .

Intrinsic characterization of multiplicity

- The idea is to take specific extensions of p and sum the rank of the parts of these extensions which are almost r -internal.
- First, what is the part which is almost r -internal of a type? Image of maximal rank which is almost r -internal.

Definition (r -reductions)

Let $p = \text{tp}(a/A)$ be stationary. Let r be a minimal type.

- An **r -reduction** of a over A is a tuple $b \in \text{dcl}(Aa)$ such that:
 - ① $\text{tp}(b/A)$ is almost r -internal, and
 - ② if $c \in \text{dcl}(Aa)$ with $\text{tp}(c/A)$ almost r -internal, then $c \in \text{acl}(Ab)$.
- An r -reduction of p is $\text{tp}(b/A)$ where b is an r -reduction of a over A .

Let q be the generic type of $\delta(x) = x^3(x - 1)$ and

p the generic type of $\begin{cases} \delta(y) = yx \\ \delta(x) = x^3(x - 1) \end{cases}$ over a field of constants F .

- A q -reduction of p is q itself.

Intrinsic characterization of multiplicity

- The idea is to take specific extensions of p and sum the rank of the parts of these extensions which are almost r -internal.
- First, what is the part which is almost r -internal of a type? Image of maximal rank which is almost r -internal.

Definition (r -reductions)

Let $p = \text{tp}(a/A)$ be stationary. Let r be a minimal type.

- An **r -reduction** of a over A is a tuple $b \in \text{dcl}(Aa)$ such that:
 - ① $\text{tp}(b/A)$ is almost r -internal, and
 - ② if $c \in \text{dcl}(Aa)$ with $\text{tp}(c/A)$ almost r -internal, then $c \in \text{acl}(Ab)$.
- An r -reduction of p is $\text{tp}(b/A)$ where b is an r -reduction of a over A .

Let q be the generic type of $\delta(x) = x^3(x - 1)$ and

p the generic type of $\begin{cases} \delta(y) = yx \\ \delta(x) = x^3(x - 1) \end{cases}$ over a field of constants F .

- A q -reduction of p is q itself.
- A $\mathcal{C}$ -reduction of q is $\text{tp}(1/F)$.

Intrinsic characterization of multiplicity

Definition

r -reductions Let $p = \text{tp}(a/A)$ be stationary. Let r be a minimal type.

- An r -**reduction** of a over A is a tuple $b \in \text{dcl}(Aa)$ such that:
 - 1 $\text{tp}(b/A)$ is almost r -internal, and
 - 2 if $c \in \text{dcl}(Aa)$ with $\text{tp}(c/A)$ almost r -internal, then $c \in \text{acl}(Ab)$.
- An r -reduction of p is $\text{tp}(b/A)$ where b is an r -reduction of a over A .

Let q be the generic type of $\delta(x) = x^3(x - 1)$ and

p the generic type of $\begin{cases} \delta(x) = x^3(x - 1) \\ \delta(y) = yx \end{cases}$.

- A q -reduction of p is q itself.
- A $\mathcal{C}$ -reduction of q is $\text{tp}(1/F)$.
- r -reductions of types are unique up to interalgebraicity and A -conjugacy $\rightarrow$ the rank of r -reductions of p are all the same

Intrinsic characterization of multiplicity

- The idea is to take specific extensions of p and sum the rank of the parts of these extensions which are almost r -internal.
- Which extensions do we consider?

Intrinsic characterization of multiplicity

- The idea is to take specific extensions of p and sum the rank of the parts of these extensions which are almost r -internal.
- Which extensions do we consider? Extensions over level sets.

Intrinsic characterization of multiplicity

- The idea is to take specific extensions of p and sum the rank of the parts of these extensions which are almost r -internal.
- Which extensions do we consider? Extensions over level sets.
- We only want to consider stationary types; a type over $\text{acl}(A)$ is always stationary.

Intrinsic characterization of multiplicity

- The idea is to take specific extensions of p and sum the rank of the parts of these extensions which are almost r -internal.
- Which extensions do we consider? Extensions over level sets.
- We only want to consider stationary types; a type over $\text{acl}(A)$ is always stationary.

Definition (Levels)

- The zeroth level of a over A is $\ell_{0,A}(a) = A$.
- The first level of a over A is $\ell_{1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(A)) \text{ is almost internal to a minimal type}\}$.

Intrinsic characterization of multiplicity

- The idea is to take specific extensions of p and sum the rank of the parts of these extensions which are almost r -internal.
- Which extensions do we consider? Extensions over level sets.
- We only want to consider stationary types; a type over $\text{acl}(A)$ is always stationary.

Definition (Levels)

- The zeroth level of a over A is $\ell_{0,A}(a) = A$.
- The first level of a over A is $\ell_{1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(A)) \text{ is almost internal to a minimal type}\}$. $\leftarrow$ this includes all $b \in \text{acl}(A)$
- The $n+1$ th level of a over A is $\ell_{n+1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(\ell_{n,A}(a))) \text{ is almost internal to a minimal type}\}$.

We say $\text{tp}(a/A)$ has m -levels if m is the least k such that $a \in \text{acl}(\ell_{k,A}(a))$.

Intrinsic characterization of multiplicity

- The idea is to take specific extensions of p and sum the rank of the parts of these extensions which are almost r -internal.
- Which extensions do we consider? Extensions over level sets.
- We only want to consider stationary types; a type over $\text{acl}(A)$ is always stationary.

Definition (Levels)

- The zeroth level of a over A is $\ell_{0,A}(a) = A$.
- The first level of a over A is $\ell_{1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(A)) \text{ is almost internal to a minimal type}\}$. $\leftarrow$ this includes all $b \in \text{acl}(A)$
- The $n+1$ th level of a over A is $\ell_{n+1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(\ell_{n,A}(a))) \text{ is almost internal to a minimal type}\}$.

We say $\text{tp}(a/A)$ has m -levels if m is the least k such that $a \in \text{acl}(\ell_k(a))$.

- If $\text{tp}(a/A)$ is almost internal to a minimal type, then $a \in \ell_{1,A}(a)$ and so $\text{tp}(a/A)$ has 1 level.

Intrinsic characterization of multiplicity

Definition (Levels)

- The zeroth level of a over A is $\ell_{0,A}(a) = A$.
- The first level of a over A is $\ell_{1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(A)) \text{ is almost internal to a minimal type}\}$. ← this includes all $b \in \text{acl}(A)$
- The $n + 1$ th level of a over A is $\ell_{n+1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(\ell_{n,A}(a))) \text{ is almost internal to a minimal type}\}$.

We say $\text{tp}(a/A)$ has m -levels if m is the least k such that $a \in \text{acl}(\ell_{k,A}(a))$.

- If $\text{tp}(a/A)$ is non-algebraic and almost internal to a minimal type, then $a \in \text{acl}(\ell_{1,A}(a))$, and so $\text{tp}(a/A)$ has 1 level.
- This is not an if and only if!

Intrinsic characterization of multiplicity

Definition (Levels)

- The zeroth level of a over A is $\ell_{0,A}(a) = A$.
- The first level of a over A is $\ell_{1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(A)) \text{ is almost internal to a minimal type}\}$. ← this includes all $b \in \text{acl}(A)$
- The $n + 1$ th level of a over A is $\ell_{n+1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(\ell_{n,A}(a))) \text{ is almost internal to a minimal type}\}$.

We say $\text{tp}(a/A)$ has m -levels if m is the least k such that $a \in \text{acl}(\ell_{k,A}(a))$.

- If $\text{tp}(a/A)$ is non-algebraic and almost internal to a minimal type, then $a \in \text{acl}(\ell_{1,A}(a))$, and so $\text{tp}(a/A)$ has 1 level.
- This is not an if and only if!
- Let r, s be orthogonal minimal types. Then for any $(a, b) \models r \otimes s$,

Intrinsic characterization of multiplicity

Definition (Levels)

- The zeroth level of a over A is $\ell_{0,A}(a) = A$.
- The first level of a over A is $\ell_{1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(A)) \text{ is almost internal to a minimal type}\}$. ← this includes all $b \in \text{acl}(A)$
- The $n + 1$ th level of a over A is $\ell_{n+1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(\ell_{n,A}(a))) \text{ is almost internal to a minimal type}\}$.

We say $\text{tp}(a/A)$ has m -levels if m is the least k such that $a \in \text{acl}(\ell_{k,A}(a))$.

- If $\text{tp}(a/A)$ is non-algebraic and almost internal to a minimal type, then $a \in \text{acl}(\ell_{1,A}(a))$, and so $\text{tp}(a/A)$ has 1 level.
- This is not an if and only if!
- Let r, s be orthogonal minimal types. Then for any $(a, b) \models r \otimes s$, $a, b \in \text{acl}(\ell_{1,A}((a, b))) \implies (a, b) \in \text{acl}(\ell_{1,A}((a, b)))$.

Intrinsic characterization of multiplicity

Definition (Levels)

- The zeroth level of a over A is $\ell_{0,A}(a) = A$.
- The first level of a over A is $\ell_{1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(A)) \text{ is almost internal to a minimal type}\}$. ← this includes all $b \in \text{acl}(A)$
- The $n + 1$ th level of a over A is $\ell_{n+1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(\ell_{n,A}(a))) \text{ is almost internal to a minimal type}\}$.

We say $\text{tp}(a/A)$ has m -levels if m is the least k such that $a \in \text{acl}(\ell_{k,A}(a))$.

- If $\text{tp}(a/A)$ is non-algebraic and almost internal to a minimal type, then $a \in \text{acl}(\ell_{1,A}(a))$, and so $\text{tp}(a/A)$ has 1 level.
- This is not an if and only if!
- Let r, s be orthogonal minimal types. Then for any $(a, b) \models r \otimes s$, $a, b \in \text{acl}(\ell_{1,A}((a, b))) \implies (a, b) \in \text{acl}(\ell_{1,A}((a, b)))$. So $r \otimes s$ is not almost internal to a minimal type, but it has 1 level.

Intrinsic characterization of multiplicity

- $\ell_{0,A}(a) = A$.
- $\ell_{1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(A)) \text{ al. int. to a minimal type}\}.$
- $\ell_{n+1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(\ell_{n,A}(a))) \text{ al. int. to a minimal type}\}.$

We say $\text{tp}(a/A)$ has m -levels if m is the least k such that $a \in \text{acl}(\ell_{k,A}(a))$.

Let q be the generic type of $\delta(x) = x^3(x - 1)$ and

p the generic type of $\begin{cases} \delta(x) = x^3(x - 1) \\ \delta(y) = yx \end{cases}.$

- A q -reduction of p is q itself, a $\mathcal{C}$ -reduction of q is $\text{tp}(1/F)$.

Intrinsic characterization of multiplicity

- $\ell_{0,A}(a) = A$.
- $\ell_{1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(A)) \text{ al. int. to a minimal type}\}$.
- $\ell_{n+1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(\ell_{n,A}(a))) \text{ al. int. to a minimal type}\}$.

We say $\text{tp}(a/A)$ has m -levels if m is the least k such that $a \in \text{acl}(\ell_{k,A}(a))$.

Let q be the generic type of $\delta(x) = x^3(x-1)$ and

p the generic type of $\begin{cases} \delta(x) = x^3(x-1) \\ \delta(y) = yx \end{cases}$.

- A q -reduction of p is q itself, a $\mathcal{C}$ -reduction of q is $\text{tp}(1/F)$.
- $\text{acl}(\ell_{0,F}(a, b)) = \text{acl}(F)$.
- $\text{acl}(\ell_{1,F}(a, b)) = \text{acl}(Fa)$.
- So $\text{tp}(a, b/\text{acl}(Fa))$ is an extension of the fiber $\text{tp}(a, b/F(a))$ and thus is $\mathcal{C}$ -internal.

Intrinsic characterization of multiplicity

- $\ell_{0,A}(a) = A$.
- $\ell_{1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(A)) \text{ al. int. to a minimal type}\}$.
- $\ell_{n+1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(\ell_{n,A}(a))) \text{ al. int. to a minimal type}\}$.

We say $\text{tp}(a/A)$ has m -levels if m is the least k such that $a \in \text{acl}(\ell_{k,A}(a))$.

Let q be the generic type of $\delta(x) = x^3(x-1)$ and

p the generic type of $\begin{cases} \delta(x) = x^3(x-1) \\ \delta(y) = yx \end{cases}$.

- A q -reduction of p is q itself, a $\mathcal{C}$ -reduction of q is $\text{tp}(1/F)$.
- $\text{acl}(\ell_{0,F}(a, b)) = \text{acl}(F)$.
- $\text{acl}(\ell_{1,F}(a, b)) = \text{acl}(Fa)$.
- So $\text{tp}(a, b/\text{acl}(Fa))$ is an extension of the fiber $\text{tp}(a, b/F(a))$ and thus is $\mathcal{C}$ -internal. So, a $\mathcal{C}$ -reduction of $\text{tp}(a, b/\text{acl}(Fa))$ is itself.
- Hence $(a, b) \in \text{acl}(\ell_{2,F}((a, b)))$.

Intrinsic characterization of multiplicity

Definition

Let $p = \text{tp}(a/A)$ be stationary. Let r be a minimal type.

- An **r -reduction** of a over A is a tuple b such that $b \in \text{dcl}(Aa)$ and $\text{tp}(b/A)$ is maximally almost r -internal.
- An r -reduction of p is $\text{tp}(b/A)$ where b is an r -reduction of a over A .
- $\ell_{n+1,A}(a) = \{b \in \text{acl}(Aa) \mid \text{tp}(b/\text{acl}(\ell_{n,A}(a))) \text{ is almost internal to a minimal type}\}$

Theorem

Let $p = \text{tp}(a/A)$ be a stationary, non-algebraic, finite rank type with $m+1$ levels. Let r be a minimal type. For $i = 0, \dots, m$, let q_i be the r -reduction of $\text{tp}(a/\text{acl}(\ell_{i,A}(a)))$. Then,

$$m_r = \sum_{i=0}^m U(q_i).$$

References I



Steven Buechler.

Vaught's conjecture for superstable theories of finite rank.

Annals of Pure and Applied Logic, 155(3):135–172, 2008.



Zoe Chatzidakis, Matthew Harrison-Trainor, and Rahim Moosa.

Differential-algebraic jet spaces preserve internality to the constants.

J. Symb. Log., 80(3):1022–1034, 2015.



Christine Eagles and Léo Jimenez.

Domination, fibrations, and splitting.

The Journal of Symbolic Logic, page 1–23, 2025.

published online at <https://doi.org/10.1017/jsl.2025.10156>.



Christine Eagles and Léo Jimenez.

Internality of autonomous algebraic differential equations, 2025.

arxiv:2409.01863.

References II



James Freitag, Léo Jimenez, and Rahim Moosa.
Finite-dimensional differential-algebraic permutation groups.
J. Inst. Math. Jussieu, 24(2):603–626, 2025.



Ehud Hrushovski.
Contributions to stable model theory.
ProQuest LLC, Ann Arbor, MI, 1986.
Thesis (Ph.D.)—University of California, Berkeley.



Rémi Jaoui, Léo Jimenez, and Anand Pillay.
Relative internality and definable fibrations.
Adv. Math., 415:Paper No. 108870, 38, 2023.



Ruizhang Jin.
The logarithmic derivative and model-theoretic analysability in differentially closed fields.
University of Waterloo, 2019.
Thesis (Ph.D.)—University of Waterloo.

References III



Prerna Bihani Juhlin.

Fine structure of dependence in superstable theories of finite rank.

ProQuest LLC, Ann Arbor, MI, 2010.

Thesis (Ph.D.)—University of Notre Dame.



Rahim Moosa and Anand Pillay.

Some model theory of fibrations and algebraic reductions.

Selecta mathematica (Basel, Switzerland), 20(4):1067–1082, 2014.



Daniel Palacin and Frank O Wagner.

Ample thoughts.

The Journal of Symbolic Logic, 78(2):489–510, 2013.



Anand. Pillay.

Geometric stability theory.

Oxford logic guides ; 32. Clarendon Press, Oxford, 1996.