A point to set principle for topological entropy and applications to relating dimension, entropy, and Lyapunov exponents

Emma Dinowitz

September 30, 2025

Introduction

In recent years the point to set principle for hausdorff dimension has emerged as an example of an application of computability theory to Geometric measure theory and fractal geometry.
In this talk we will discuss how to prove an analogous point to set principle for topological entropy in dynamics, and how many classical results in the dimension theory of dynamical systems about the relation between dimension, entropy and lyapunov exponents can be proven by relating the two pointwise quantities.
We will also discuss 2 open problems in the dimension theory of nonuniformly hyperbolic dynamical systems which have been proved using this method.

Hausdorff Measure and Dimension

Definition (Hausdorff Measure)

For a metric space (X, d) and $Y \subset X$:

$$\mathcal{H}^s_\delta(Y) = \inf_{Y \subset \cup E_i, |E_i| < \delta} \sum_i |E_i|^s$$

$$\mathcal{H}^s(Y) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(Y)$$

Lemma

For any $Y \subset X$, there exists a unique $s_0 \in [0, \infty]$ such that:

- $\mathcal{H}^s(Y) = 0$ *for all* $s > s_0$
- $\mathcal{H}^s(Y) = \infty$ *for all* $s < s_0$

Hausdorff Dimension

Definition (Hausdorff Dimension)

The Hausdorff dimension of A, written $\dim_H(A)$, is defined to be the unique s_0 from the previous lemma.

Example

For the metric $d(x, y) = \frac{1}{2}^{\min\{n | x_n \neq y_n\}}$

$$\dim_H(2^{\mathbb{N}}) = 1$$

Example

For the standard middle third cantor set $\mathcal{C}$

$$\dim_H(\mathcal{C}) = \frac{\ln(2)}{\ln(3)}$$

(oracle) Kolmogorov complexity

A universal oracle turing machine M^A is a universal turing machine for each oracle A.
We define the A Kolmogorov complexity of finite string σ (written $K^A(\sigma)$) to be the minimum length $|\tau|$ of a program τ with $M^A(\tau) = \sigma$
We now extend this to 2 measures of information content for infinite binary strings $x \in 2^{\mathbb{N}}$

$$\dim^A(x) = \liminf_{n\to\infty} \frac{K^A(x|n)}{n}$$

$$\mathrm{Dim}^A(x) = \limsup_{n\to\infty} \frac{K^A(x|n)}{n}$$

Martin Lof random points for $2^{\mathbb{N}}$

A point x is called A-martin Lof random if it is not in any A-computably measure 0 set

Theorem

If $x \in 2^{\mathbb{N}}$ is A Martin Lof random then $K^A(x|n) \geq n - C$

Corollary

For $x \in 2^{\mathbb{N}}$ A-Martin Lof random we have $\dim^A(x) = Dim^A(x) = 1$

Generalizing

How do we generalize this to arbitrary complete separable metric spaces?
An A-computable metric space is a complete separable metric space so that there is a dense list of points x_σ (indexed by finite binary strings) so that the infinite matrix $d(x_i, x_j)$ is A-computable
We define the A kolmogorov complexity of a point $x \in X$ at scale δ by

$$K_\delta^A(x) = \min_{x_\sigma \in B_\delta(x)} K^A(\sigma)$$

$$\dim^A(x) = \liminf_{\delta \to 0} \frac{K_\delta^A(x)}{-\log_2(\delta)}$$

$$\mathrm{Dim}^A(x) = \limsup_{\delta \to 0} \frac{K_\delta^A(x)}{-\log_2(\delta)}$$

point to set principle for hausdorff dimension

Theorem (Point to set principle for subsets of separable metric spaces)

$$\dim_H(Y) = \min_{A \subset \mathbb{N}} \sup_{x \in Y} \dim^A(x)$$

$$\dim_P(Y) = \min_{A \subset \mathbb{N}} \sup_{x \in Y} Dim^A(x)$$

Examples

Example

For the metric $d(x, y) = \frac{1}{2}^{\min\{n | x_n \neq y_n\}}$

$$\dim_H(2^{\mathbb{N}}) = 1$$

Note that $K^A(x|n) \leq n$ for all x, which yields $\dim^A(x) \leq 1$ for all A, x. Conversely an A Martin lof random has $\dim^A(x) = 1$. Therefore $\dim_H(2^{\mathbb{N}}) = 1$

Examples

Example

For the standard middle third cantor set $\mathcal{C}$, $\dim_H(\mathcal{C}) = \frac{\ln(2)}{\ln(3)}$
We use the characterization of $\mathcal{C}$ as ternary strings only containing the symbols $0, 2$.
Let $\delta = 3^{-k}$. Then the complexity of the first k ternary entries of x is the same as if you view it as a binary string.

$$K^A_{3^{-k}}(x) = K^A(x|k) \leq k \implies \frac{K^A_{3^{-k}}(x)}{-\log_2(3^{-k})} \leq \frac{k}{\log_2(3)k} = \frac{\ln(2)}{\ln(3)}$$

Similarly if you let the binary entries of x be A-Martin Lof random then

$$K^A_{3^{-k}}(x) = K^A(x|k) \geq k - C \implies \frac{K^A_{3^{-k}}(x)}{-\log_2(3^{-k})} \geq \frac{k - C}{\log_2(3)k} \to \frac{\ln(2)}{\ln(3)}$$

Bowen topological entropy

Let X be a metric space and $T : X \to X$ a continuous map. The *ε-size* of a set $E \subset X$ is the dyadic value 2^{-s} where

$$s \;=\; \sup\Big\{\, n \geq 0 : \operatorname{diam}\big(T^i E\big) \;\leq\; \varepsilon \text{ for } 0 \leq i < n \Big\}.$$

Mimicking the construction of Hausdorff measure, define

$$m^s_\delta(Y,\varepsilon) = \inf_{\substack{Y \subset \bigcup_i U_i \\ \varepsilon\text{-size}(U_i) < \delta}} \sum_i \big(\varepsilon\text{-size}(U_i)\big)^s$$

$$m^s(Y,\varepsilon) \;:=\; \lim_{\delta \to 0^+} m^s_\delta(Y,\varepsilon)$$

There is a critical exponent s_0 such that

$$m^s(Y,\varepsilon) = \infty \quad \text{for } s < s_0, \qquad m^s(Y,\varepsilon) = 0 \quad \text{for } s > s_0.$$

$$h_{Top}(T,Y,\varepsilon) \;:=\; \inf\{s : m^s(Y,\varepsilon) = 0\} \;=\; \sup\{s : m^s(Y,\varepsilon) = \infty\}.$$

$$h_{Top}(T,Y) \;:=\; \lim_{\varepsilon \to 0^+} h_{Top}(T,Y,\varepsilon),$$

lower orbit complexity

How can we make a pointwise information quantity for dynamical systems?
Instead of metric balls we use Bowen balls

$$B_n(x, T, \varepsilon) = \{y \in X \mid d(T^i x, T^i y) < \varepsilon, 0 \leq i \leq n\}$$

$$K_n^A(x, T, \varepsilon) = \min_{x_\sigma \in B_n(x, T, \varepsilon)} K^A(\sigma)$$

$$\underline{K}^A(x, T, \varepsilon) = \liminf_{n \to \infty} \frac{K_n^A(x, T, \varepsilon)}{n}$$

$$\underline{K}^A(x, T) = \lim_{\varepsilon \to 0} \underline{K}^A(x, T, \varepsilon)$$

(Note we can equivalently define this notion using ε pseudo orbits of length n instead)

point to set principle statement

Theorem

For an arbitrary subset $Y \subset X$

$$h_{Top}(Y, T) = \min_{A \subset \mathbb{N}} \sup_{x \in Y} \underline{K}^A(x)$$

There are also several other dynamical dimension quantities that admit point to set principles of this form.

null s-covers

Definition (Null s-cover)

Let $Y \subset X$. A *null s-cover* of Y is a set $E \subset \mathbb{N}^3$ satisfying

1. $$\sum_{(i,n,p)\in E} 2^{-sn} < \infty;$$
2. for every $k, p \in \mathbb{N}$, the family
$$\left\{ B_n(s_i, 2^{-p}) \ : \ (i,n,p) \in E, \ n \geq k \right\}$$
is a cover of Y.

point to set principle proof outline

Lemma

For a continuous map $T : X \to X$ *and* $Y \subset X$,

$$h_{Top}(T, Y) \;=\; \inf\{\, s \;:\; Y \textit{ admits a null } s\textit{-cover}\}.$$

An *A-effective* null s-cover is a null s-cover that is A-recursively enumerable.

Definition (Effective topological entropy)

For a continuous map T and $Y \subset X$ the *effective topological entropy* is

$$h^A_{eff}(T, Y) \;=\; \inf\{\, s \;:\; Y \text{ admits an } A\textit{-effective} \text{ null } s\text{-cover}\}.$$

Because fewer s-covers are allowed in the effective setting, $h_{Top}(T, Y) \leq h^A_{eff}(T, Y)$ for all A.

Theorem (Galatolo, Hoyrup, Rojas)

Let X be an A-computable metric space and $T : X \to X$ a continuous map. For every $Y \subset X$, $A \subset \mathbb{N}$

$$h^A_{eff}(T, Y) = \sup_{x \in Y} \underline{K}^A(x, T),$$

Theorem

Given a set Y there is an oracle A so that

$$h_{Top}(Y) = h^A_{eff}(Y)$$

Combining the above 2 theorems is the point to set principle

relating entropy to dimension in symbolic spaces

Let $(\mathcal{A}^{\mathbb{N}}, \sigma)$ be the full shift on the finite or countable alphabet $\mathcal{A}$ with the left shift dynamical system σ. Consider the metric $d(x, y) = \beta^{\min\{n | x_n \neq y_n\}}$, $\beta \in (0, 1)$

Theorem

For a subset $Y \subset \mathcal{A}^{\mathbb{N}}$ *we have* $\dim_H(Y) = \frac{h_{Top}(Y,\sigma)}{-\log(\beta)}$

Note that $B_n(x, \beta^k) = B_{\beta^{n+k}}(x)$. Therefore

$$\frac{K_n^A(x, T, \beta^k)}{n} = \frac{C^A_{\beta^{n+k}}(x)}{n} = \frac{C^A_{\beta^{n+k}}(x)}{(n+k)(-\log\beta)}(-\log\beta)\frac{n+k}{n}$$

$$\liminf_{n\to\infty} \frac{K_n^A(x, T, \beta^k)}{n} = -\log\beta \liminf_{n\to\infty} \frac{C_{\beta^{n+k}}(x)}{-(n+k)\log\beta} = -\log(\beta)\dim^A(x)$$

Take A to be an oracle strong enough for both point to set principles

hyperbolic dynamical systems

A hyperbolic dynamical system is a closed invariant subset of a diffeomorphism f on a manifold M so that there is a splitting of the tangent bundle

$$TM = E^u \oplus E^s$$

so that

- E^s is the stable direction, with vectors contracting exponentially under f^n,

$$\|df^n(v^s)\| \leq Ce^{-\lambda n}\|v^s\|, \quad \forall n \geq 0 \quad \text{(stable forward contraction)}$$

- E^u is the unstable direction, with vectors expanding exponentially in forward time and contracting in backwards time under f^n.

$$\|df^{-n}(v^u)\| \leq Ce^{-\lambda n}\|v^u\|, \quad \forall n \geq 0 \quad \text{(unstable backwards contraction)}$$

hyperbolic flows

Let f^t be a flow on a 3-dimensional manifold M.

We call f^t an uniformly hyperbolic flow on a closed invariant set Λ if there exists a splitting of the tangent bundle:

$$T\Lambda = E^s \oplus X \oplus E^u,$$

where:

- X is the flow direction (tangent to f^t),
- E^s is the stable direction, with vectors contracting exponentially under f^t,

$$\|df^t(v^s)\| \leq Ce^{-\lambda t}\|v^s\|, \quad \forall t \geq 0 \quad \text{(stable forward contraction)}$$

- E^u is the unstable direction, with vectors expanding exponentially in forward time and contracting in backwards time under f^t.

$$\|df^{-t}(v^u)\| \leq Ce^{-\lambda t}\|v^u\|, \quad \forall t \geq 0 \quad \text{(unstable backwards contraction)}$$

stable and unstable manifolds

We can integrate the vector fields $E^u(x), E^s(x)$ to obtain local stable/unstable manifolds $W^u_{loc}(x), W^s_{loc}(x)$.
The expanding/contracting conditions on the vector fields integrate to expansion/contraction of the local manifolds.
The local expansion/contraction is controlled by the functions $-\log(Df|E^u), \log(Df|E^s)$.

- If $y \in W^u_{loc}(x)$ then

$$d^u(f^{-N}(x), f^{-N}(y)) \leq e^{\sum_{n=0}^{N-1} \log(Df|E^u(f^n x))} d^u(x, y)$$

- If $y \in W^s_{loc}(x)$ then

$$d^s(f^N(x), f^N(y)) \leq e^{-\sum_{n=0}^{N-1} \log(Df|E^s(f^n x))} d^s(x, y)$$

Lyapunov exponents

1. **Forward-time stable:**

$$\lambda_+^s(x) = \lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} \log(Df|E^s(f^n x))$$

2. **Forward-time unstable:**

$$\lambda_+^u(x) = \lim_{N\to+\infty} \frac{-1}{N} \sum_{n=0}^{N-1} \log(Df|E^u(f^n x))$$

3. **Backward-time stable:**

$$\lambda_-^s(x) = \lim_{N\to+\infty} \frac{-1}{N} \sum_{n=0}^{N-1} \log(Df|E^s(f^{-n} x))$$

4. **Backward-time unstable:**

$$\lambda_-^u(x) = \lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} \log(Df|E^u(f^{-n} x))$$

Lyapunov exponents

We define the level set of lyapunov exponents

$$L(\alpha,\beta) = \{x \in X \mid \lambda^s_+(x) = \lambda^s_-(x) = \alpha, \lambda^u_+(x) = \lambda^u_-(x) = \beta\}$$

These are the points with expansion/contraction happening at the designated rates.

Theorem

In a uniformly hyperbolic dynamical system

$$\dim_H(L(\alpha,\beta)) = h_{top}(L(\alpha,\beta), f)/\beta + h_{top}(L(\alpha,\beta), f^{-1})/\alpha$$

We will give a proof of the upper bound direction theorem which generalizes to the nonuniformly hyperbolic case

$$\dim_H(L(\alpha,\beta)) \leq h_{top}(L(\alpha,\beta), f)/\beta + h_{top}(L(\alpha,\beta), f^{-1})/\alpha$$

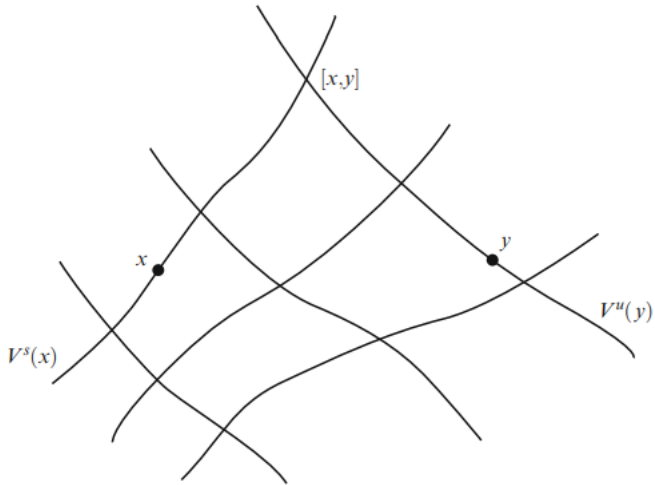
local product structure

Given two close enough points x, y in a uniformly hyperbolic system, the local product structure $[x, y]$ is defined by

$$[x, y] = W^u_{loc}(x) \cap W^s_{loc}(y)$$

It is the unique point which travels with x in backwards time and travels with y in forward time

Local product structure



Using local product structure

The upper bound $\dim_H(L(\alpha,\beta)) \leq h_{top}(L(\alpha,\beta),f)/\alpha + h_{top}(L(\alpha,\beta),f^{-1})/\beta$ relies in an essential way on the local product function being lipschitz
Suppose our strategy is to pick a close periodic orbit y and use it to split x into positive and negative time parts $[y,x],[x,y]$ respectively.
Now suppose we show

$$\dim^A([y,x]) \leq h_{top}(L(\alpha,\beta),f)/\alpha, \dim^A([x,y]) \leq h_{top}(L(\alpha,\beta),f^{-1})/\beta$$

. Having our local product structure map being Lipschitz allows us to conclude the desired upper bound

$$\dim^A(x) \leq \dim^A([y,x]) + \dim^A([x,y])$$

Theorem (Hasselblatt)

Uniformly hyperbolic dynamical systems have Lipschitz local product structure

Finding $[x, y], [y, x]$ for y periodic

We now discuss how to locate $[x, y]$ and $[y, x]$ for y periodic orbit.

Lemma

If x, y are on the same unstable manifold segment,

$$d^u(f^{-N}(x), f^{-N}(y)) \leq e^{\sum_{n=0}^{N-1} \log(Df|E^u(f^n x))} d^u(x, y)$$

$[y, x]$ follows x in forwards time and the stable manifold segment between $f^n x$ and $f^n y$ gets exponentially shortened as time goes on.

We take N so that the unstable lyapunov exponent converges to precision epsilon

$$N(\beta - \varepsilon) \leq -\sum_{n=0}^{N-1} \log(Df|E^u(f^n x)) \leq N(\beta + \varepsilon)$$

If we can find a point v on the same local unstable manifold as $f^N([y, x])$ then the exponential contraction of the unstable manifold in backwards time lets us find $[y, x]$ to precision $e^{-N(\beta-\varepsilon)} d^u(v, f^N[y, x])$

The role of expansivity

We then take a nearby periodic orbit z and consider $[f^N[y,x],z]$
This point is on the local unstable manifold of $f^N[y,x]$
We now claim we can locate $[f^N[y,x],z]$ to arbitrary precision using only the data y,z and precision δ_0 orbit data for x for T time.
Expansivity (for discrete dynamics) says that there is some precision δ_0 so that no 2 distinct points can have orbits that δ_0 shadow each other for all time
Using the specified finite amount of data we can specify the entire orbit of $[f^N[y,x],z]$.
Using this we can compute the point that is described by that orbit data.

Doing this without Lipschitz Local product structure

The main obstruction to finding points nearby x in the non uniformly hyperbolic case is not having Lipschitz local product structure.

Theorem

Suppose f^t is geodesic flow on T^1M for M surface of nonpositive curvature. Then

$$\dim_H(L(\alpha)) = 1 + 2h_{top}(L(\alpha))/\alpha$$

(Note that in this case the positive and negative exponents are the same so this is the same as $L(\alpha, \alpha)$ in the previous case)
The one comes from the flow direction.
Idea: instead of using one periodic orbit y to get approximations of $[y, x], [x, y]$, use a net of periodic orbits to compute Hausdorff metric approximations of $W^{cs}(x), W^{cu}(x)$.
The intersection of these is the orbit segment $f^t x$
This is a theorem that was only conjectured for geodesic flows on surfaces of nonpositive curvature, the lower bound was proven using symbolic dynamics modeling.

dimension of points of 0 lyapunov exponent

Theorem

For f^t geodesic flow on surfaces of nonpositive curvature

$$\dim_H(L(0)) = 3$$

This was believed to be true because for $\alpha > 0$ small enough $\dim_H(L(\alpha)) = 3$ using invariant measure methods. However there are no invariant measures of positive entropy concentrated on $L(0)$, leading to an inability to prove that $\dim_H(L(0)) = 3$ using those methods.

For $\alpha > 0$ small enough we have a measure of maximal entropy μ_α on $L(\alpha)$. For an $A-$Martin Lof random point for μ_α we have

$$\lim_{t\to\infty} \frac{K_t(x,\varepsilon,f)}{\int_0^t \lambda^u(f^t x)dx} = 1$$

$$\lim_{t\to\infty} \frac{K_t(x,\varepsilon,f^{-1})}{\int_0^t \lambda^s(f^{-t} x)dx} = 1$$

The goal is to construct a point of exponent 0 which maintains these properties in the limit by concatenating long enough orbit segments from random points for a sequence of measures

using point to set principles

Using the point to set principle for dimension it suffices to construct for every oracle A a point $x \in L(0)$ with $\dim^A(x) = 3$

Lemma

If an orbit $f^t(x)$ has $\dim^A(f^t(x)) \geq s$ for all t in some interval about 0 then there is some t with $\dim^A(f^t x) \geq s + 1$

So for each oracle it suffices to construct a point with $\dim^A(f^t x) \geq 2$ for all t in some interval about 0.

Idea of construction

(nonuniform) Specification property: Suppose you have a list of orbit segments (x_i, T_i) you would like a point to shadow and precision you would like to shadow each orbit segment at. There is a list of connecting lengths (y_i, S_i) (where S_i depend on the regions of space the (x_i, T_i) stay in and the precisions but not on the lengths T_i or on n) and a point x which follows $(x_1, T_1), (y_1, S_1), ..., (y_{n-1}, S_{n-1}), (x_n, T_n)$
This gives you a method of gluing orbit segments together to make a new orbit.

The structure of the flow

The flow can be organized as follows

- There is a set Sing of points whose orbits are entirely 0 curvature. The complement is called the regular set.
- We have a sequence of uniformly hyperbolic sets Λ_i so that for any $\alpha > 0$

$$\dim_H(L(\alpha) \cap \Lambda_i) \to \dim_H(L(\alpha)), h_{Top}(L(\alpha) \cap \Lambda_i) \to h_{top}(L(\alpha))$$

- Each of these Λ_i has an associated markov flow Σ_i consisting of a shift of finite type and a Holder continuous roof function

$$\pi_i : (\Sigma_i)_r \to \Lambda_i$$

Markov partitions

A markov parition for a hyperbolic flow is a symbolic system and a roof function (which tells you how to switch between discrete and continuous time).
We prove a Kolmogorov complexity preservation lemma for markov partitions

Lemma

Suppose $\pi : \Sigma_r \to \Lambda$ is an A-computable markov partition for a hyperbolic flow (Λ, f^t). For a point $x \in \Sigma_r$ and δ smaller than the expansivity radius we have

$$K_t(x) = K_t(\pi(x), \delta) + O(1)$$

measures of maximal entropy on level sets of lyapunov exponents

Consider a set $L(\alpha) \cap \Lambda_i$. Then there is a unique measure of maximal entropy μ on it

$$h_\mu = h_{top}(L(\alpha) \cap \Lambda_i)$$

We are going to use this measure to produce Martin Lof random points to glue together for our specification process.
We make a $\mathbb{Z} - \{0\}$ indexed sequence of measures where μ_i is the measure of maximal entropy on $L(\alpha_{|i|}) \cap \Lambda_{|i|}$ for some sequence $\alpha_i \downarrow 0$, Λ_i large enough.

Choices of orbit segments

We make a $\mathbb{Z} - \{0\}$ indexed sequence of measures where μ_i is the measure of maximal entropy on $L(\alpha_{|i|}) \cap \Lambda_{|i|}$ for some sequence $\alpha_i \downarrow 0$, Λ_i large enough.
Take the associated sequence of measures in Σ_i and take a Martin lof random point for $\oplus_{i \in \mathbb{Z}-\{0\}} \mu_i$

Lemma (van Lambalgen's lemma)

x is $y \oplus A$ μ_1 ML random and y is $x \oplus A$ μ_2 ML random iff (x, y) is $A-$ $\mu_1 \oplus \mu_2$ ML random

Therefore if we take an infinite product measure we can say $(x_i)_{i \in I}$ is A-$\oplus_{i \in I} \mu_i$ ML random iff x_j is $(A \oplus \oplus_{i \in I - \{j\}} x_i)$-$\mu_j$ ML random

Relating location data to orbit data

Lemma

If (x, T) stays away from $B_\eta(Sing)$ then given point $y \in B_\delta(x)$ and precision ε,

$$y \in B_{T^u(x,\delta,\varepsilon)}(x, f, \varepsilon) \cap B_{T^s(x,\delta,\varepsilon)}(x, f^{-1}, \varepsilon)$$

$$\int_0^{T^u(x,\delta,\varepsilon)} \lambda^u(f^t x)dt = -\ln(\delta) + O(1)$$

$$\int_0^{T^s(x,\delta,\varepsilon)} \lambda^s(f^t x)dt = -\ln(\delta) + O(1)$$

$$\frac{\int_{-T}^0 \lambda^s(f^t x)dt}{T} \to \lambda^s(x), \frac{\int_0^T \lambda^u(f^t x)dt}{T} \to \lambda^u(x)$$

Outline of proof

This lemma lets you relate location data to data of how points travel

Lemma

Given a point obtained through specification

$$K^A_{t^+,t^-}((x,f,\varepsilon_{i^+}),(x,f^{-1},\varepsilon_{i^-}))$$

$$= K^A_{T_1,\ldots,T_{i^+-1},t^+-\sum_{k=1}^{i^+-1}T_i,T_{-1},\ldots,T_{-(i^--1)},t^--\sum_{k=1}^{i^--1}T_{-i}}(x_1,\ldots,x_{i^+},x_{-1},\ldots,x_{i_-})+o(t^+)+o(t^-)$$

This lets you turn the complexity of a point obtained through specification into the complexity of it's constituent parts.

Combine this with the lemma about kolmogorov complexity preservation for markov partitions and you can work entirely symbolically!

Lemma

Let x_i be relatively A-μ_iMartin lof random on a symbolic space. Then

$$K^A_{n_1,\ldots,n_j,m_1,\ldots,m_k}(x_{-1},\ldots,x_{-j},x_1,\ldots,x_k)$$

$$=\sum_{i=1}^{j} n_i h_{-i}+\sum_{i=1}^{k} m_i h_i+\sum_{i=1}^{j} o(n_i)+\sum_{i=1}^{k} o(m_i)$$

doing this lets you pass to the limit