Problem Reducibility of Weakened Ginsburg–Sands Theorem

Heidi Benham

University of Connecticut

Online Logic Seminar
April 24, 2025

Inspiration

The following theorem in point-set topology is due to Ginsburg and Sands (1979).

Theorem (Ginsburg and Sands, 1979)

Every infinite topological space has a subspace homeomorphic to one of the following topologies on ω:

1. *Indiscrete (only $\emptyset$ and ω are open)*
2. *Initial segment (the open sets are $\emptyset$, ω and all intervals [0,n] for $n \in \omega$)*
3. *Final segment (the open sets are $\emptyset$, ω and all intervals $[n, \infty)$ for $n \in \omega$)*
4. *Discrete (all subsets are open)*
5. *Cofinite (the open sets are $\emptyset$, ω, and all cofinite subsets of ω)*

Proof Sketch

Fix an infinite topological space X and let $\mathrm{cl}(x)$ denote the closure of the singleton $\{x\}$. The proof of the Ginsburg–Sands Theorem is done in the following way.

- Define the equivalence relation $\sim$ on X by

$$x \sim y \Longleftrightarrow \mathrm{cl}(x) = \mathrm{cl}(y)$$

  for all $x, y \in X$.
- If there is an infinite equivalence class, it forms an infinite indiscrete subspace.
- If all equivalence classes are finite, there infinitely many of them.
- Pick a representative from each class. This is an infinite T_0 subspace, so assume X was T_0 to begin with.

- It follows that the relation $\leq_X$ given by

$$x \leq_X y \iff x \in \mathrm{cl}(y)$$

  is a partial order.
- By CAC (Chain–Antichain Principle) there is either an infinite chain or infinite antichain
- If there is an infinite chain, then, by ADS (Ascending–Descending Sequence Principle) there is either:
  - An infinite ascending sequence, which gives us a subspace homeomorphic to the final segment topology, or
  - An infinite descending sequence which gives us a subspace homeomorphic to the initial segment topology
- If it is an infinite antichain, then it forms an infinite T_1 subspace, so assume X is T_1
- If there is no infinite cofinite subspace, we form an infinite discrete subspace by induction

Motivation for Project

Looking at the proof, there are several natural ways to break this theorem up:

- Use of the closure operator
- Use of CAC (separating T_0 from T_1)
- Use of ADS (initial and final segment)
- The T_1 case and induction

Some natural questions that arose were:

Questions

- What axioms are necessary to prove the existence of the closure operator for a topological space?
- Were CAC and ADS necessary?
- Can the induction in the T_1 case be simplified?

Computable Topology

In order to study the Ginsburg–Sands Theorem in Reverse Mathematics, we restrict to a special type of topological space, first formalized for the language of second order arithmetic by Dorais.

Definition (Dorais, 2011)

A **countable, second countable (CSC) space** is a triple $(X, \mathcal{U}, k)$ where X is a countable set, $\mathcal{U}$ is a countable sequence $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$ of subsets of X, and k is a function $k : X \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, such that

- for all $x \in X$, there is $i \in \mathbb{N}$ such that $x \in U_i$
- if $x \in U_i \cap U_j$, then $x \in U_{k(x,i,j)} \subseteq U_i \cap U_j$.

Full Ginsburg–Sands

We now have a way to define the Ginsburg–Sands Theorem as a reverse mathematical principle.

Definition

GS is the statement that every infinite CSC space has an infinite subspace which is indiscrete; has the initial segment topology; has the final segment topology; is discrete; or has the cofinite topology.

Theorem (Benham, DeLapo, Dzhafarov, Solomon, and Villano)

The following are equivalent over RCA_0*:*

1. ACA_0
2. GS
3. *The statement that for every CSC space, the closure relation exists.*

Restrictions of GS

Definition

- $\mathrm{GS}^{\mathrm{cl}}$ is the restriction of GS to CSC spaces for which the closure relation exists
- wGS is the statement that every infinite CSC space has an infinite subspace which is indiscrete; has the initial segment topology; has the final segment topology; or is T_1
- $\mathrm{wGS}^{\mathrm{cl}}$ is the restriction of wGS to CSC spaces for which the closure relation exists
- GST_1 is the restriction of GS to T_1 CSC spaces

The following implications are immediate:

- $\mathrm{GS} \rightarrow \mathrm{GS}^{\mathrm{cl}} \rightarrow \mathrm{wGS}^{\mathrm{cl}}$
- $\mathrm{GS} \rightarrow \mathrm{wGS} \rightarrow \mathrm{wGS}^{\mathrm{cl}}$
- $\mathrm{GS} \rightarrow \mathrm{GST}_1$

Theorem (BDDSV)

The following are equivalent over RCA_0*:*

1. ACA_0
2. GS
3. wGS

Theorem (BDDSV)

The following are equivalent over RCA_0*:*

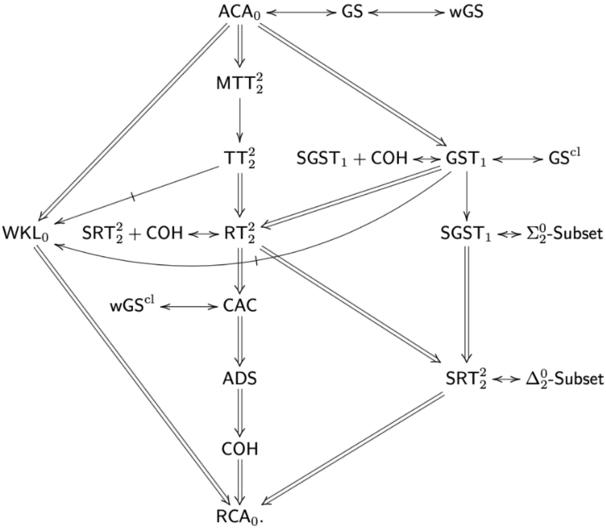
1. CAC
2. $\mathsf{wGS}^{\mathsf{cl}}$

Theorem (BDDSV)

The following are equivalent over RCA_0*:*

1. GST_1
2. $\mathsf{GS}^{\mathsf{cl}}$

Reverse Math Zoo of Ginsburg–Sands



Observation

The proof that $\mathsf{RCA}_0 \vdash \mathsf{CAC} \to \mathsf{wGS}^{\mathsf{cl}}$ uses both CAC and ADS.

- Recall from original Ginsburg–Sands Theorem proof, this would involve:
  - Using CAC to separate T_1 case from the non-T_1 case
  - Using ADS to separate the final segment topology from the initial segment topology
- This does not matter when we look at subsystems of second order arithmetic
- This could tell us something interesting about the Weihrauch degree of $\mathsf{wGS}^{\mathsf{cl}}$

Problems, Computable Reducibility, Weihrauch Reducibility

Many principles of second order arithmetic can be thought of as instance-solution problems.

Definition

An *(instance-solution) problem* is a partial function $\mathsf{P} : \mathcal{A} \to \mathcal{P}(\mathcal{B})$ for some sets $\mathcal{A}$ and $\mathcal{B}$. Elements of $\mathrm{dom}(\mathsf{P})$ are P-*instances* and elements of $\mathsf{P}(x)$ for $x \in \mathcal{A}$ are P-*solutions* to x.

Any theorem of the form $\forall x(\varphi(x) \to \exists y \psi(x, y))$ can be thought of as an instance-solution problem.

Definition

A problem P is *computably reducible* to a problem Q ($\mathsf{P} \leq_c \mathsf{Q}$) if every P-instance X computes a Q-instance $\widehat{X}$ such that if $\widehat{Y}$ is any Q-solution to $\widehat{X}$, then $X \oplus \widehat{Y}$ computes a P-solution to X.

There is also a uniform version of this

Definition

A problem P is *Weihrauch reducible* to a problem Q ($\mathsf{P} \leq_W \mathsf{Q}$) if there exist Turing functional Φ and Ψ such that whenever X is a P-instance, $\Phi(X)$ is a Q-instance and if $\widehat{Y}$ is any Q-solution to $\Phi(X)$, then $\Psi(X \oplus \widehat{Y})$ is a P-solution to X.

There are also strong versions of each of these in which the original P-instance X is not referenced at the last step to compute a P-solution to X.

Note that both $\mathsf{wGS}^{\mathsf{cl}}$ and CAC can be viewed as problems.

Theorem (Benham)

$\mathsf{wGS}^{\mathsf{cl}} \not\leq_c \mathsf{CAC}$

Showing this would be greatly simplified if we could find aproblem that more closely resembles CAC that is Weihrauch-equivalent to $\mathsf{wGS}^{\mathsf{cl}}$.

Definition

- QADAC is the problem the instances of which are infinite quasi-orders $\leq$ on ω and the solutions of which are infinite sets S that are either an antichain under $\leq$, a clique under $\leq$, an ascending sequence under $\leq$, or a descending sequence under $\leq$.
- ADAC is the problem the instances of which are infinite partial orders $\leq$ on ω and the solutions of which are infinite sets S that are either an antichain under $\leq$, an ascending sequence under $\leq$, or a descending sequence under $\leq$.

The following implications over RCA_0 are immediate.

$$\mathsf{QADAC} \rightarrow \mathsf{ADAC} \rightarrow \mathsf{CAC} \rightarrow \mathsf{ADS}.$$

Theorem (Benham)

QADAC $\equiv_{sW}$ $\mathsf{wGS}^{\mathsf{cl}}$.

Proof Idea: We have the following correspondences:

1. Clique $\leftrightarrow$ indiscrete topology
2. Antichain $\leftrightarrow$ T_1 topology
3. Descending sequence $\leftrightarrow$ initial segment topology
4. Ascending sequence $\leftrightarrow$ final segment topology

Theorem (Benham)

QADAC $\leq_c$ ADAC.

Proof Sketch:

- If the QADAC-instance contains an infinite equivalence class, we can make an ADAC-instance by making a partial order that consists of only the elements of the equivalence class that is ordered so that every thing is incomparable. Then the only solution is an antichain, and this precisely corresponds to the clique in the QADAC-instance
- Otherwise, thin the QADAC-instance so that there are no nontrivial equivalence classes. This is now a partial order and thus an ADAC-instance. Any ADAC-solution to this partial order is a QADAC-solution to the original quasiorder

Conjecture

QADAC $\nleq_W$ ADAC.

Definition

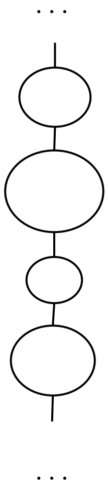
QADS is the problem the instances of which are infinite quasi-linear orders $\leq$ on ω and the solutions of which are infinite sets S that are either an clique under $\leq$, an ascending sequence under $\leq$, or a descending sequence under $\leq$.

Theorem (Benham)

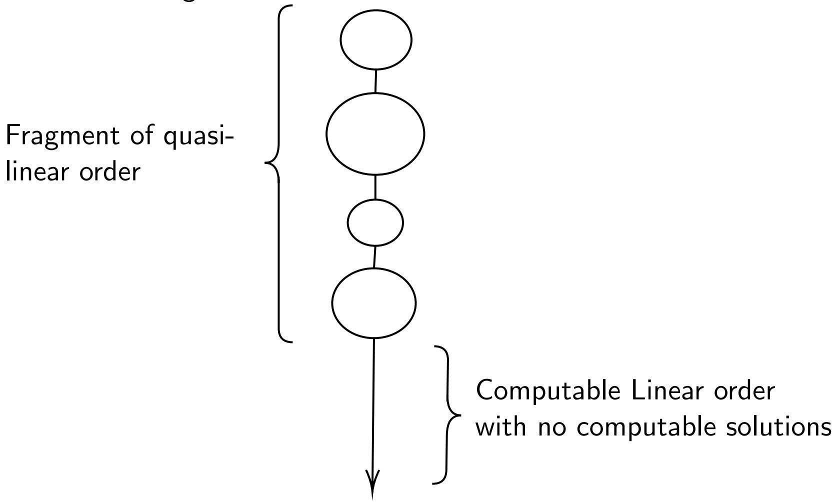
QADS $\nleq_W$ ADS.

Proof Sketch

We start by building a quasi-linear order that looks something like:



With this quasi-linear order, we build a new quasi-linear order that looks something like:



Theorem (Benham)

ADAC $\not\leq_c$ CAC.

In order to show this, we use a definition introduced by Patey:

Definition (Patey, 2016)

A problem P *admits preservation of p among k* hyperimmunities if for each collection of k hyperimmune sets and each computable P-instance X there is P-solution Y to X such that p of the k hyperimmune sets are Y-hyperimmune.

Lemma (Benham)

CAC *preserves two among three hyperimmunities.*

Proof Sketch:

- To show this, we fix a computable partial order and 3 hyperimmune sets
- Via forcing, we construct a chain C and an antichain A such that each of the hyperimmune sets are hyperimmune relative to A or C
- By the Pigeonhole Principle, this means that there are 2 of the hyperimmune sets are either A-hyperimmune or C-hyperimmune

Lemma (Benham)

ADAC *does not preserve two among three hyperimmunities. That is, there is an instance of* ADAC *such that, there are three hyperimmune sets such that two of the three are not hyperimmune relative to any solution to the instance.*

Proof Sketch:

- Prove that there is a computable weakly stable partial ordering such that each pairwise union of the set of small elements, the set of large elements, and the set of isolated elements are hyperimmune
- The pairwise unions of these sets are also hyperimmune
- An ADAC-solution to this partial order would be an antichain, an ascending sequence, or a descending sequence

- An antichain would be a subset of the pairwise unions containing the set of isolated points
- An ascending sequence would be a subset of the pairwise unions containing the set of small elements
- A descending sequence would be a subset of the pairwise unions containing the set of large elements
- In any of these cases, there will be two out of three sets that are not hyperimmune relative to the solution

Proof Sketch of ADAC $\not\leq_c$ CAC

- Suppose to the contrary
- We use the same hyperimmune sets and partial order from the previous theorem
- Then, any CAC-instance computable from this partial order has a CAC-solution C, which, together with the partial order, computes an ADAC-solution A
- Two of the hyperimmune sets are C-hyperimmune
- Thus, no C-computable function dominates either of the principle functions these sets
- For any ADAC-solution A computable from C, any A-computable function also cannot dominate these principle functions, which is a contradiction

Wrapping it Up

We can now prove the following theorem:

Theorem

$\mathsf{wGS}^{\mathrm{cl}} \not\leq_c \mathsf{CAC}$.

We have that ADAC $\leq_c$ QADAC. Thus, we must have that QADAC $\not\leq_c$ CAC for, otherwise, by transitivity, ADAC $\leq_c$ CAC, which is a contradiction. Since $\mathsf{wGS}^{\mathrm{cl}} \equiv_{sW}$ QADAC, it is also true that $\mathsf{wGS}^{\mathrm{cl}} \not\leq_c$ CAC.

References

- H. Benham, A. DeLapo, D. D. Dzhafarov, R. Solomon, and J. D. Villano, *The Ginsburg–Sands theorem and computability theory*, Adv. Math., vol. 444, p. 109618, 2024.
- François G. Dorais, *Reverse mathematics of compact countable second countable spaces*, preprint, arXiv:1110.6555, 2011.
- John Ginsburg and Bill Sands, *Minimal infinite topological spaces*, Amer. Math. Monthly **86** (1979), no. 7, 574–576.
- Ludovic Patey, *Les mathématiques à rebours de théorèmes de type Ramsey*. PhD thesis, Université Paris Diderot, 2016.

Questions?

Thank you!