Relation Algebra and Sumset Problems in Abelian Groups

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An abstract relation algebra (RA) $\langle A, +, \cdot, -, ;, \breve{}, 1' \rangle$ has operations intended to mimic those of a concrete relation algebra

$$\langle \mathcal{P}(U \times U), \cup, \cap, ^c, \circ, ^{-1}, \mathrm{Id} \rangle$$

An RA is *representable* if it embeds in a concrete RA
Representability is undeciable for finite RAs
Restrict the search space to find representations

Representations “are” edge-colorings of complete graphs
K_n with c colors, $\sim c^{n^2}$ colorings
First restriction: consider only cyclic colorings

$$\mathbb{Z}_8 : \quad \{1, 4, 7\}, \{2, 3, 5, 6\}$$

reduces complexity to $\sim c^n$ but requires communtativity of the RA

Second restriction: cyclic reps over $\mathbb{Z}_p$, p prime, where the color classes are the cosets of some multiplicative subgroup of $\mathbb{Z}_p^\times$

requires commutativity and $\mathbb{Z}_c \hookrightarrow \mathrm{Aut}$

Significant upside: we get decidability

For an RA saisfying the above, we can check for this type of rep in $\mathcal{O}\left(\frac{c^8}{\log c}\right)$ time

Outline of proof

1. We need check only primes $p \leq c^4 + 5$ [A., 2017]

   Argument uses discrete Fourier analysis

   One can count solutions (x, y, z) inside $A \subset \mathbb{Z}_p^\times$ to the equation $x + y = z$ via the exponential sum

$$\frac{1}{p} \sum_{k=0}^{p-1} \sum_{x \in A} \sum_{y \in A} \sum_{z \in A} e^{\frac{-2\pi i k}{p}(x+y-z)} \qquad (1)$$

   (1) can be rearranged to get

$$\frac{1}{p} \sum_{k=0}^{p-1} \widehat{\mathrm{Ch}}_A(k)^2 \cdot \widehat{\mathrm{Ch}}_A(-k) \qquad (2)$$

Proof con't

Pull the $k = 0$ term out of (2) to get

$$\delta^3 p^2 + \frac{1}{p} \sum_{k=1}^{p-1} \widehat{\mathrm{Ch}}_A(k)^2 \cdot \widehat{\mathrm{Ch}}_A(-k) \tag{3}$$

and then bound the "error term."

By a slight modification, generalize to $ax + by = cz$.

Proof con't

2. Each prime can be checked in $\mathcal{O}(p)$ [A., Ylvisaker 2019]
   Checking $H_0 + H_i \supseteq H_j$ is sufficient

$$H_0 + H_i \supseteq H_j \Leftrightarrow (g^j - H_0) \cap H_i \neq \emptyset$$

3. Using estimates on primes, we can check all primes under $c^4 + 5$ in $\mathcal{O}\left(\frac{c^8}{\log c}\right)$. [A., 2019]

Examples

All known reps of Ramsey algebras (one tiny exception!) are these multiplicative subgroup reps

RA 34_{65} – Three colors: red, blue, green

blue-blue-green and green-green-blue forbidden

Look for a cyclic RA with colors $a_1, \ldots, a_n$ such that triangles of the form a_i-a_i-$a_{n/2}$ are forbidden.

Map blue to H_0 and green to $H_{n/2}$.

Let $p = 3597$, and let H_0 be the multiplicative subgroup of $\mathbb{Z}_p^\times$ of index $n = 24$.

Then a_i-a_i-a_{i+12} are the only forbidden triangles.

Another Decidability Result

If an RA has no mandatory 3-cycles, i.e., “rainbow triangles,” then it is decidable whether the RA is representable. [Maddux 2006]

This is another strong restriction.

Reps over abelian groups

The problem of representability can be stated in terms of sumsets.

Given G, does there exist a partition

$$G = \{0\} \cup A \cup B$$

such that

- $A = -A$
- $B = -B$
- $A + A = G$
- $A + B = G \setminus \{0\}$
- $B + B = \{0\} \cup A$???

This question is open, but solved for cyclic groups [ABCCC 2024]

Generalize to n shades of blue

For which G does there exist a partition

$$G = \{0\} \cup A \cup B_1 \cup \cdots \cup B_n$$

such that

- $A = -A$
- $B_i = -B_i$
- $A + A = G$
- $A + B_i = G \setminus \{0\}$
- $B_i + B_i = \{0\} \cup A$
- $B_i + B_j = A \quad (i \neq j)$???

Sum-free sets

In this case, $B_1 \cup \cdots \cup B_n$ is a sum-free set.

For every finite abelian G, there is a subset X of G of order $\alpha|G|$, where $\alpha \in [2/7, 1/2]$, such that X is sum-free. [Green, Ruza 2005]

There is NO $\alpha > 0$ such that every finite group G has a product-free subset of order at least $\alpha|G|$. [Gowers 2008]

In fact, one cannot even replace $\alpha|G|$ by $\alpha|G|^{8/9}$.

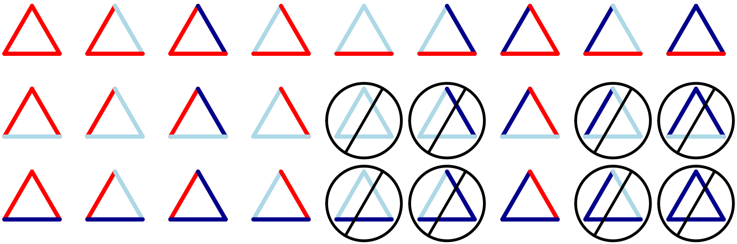
Sum-free sets in $\mathbb{Z}_2^n$ – the state of the art

Consider $G = \mathbb{Z}_2^{3k+1}$.

Define $A = \{x \in G : x \text{ has between 1 and } 2k \text{ ones}\}$
and
$B = \{x \in G : x \text{ has more than } 2k \text{ ones}\}$.

B can be split randomly into $B_1, \ldots, B_n$ if $2^{3k+1} > n^{6+o(1)}$.
[ALMMPSX 2022]

The Special Case $n = 2$



The Special Case $n = 2$

The case $n = 2$ has received extra attention.

The smallest known rep is over $\mathbb{Z}_2^{10}$, 1024 points.

The smallest known cyclic rep is over $\mathbb{Z}_{751181}$.

The best known lower bound for general reps is 26 points.

The best known lower bound for cyclic reps is 121 points.

An Open Problem

Find a group G and partition $G = \{0\} \cup A \cup B \cup C$ such that

- $A = -A$, $B = -B$, $C = -C$
- $A + A = G$
- $A + B = G \setminus \{0\}$
- $A + C = G \setminus \{0\}$
- $B + B = \{0\} \cup A \cup B$
- $B + C = A$
- $C + C = \{0\} \cup A$.

This is RA 33_{65}.

Thank you for your attention.