A Wave Equation Viewed as an Ordinary Differential Equation

T.A. BURTON, JOZSEF TERJÉKI and BO ZHANG

1. Introduction.

Through dynamical system theory, many properties of evolution equations are found to be parallel to those of special ordinary differential equations. The theory of inertial manifolds (cf. [11]) establishes deep theoretical connections between infinite dimensional and finite dimensional dynamical systems in terms of limit sets which are exponentially asymptotically stable. Central to so much of the application of this theory is the use of energy methods or the equivalent use of Liapunov functions.

This work takes a close look at six very well known classical problems associated with the ordinary differential equation

$$u'' + f(t)g(u) = 0, \quad ug(u) > 0 \text{ if } u \neq 0, \quad f(t) \geq 0, \tag{1}$$

and shows that these problems have parallels for the equation

$$u_{tt} = f(t)g(u_x)_x, \qquad u(t,0) = u(t,1) = 0, \tag{2}$$

both in terms of results and methods of solution. These problems concern oscillation, continuation of solutions, decay of solutions, limit circle considerations, and limiting behavior of solutions.

The study actually began in [1] where it was noted that there were striking similarities between the classical Liénard equation

$$u'' + f(u)u' + g(u) = 0, \quad f(u) > 0, \quad ug(u) > 0 \text{ if } u \neq 0, \tag{L}$$

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and several forms of the damped wave equation such as

$$(W) \qquad u_{tt} = g(u_x)_x - f(u)u_t, \qquad u(t,0) = u(t,1) = 0.$$

In particular:

(i) Each of (L) and (W) has a natural Liapunov function with derivative which is negative semi-definite.

(ii) Each of (L) and (W) has a Liénard transformation, the transformed form of which has a natural Liapunov function whose derivative is negative semi-definite.

(iii) A combination of the Liapunov function in (i) and (ii) produces a Liapunov function whose derivative is negative definite.

(iv) The forms of the Liapunov functions for (L) and (W) are very similar, as are the consequences derivable from them.

Here, we continue that type of study, selecting a Liapunov function for (1), converting it to a Liapunov function for (2), and deducing parallel results for oscillation, continuation, and other qualitative behavior of solutions.

2. Oscillation.

Wintner [14] considered the linear equation

$$(3) \qquad u'' + f(t)u = 0$$

and generalized the following idea. Suppose that $f : [0,\infty) \to [0,\infty)$ is continuous and $\int_0^\infty f(t)dt = \infty$; then every solution oscillates. He proved this by assuming that a solution $u(t)$ has no zero past some t_0 and formed a Chetayev type Liapunov function

$$V(t) = u'(t)/u(t) \text{ for } t > t_0.$$

Then

$$V'(t) = [uu'' - (u')^2]/u^2 = -f(t) - V^2(t),$$

a Riccati equation having a solution reaching negative infinity in finite time $t > t_0$.

To extend the result to (2) we must first decide how to define oscillation for (2). Recall that a solution $u(t)$ of (1) is oscillatory if there is a sequence $\{t_n\} \uparrow \infty$ with $u(t_n) = 0$, $u(t) \not\equiv 0$. But a reading of classical oscillation papers reveals that for $f(t) \geq 0$ the property of most interest was the equivalent fact that $u''(t) = -f(t)g(u(t))$ oscillated. If we take that as a definition, then Wintner's proof works for (2).

In fact, such arguments showing oscillations have been equally effective for delay equations and instead of (2) we deal here with

$$(2^*) \qquad \begin{cases} u_{tt} = f(t)g(u_x(t-h,x))_x \\ u(t,0) = u(t,1) = 0 \end{cases}$$

where h is a nonnegative constant. It may be noted that if $h > 0$ then (2^*) can be solved by the method of steps, but it requires very smooth initial functions.

Definition 1. A solution of (2^*) is oscillatory if there are sequences $\{t_n\} \uparrow \infty$ and $\{x_n\} \subset (0,1)$ such that $g(u_x(t_n, x_n))_x$ and $g(u_x(t_{n+1}, x_{n+1}))_x$ have opposite sign.

The reader may consider a vibrating string and conclude that Def. 1 is what we would intuitively mean by the string vibrating.

Theorem 1. *Suppose that for each $t_1 \geq 0$, the only solution of (2^*) satisfying $u(t_1+\theta, x) = u_t(t_1+\theta, x) \equiv 0$ for $-h \leq \theta \leq 0$ is the zero solution. Assume that $f(t) \geq 0$, that $\int_0^\infty f(s)ds = \infty$, and that $g'(r) \geq g_0 > 0$. Let $u(t,x)$ satisfy (2^*) on $[0,\infty)$ and be nonoscillatory. Then $u(t,x)$ is zero.*

Proof. Taking into account the boundary conditions, we follow Wintner and write

$$V(t) = \int_0^1 u_t(t,x)dx \Big/ \int_0^1 u(t-h,x)dx$$

for an assumed nonoscillatory solution u. This means that there is a $t_0 \geq 0$ such that u_{xx} has one sign for $t \geq t_0$. Suppose, to be definite, that $u_{xx}(t,x) \leq 0$ for $t \geq t_0$. Since $u(t,0) = u(t,1) = 0$ we will suppose that $u(t,x) \geq 0$ on $[t_0, \infty)$. From (2^*) we have $u_{tt}(t,x) \leq 0$ and so $u_t(t,x)$ is decreasing on $[t_0+h, \infty)$ for each fixed $x \in [0,1]$. Hence, $u_t(t,x) \geq 0$ for all $t \geq t_0+h$ and $x \in [0,1]$. If $u(t_1,x_1) = 0$ for some $t_1 > t_0+h$ and $x_1 \in (0,1)$ then $u_{xx}(t_1,x) \leq 0$ and $u(t_1,x) \geq 0$ imply that $u(t_1,x) = 0$ for all $x \in [0,1]$. Consequently, $u(t,x)$ has a minimum at $t = t_1$ for all fixed x, so $u_t(t_1,x) = 0$ for all $x \in [0,1]$. Therefore we conclude that either $u(t,x) > 0$ for all $t > t_0$ and $x \in (0,1)$ or $u(t,x) \equiv 0$, $u_t(t,x) \equiv 0$ for all $x \in [0,1]$ and all large t. By the assumed uniqueness, $u(t,x) \equiv 0$.

Now assume that $\int_0^1 u(t,x)dx > 0$ for $t \geq t_0$ so $V(t)$ is defined and $V(t) \geq 0$ for $t \geq t_0 + h$. We then have

$$\begin{aligned} V'(t) &= \Big[\int_0^1 u(t-h,x)dx \int_0^1 f(t)g(u_x(t-h,x))_x dx \\ &\quad - \int_0^1 u_t(t,x)dx \int_0^1 u_t(t-h,x)dx\Big] \Big/ \Big[\int_0^1 u(t-h,x)dx\Big]^2 \\ &= \Big[f(t)\int_0^1 g'(u_x(t-h,x))u_{xx}(t-h,x)dx \Big/ \int_0^1 u(t-h,x)dx\Big] \\ &\quad - \Big\{\int_0^1 u_t(t,x)dx \int_0^1 u_t(t-h,x)dx \Big/ \Big[\int_0^1 u(t-h,x)dx\Big]^2\Big\}. \end{aligned}$$

Now $g'(r) \geq g_0 > 0$ and $u(t,x) \geq 0$, $u_{xx} \leq 0$, so for fixed $t \geq t_0 + h$ we have

$$\begin{aligned}
-\int_0^1 g'(u_x(t-h,x))u_{xx}(t-h,x)dx / \int_0^1 u(t-h,x)dx \\
\geq g_0 \int_0^1 |u_{xx}(t-h,x)|dx / \int_0^1 u(t-h,x)dx \\
\geq g_0 \int_0^1 |u_{xx}(t-h,x)|dx / u(c)
\end{aligned}$$

where $u(c) = \sup_{0\leq x\leq 1} u(t-h,x) > 0$ (t is fixed). But $u(t,0) = u(t,1) = 0$ so there is an $\xi \in (0,1)$ with $u_x(t-h,\xi) = 0$. This means that

$$\begin{aligned}
\int_0^1 |u_{xx}(t-h,x)|dx \geq \sup_{0\leq x\leq 1} |u_x(t-h,x)| \geq \int_0^1 |u_x(t-h,x)|dx \\
\geq \sup_{0\leq x\leq 1} |u(t-h,x)| = u(c).
\end{aligned}$$

Hence,

$$g_0 \int_0^1 |u_{xx}(t-h,x)|dx/u(c) \geq g_0.$$

Moreover, since $u_{tt}(t,x) \leq 0$, $u_t(t,x) \geq 0$, for $t \geq t_0 + h$, we have

$$\int_0^1 u_t(t,x)dx \leq \int_0^1 u_t(t-h,x)dx \text{ for } t \geq t_0 + 2h$$

and

$$-\int_0^1 u_t(t,x)dx \int_0^1 u_t(t-h,x)dx \leq -\Big(\int_0^1 u_t(t,x)dx\Big)^2.$$

Thus,

$$V'(t) \leq -f(t)g_0 - V^2$$

a Riccati equation with $V(t_1) = -\infty$ for some $t_1 > t_0$. It then follows that there is a $t_1 > t_0 + 2h$ with $\int_0^1 u(t_1,x)dx = 0$ and so $u(t_1,x) \equiv 0$ on $[0,1]$, as required.

Remark. Zlámal [15] generalized Wintner's theorem and showed that if

$$(*) \qquad \begin{cases} \text{there exists a function } w(t) > 0 \text{ such that} \\ \int_0^\infty w(t)f(t)dt = \infty \text{ and } \int_0^\infty (w'(t))^2 w^{-1}(t)dt < \infty \end{cases}$$

then solutions of (3) oscillate. Our theorem also remains valid if we assume (*) instead of $\int_0^\infty f(t)dt = \infty$. To see this we have to finish the proof in a different way: From the inequality $V'(t) \le -f(t)g_0 - V^2$ we get for $T > t_2 \ge t_0 + 2h$ that

$$\begin{aligned}
g_0 \int_{t_2}^{T} f(t)w(t)dt &\le -w(T)V(T) + w(t_2)V(t_2) + \int_{t_2}^{T} w'(t)V(t)dt \\
&- \int_{t_2}^{T} w(t)V^2(t)dt \le w(t_2)V(t_2) \\
&+ \left(\int_{t_2}^{\infty} (w'(t))^2 w^{-1}(t)dt\right)^{1/2} \left(\int_{t_2}^{T} w(t)V^2(t)dt\right)^{1/2} \\
&- \int_{t_2}^{T} w(t)V^2(t)dt \\
&\le w(t_2)V(t_2) + \int_{t_2}^{\infty} (w'(t))^2 w^{-1}(t)dt/4 \\
&- \left[\frac{1}{2}\left(\int_{t_2}^{\infty} (w'(t))^2 w^{-1}(t)dt\right)^{1/2} - \left(\int_{t_2}^{T} w(t)V^2(t)dt\right)^{1/2}\right]^2 \\
&\le w(t_2)V(t_2) + \int_{t_2}^{T} (w'(t))^2 w^{-1}(t)dt/4 < \infty,
\end{aligned}$$

a contradiction as $T \to \infty$.

3. Continuation of solutions

Frequently, in oscillation problems concerning (1), the function $f(t)$ is allowed to become negative some of the time. But then special care must be taken concerning the growth of g to be sure that a solution will not have finite escape time. In [2] it was shown that if $f(t_1) < 0$ for some $t_1 > 0$ and if

$$G(x) := \int_0^x g(s)ds,$$

then (1) has a solution not continuable to $t = \infty$ provided that either

(a) $\int_0^\infty [1 + G(x)]^{-1/2} dx < \infty$ or

(b) $\int_0^{-\infty} [1 + G(x)]^{-1/2} dx > -\infty$.

A partial converse was also obtained. Here, a similar result for (2) holds. It is to be noted that (1) can have a solution defined for $t \ge 0$ having $u(t) > 0$ and $u(t) \to \infty$; thus the conditions (a) and (b) are separate. The behavior of $g(u)$ for $u < 0$ is immaterial. But for (2); because of the boundary condition and the inequality $\int_0^1 \pi^2 u^2 dx \le \int_0^1 u_x^2 dx$, if $u(t,x) \to +\infty$, then $g(u_x) \to \pm\infty$. Hence, the parallel result for (2) will involve g for both positive and negative values of its argument.

Theorem 2. *Suppose that there is a $t_1 > 0$ with $f(t_1) < 0$ and suppose that there is a convex downward function $\overline{g} : R^+ \to R^+$ such that $xg(x) \ge \overline{g}(x^2)$. For*

$$\overline{G}(u) = \int_0^u \overline{g}(\xi)d\xi, \ \textit{if} \ \int_0^\infty [1 + \overline{G}(x)]^{-1/2} dx < \infty,$$

then there are initial conditions for (2) such that any solution having those initial conditions can not be defined for all $t \geq t_1$.

Proof. Since $f(t_1) < 0$ and $f(t)$ is continuous, there are positive constants δ, m, M such that $-M \leq f(t) \leq -m < 0$ if $t_1 \leq t \leq t_1 + \delta$. Let $u(t,x)$ be a solution of (2) and define $z(t) = \int_0^1 u^2(t,x)dx$ and $y(t) = 2\int_0^1 u(t,x)u_t(t,x)dx$. We then have the system of ordinary differential equations

$$\begin{cases} z' = y \\ y' = 2\int_0^1 u_t^2(t,x)dx - 2f(t)\int_0^1 u_x g(u_x)dx. \end{cases} \tag{4}$$

Denote by $(z(t), y(t))$ a solution of (4) satisfying $z(t_1) = 1$, $y(t_1) = y_1$ with y_1 large and to be determined later. So long as $(z(t), y(t))$ is defined on $[t_1, t_1 + \delta]$ we have both $y(t)$ and $z(t)$ monotonically increasing. From (4) we obtain

$$\begin{aligned} 2yy' &= 2y\int_0^1 u_t^2(t,x)dx - 2f(t)2y(t)\int_0^1 u_x g(u_x)dx \\ &\geq 4my(t)\overline{g}\left(\int_0^1 u_x^2 dx\right) \\ &\geq 4my(t)\overline{g}(z(t)), \end{aligned}$$

using Jensen's inequality and then Wirtinger's inequality, so that

$$\begin{aligned} y^2(t) &\geq y^2(t_1) + 4m\int_{t_1}^t z'(s)\overline{g}(z(s))ds \\ &= y^2(t_1) + 4m\overline{G}(z(t)) - 4m\overline{G}(z(t_1)) \end{aligned}$$

and

$$z'(t) = y(t) \geq [y^2(t_1) - 4m\overline{G}(1) + 4m\overline{G}(z(t))]^{1/2}.$$

Divide both sides by the right-hand side and integrate from t_1 to t to obtain

$$\int_1^{z(t)} [y^2(t_1) - 4m\overline{G}(1) + 4m\overline{G}(z)]^{-1/2}dz \geq t - t_1.$$

That is

$$\int_0^{\infty} [y^2(t_1) - 4m\overline{G}(1) + 4m\overline{G}(z)]^{-1/2}dz \geq t - t_1.$$

If $y^2(t_1) \geq 4m\overline{G}(1) + 4m$, then we have that

$$[y^2(t_1) - 4m\overline{G}(1) + 4m\overline{G}(z)]^{-1/2} \leq (4m)^{-1/2}(1 + \overline{G}(z))^{-1/2} \in L^1[0,\infty).$$

On the other hand, for fixed $z \in (0,\infty)$ we have

$$[y^2(t_1) - 4m\overline{G}(1) + 4m\overline{G}(z)]^{-1/2} \to 0 \text{ as } y^2(t_1) \to \infty.$$

Therefore, by the Lebesgue dominated convergence theorem it follows that

$$\int_0^\infty [y^2(t_1) - 4m\overline{G}(1) + 4m\overline{G}(z)]^{-1/2} dz \to 0$$

as $y^2(t_1) \to \infty$. Consequently, we may take $y^2(t_1)$ so large that

$$\int_0^\infty [y^2(t_1) - 4m\overline{G}(1) + 4m\overline{G}(z)]^{-1/2} dz < \delta.$$

That is, $z(t) \to \infty$ before t reaches $t_1 + \delta$.

4. Instability

Section 3 deals with a drastic type of instability. But if $f(t) < 0$ for all $t \geq 0$, then a more gentle type of instability can occur.

As motivation we again consider equation (1) and suppose that $f(t) \leq -f_0 < 0$ on $[0, \infty)$. Then the classical theory of Chetayev (cf. [6; p. 27], for example) leads to the Liapunov function $V = uv$ for the system $\{u' = v,\ v' = -f(t)g(u)\}$ so that $V' = uv' + u'v = v^2 - f(t)g(u)u \geq v^2 + f_0 ug(u)$. Therefore, V vanishes on $u = 0$ and on $v = 0$, with $V' > 0$ on the set $uv > 0$. Thus, the zero solution is unstable.

We now give a very simple parallel for (2).

Theorem 3. *If $f(t) \leq 0$ for $t \geq 0$ $ug(u) \geq 0$ for all $u \in R$, then the solution $u = 0$ is unstable.*

Proof. Let $u(t, x)$ be a solution of (2) on $[0, \infty)$ with $u(0, x) \geq 0$, $u_t(0, x) \geq 0$, and $\int_0^1 u(0, x)u_t(0, x)dx > 0$. Define $\{u_t = v,\ v_t = f(t)g(u_x)_x\}$ and

$$V(t) = \int_0^1 u(t, x)v(t, x)dx$$

so that

$$\begin{aligned} V'(t) &= \int_0^1 u_t v dx + \int_0^1 u v_t dx = \int_0^1 v^2 dx + \int_0^1 f(t)(g(u_x))_x u dx \\ &= \int_0^1 v^2 dx - \int_0^1 f(t)g(u_x)u_x dx \geq \int_0^1 u_t^2 dx. \end{aligned}$$

Suppose that $u = 0$ is stable. Then for a given $\epsilon > 0$ and $t_1 \geq 0$ there is a $\delta > 0$ such that $\int_0^1 u_t^2(t_1, x)dx < \delta^2$ and $\int_0^1 u^2(t_1, x)dx < \delta^2$ imply that any solution $u(t, x)$ satisfying those initial conditions will satisfy $\int_0^1 u^2(t, x)dx < \epsilon^2$ and $\int_0^1 u_t^2(t, x)dx < \epsilon^2$ for $t \geq t_1$. Now

$$\begin{aligned} V(t_1) \leq V(t) &\leq \Big(\int_0^1 u^2 dx \int_0^1 u_t^2 dx\Big)^{1/2} \\ &\leq \epsilon\Big(\int_0^1 u_t^2\Big)^{1/2}. \end{aligned}$$

Hence

$$(V(t_1)/\epsilon)^2 \leq \int_0^1 u_t^2(t,x)dx.$$

Then

$$V'(t) \geq (V(t_1)/\epsilon)^2$$

and so

$$\epsilon^2 \geq V(t) \geq V(t_1) + (V(t_1)/\epsilon)^2(t - t_1)$$

for all $t \geq t_1$ is a contradiction.

Because of the special form of this equation, the result is actually stronger than its ODE counterpart using the Chetayev theorem. We now give a simple generalization of Chetayev's theorem to abstract equations.

Consider the ordinary differential equation

$$u'(t) = F(t, u(t)), \quad F(t, 0) = 0, \tag{5}$$

in a Banach space X with norm $|\cdot|_X$.

Theorem 4. *Let A be an open subset of X with $O \in \partial A$ and let $B > 0$. Suppose that $V : \overline{A} \to R^+$, that $V(u)$ is bounded on $\{u \in A : |u|_X \leq B\}$, that $V(u) > 0$ for $u \in A$ and $|u|_X \leq B$, and that $V(u) = 0$ for $u \in \partial A$ and $|u|_X \leq B$. In addition, suppose that $V'_{(5)}(u(t)) \geq \alpha(t)W(u(t))$ for $u \in A$ and $|u|_X \leq B$ where $\alpha(t) \geq 0$, $\int_0^\infty \alpha(t)dt = \infty$, and $W(u) \geq 0$ for $u \in A$. Moreover, suppose that for any $\mu > 0$ there exists $\tilde{\mu} > 0$ such that $[u \in A,\ V(u) \geq \mu]$ imply that $W(u) \geq \tilde{\mu}$. Then the zero solution of (5) is unstable.*

Proof. If the theorem is false, then for $\epsilon = B/2$ there is a $\delta > 0$ such that $|u_0|_X < \delta$ and $t > 0$ imply that $|u(t, 0, u_0)|_X < \epsilon$, where $u(t, 0, u_0)$ is a solution satisfying $u(0, 0, u_0) = u_0$; we also let $u(t, 0, u_0) = u(t)$. Choose $u_0 \in A$, $|u_0|_X = \delta/2$. Then $V(u_0) > 0$ and so long as $u(t, 0, u_0) \in A$ we have $V'(u(t, 0, u_0)) > 0$ so that

$$V(u(t, 0, u_0)) \geq V(u_0) > 0. \tag{6}$$

This means that $u(t) \in \{\xi \in A : |\xi|_X \leq B\}$ and there is a $\tilde{\mu} > 0$ such that $W(u(t)) \geq \tilde{\mu}$ by (6). This yields

$$V'_{(5)}(u(t)) \geq \alpha(t)W(u(t)) \geq \alpha(t)\tilde{\mu}$$

for all $t > 0$. An integration yields a contradiction to V being bounded on A whenever $|u|_X \leq B$. This completes the proof.

The reader may verify the conditions of Theorem 4 for (2), $X = H_0^1 \times H^0$, $A = \{(u, v) \in X | \int_0^1 uv\, dx > 0\}$, $B = 1$, $\alpha(t) \equiv 1$, $W(u, v) = \int_0^1 v^2 dx$ and $V(u(t), v(t)) = \int_0^1 u(t)v(t)dx$. Jensen's inequality is used in this exercise.

5. Limiting behavior

In 1893 Kneser [11] considered (3) with $f(t) \le 0$ and gave conditions to ensure the "Kneser condition" that every solution $u(t)$ satisfies $u(t) \to 0$ or $|u(t)| \to \infty$. In 1962 Utz [13], motivated by Kneser's work, considered

$$u'' = f(t)u^{2n-1}, \quad n \text{ a positive integer} \tag{7}$$

and proved the following result.

Theorem (Utz). *Let $f(t) > 0$ and continuous on $[0, \infty)$ and suppose that for each u_0, u_0' there is a unique solution on $[t_0, \infty)$ for each $t_0 \ge 0$. Then (7) has a solution $u(t) \not\equiv 0$ such that $u(t) \to 0$ and $u'(t) \to 0$, both monotonically, as $t \to \infty$.*

In view of our Theorem 2 and the continuation assumption, this result is valid only for $n = 1$; that is, (7) must be linear. Moreover, more must be added to the conditions on $f(t)$ to obtain the "Kneser condition" since $u(t) = 1 + e^{-t}$ is a solution of

$$u'' = [1/(1 + e^t)]u$$

and it tends to 1 as $t \to \infty$. In fact, equation (3), in the case $f(t) \le 0$, has the Kneser property if and only if $\int_0^\infty sf(s)ds = \infty$ ([7; p. 103] and [10; Lemma 1]). This assertion is valid for the nonlinear case too as can be seen in the same way as in [7] when things are defined as follows. Let $h : [0, \infty) \times R \to R$ be continuous and locally Lipschitz in the second variable, $h(t, u)u > 0$ for $u \ne 0$, and suppose in addition that $h(t, u)$ is monotone increasing with respect to u for fixed t. If $\int_0^\infty th(t, c)dt < \infty$ for some $c > 0$, then $u'' = h(t, u)$ has a solution $u(t)$ such that $u(t) > 0$, $u'(t) < 0$, $u'(t) \to 0$, $u(t) \to \alpha$ as $t \to \infty$.

This will motivate the next result for (2) in that we, therefore, see that more is needed on $f(t)$.

Theorem 5. *Let $f(t) \le 0$, $g(u)/u \ge \alpha$ if $u \ne 0$ for some $\alpha > 0$, and let $\int_0^\infty tf(t)dt = -\infty$. If $u(t, x)$ is a solution of (2) on $[0, \infty)$, then either*

(a) $\int_0^1 u^2(t, x)dx \to 0$ as $t \to \infty$ or

(b) $\int_0^1 u^2(t, x)dx \to \infty$ as $t \to \infty$.

Proof. Let $u(t) = u(t, x)$ be a solution of (2) on $[0, \infty)$. Then

$$(d/dt)\int_0^1 u^2(t, x)dx = \int_0^1 2uu_t dx$$

and

$$\begin{aligned}(d^2/dt^2)\int_0^1 u^2(t, x)dx &= 2\int_0^1 [u_t^2 + uu_{tt}]dx \\ &= 2\int_0^1 [u_t^2 - 2f(t)u_x g(u_x)]dx \ge 0\end{aligned}$$

after use of (2) and an integration by parts. This implies that either

$$\lim_{t\to\infty}\int_0^1 u^2dx = \infty$$

or (since the quantity is nonnegative)

$$\lim_{t\to\infty}\int_0^1 u^2dx = c,$$

where c is a nonnegative constant. We claim that c = 0 in the latter case.

Suppose that $c > 0$. Then there is a $t_1 > 0$ such that $\int_0^1 u^2dx \geq c/2$ on $[t_1, \infty)$.

(i) If there is a $t_2 \geq 0$ such that

$$(d/dt)\int_0^1 u^2dx\big|_{t=t_2} = 2\int_0^1 u(t_2,x)u_t(t_2,x)dx =: \beta > 0,$$

then it follows readily that

$$\int_0^1 u^2dx \geq \beta(t-t_2) \to \infty \text{ as } t\to\infty$$

(since the derivative is an increasing function), a contradiction.

(ii) If $(d/dt)\int_0^1 u^2(t,x)dx \leq 0$ on $[0,\infty)$, then

$$(d/dt)\int_0^1 u^2(t,x)dx = (d/dt)\int_0^1 u^2(t,x)dx\big|_{t=0}$$
$$+\,2\int_0^t\left(\int_0^1 u_t^2(s,x)dx - f(s)\int_0^1 u_x(s,x)g(u_x(s,x))dx\right)ds$$

(since $\int_0^t F''(s)ds = F'(t) - F'(0)$). As the left side is nonpositive, this implies that

$$\int_0^\infty\left(\int_0^1 u_t^2(s,x) - f(s)\int_0^1 u_x(s,x)g(u_x(s,x))dx\right)ds < \infty.$$

Clearly,

$$(d/dt)\int_0^1 u^2(t,x)dx =: F'(t) \to 0 \text{ as } t\to\infty;$$

for $F'(t) \leq 0$, $F''(t) \geq 0$, so if $F'(t) \leq -d < 0$ for all $t \geq t_2$, then $F(t) - F(t_2) \leq -d(t-t_2)$ yielding $F(t) \to -\infty$, a contradiction to $F(t) = \int_0^1 u^2dx \geq 0$. Thus, we have

$$-2\int_0^1 uu_tdx = 2\int_t^\infty\left(\int_0^1 u_t^2(s,x)dx - f(s)\int_0^1 u_x(s,x)g(u_x(s,x))dx\right)ds.$$

Let $\phi(x) = u(0, x)$, $\psi(x) = u_t(0, x)$, and then integrate the last expression from 0 to t and obtain

$$\begin{aligned}\int_0^1 \phi^2(x)dx = &\int_0^1 u^2(t,x)dx \\ &+ 2\int_0^t \int_w^\infty \Big(\int_0^1 u_t^2(s,x)dx - f(s)\int_0^1 u_x(s,x)g(u_x(s,x)\Big)dx\,ds\,dw \\ &= \int_0^1 u^2(t,x)dx + 2\int_0^t\int_w^\infty\int_0^1 u_t^2(s,x)dx\,ds\,dw \\ &- 2\int_0^t sf(s)\int_0^1 u_x(s,x)g(u_x(s,x))dx\,ds \\ &- 2t\int_t^\infty f(s)\int_0^1 u_x(s,x)g(u_x(s,x))dx\,ds \\ &\geq -2\alpha\int_0^t sf(s)\int_0^1 u_x^2(s,x)dx\,ds.\end{aligned}$$

But $c/2 \leq \int_0^1 u^2dx$ on $[t_1, \infty)$ so $c/2 \leq \int_0^1 u^2dx \leq \int_0^1 u_x^2dx$, together with $\int_0^\infty sf(s)ds = -\infty$ now yields $\int_0^1 \phi^2(x)dx = \infty$, a contradiction. This completes the proof.

6. Decay of solutions and limit circle

Another classical problem is concerned with giving conditions on $f(t)$ in (3) to ensure that all solutions tend to zero. The literature is vast, but one may loosely state that it is sufficient to ask that $f(t) \to \infty$ monotonically and that $f'(t)/f^{3/2}(t)$ be bounded (cf. [3]). (It is not sufficient that $f(t) \to \infty$, as may be seen in [9].) But if one asks a bit more, then a trivial proof is available [4]. It goes as follows.

First, define a Liouville transformation $s = \int_0^t \sqrt{f(v)}dv$, $u(t) = w(s)$ and map (3) into $\ddot{w}(s) + [f'(t)/2f^{3/2}(t)]\dot{w}(s) + w(s) = 0$, where $\cdot = d/ds$. Let $\mu(s) = f'(t)/4f^{3/2}(t)$ and then define a system

$$\begin{cases} \dot{w} = z - \mu(s)w \\ \dot{z} = -w - \mu\dot{w} + \dot{\mu}w. \end{cases}$$

Define a Liapunov function

$$V(s) = w^2 + z^2$$

and obtain

$$\begin{aligned}\dot{V}(s) &\leq -2\mu[w^2 + z^2] + |\mu^2 + \dot{\mu}|(z^2 + w^2) \\ &= [-2\mu + |\mu^2 + \dot{\mu}|]V(s)\end{aligned}$$

so that if

$$\int_0^\infty [-2\mu(s) + |\mu^2(s) + \dot{\mu}(s)|]ds = -\infty,$$

then every solution tends to zero.

Precisely the same sort of thing works for (2) and it also leads to a limit circle result. Preparatory to proving that theorem, recall that $\int_0^1 u_x^2 dx \geq |u|_\infty^2 \geq \int_0^1 u^2 dx$ when $u(t,0) = u(t,1) = 0$. Thus, when $rg(r) \geq \alpha r^2$ we will have $G(x) = \int_0^x g(s)ds$ and $\int_0^1 G(u_x)dx \geq (\alpha/2)\int_0^1 u_x^2 dx \geq (\alpha/2)|u|_\infty^2$.

Theorem 6. *Let $g'(r) \geq 0$ for all r, $f(t) > 0$, $wg(w) \geq \alpha w^2$ for some $\alpha > 0$, $\alpha\pi^2 \leq 1$. Suppose that for $s = \int_0^t \sqrt{f(v)}dv$, for $a > 0$ and large, and for $\mu(s) = f'(t)/4f^{3/2}(t)$ we have*

$$\int_0^\infty [-2\mu(s) + (|\dot{\mu}(s) + \mu^2(s)|/\alpha\pi^2)]ds = -\infty \tag{8}$$

and $2\mu(s) \geq |\dot{\mu}(s) + \mu^2(s)|/\alpha\pi^2$ for t sufficiently large. Then any solution $u(t,x)$ of (2) defined on $[0,\infty)$ satisfies

$$\int_0^1 G(u_x(t,x))dx \to 0 \text{ as } t \to \infty.$$

Proof. First, the Liouville transformation

$$s = \int_0^t \sqrt{f(v)}dv \text{ and } w(s,x) = u(t,x) \tag{9}$$

yields

$$u_t = w_s(ds/dt) = w_s\sqrt{f(t)}$$

and

$$u_{tt} = w_{st}\sqrt{f(t)} + w_s\big(f'(t)/2\sqrt{f(t)}\big)$$

so

$$u_{tt} = w_{ss}f(t) + w_s\big(f'(t)/2\sqrt{f(t)}\big).$$

Thus, (2) becomes

$$w_{ss} = g(w_x)_x - [f'(t)/2f^{3/2}(t)]w_s, \quad w(s,0) = w(s,1) = 0.$$

And this is equivalent to the system

$$\begin{cases} w_s = z - [f'(t)/4f^{3/2}(t)]w \\ z_s = g(w_x)_x - [f'(t)/4f^{3/2}(t)]w_s + (d/ds)[f'(t)/4f^{3/2}(t)]w. \end{cases}$$

This can be written as

$$\begin{cases} w_s = z - \mu(s)w \\ z_s = g(w_x)_x - \mu(s)w_s + \dot{\mu}w. \end{cases} \tag{10}$$

With $G(r) = \int_0^r g(s)ds$, define a Liapunov function

$$V(s) = \int_0^1 [2G(w_x) + z^2]dx$$

and obtain the derivative of V along a solution as

$$\begin{aligned}\dot{V} &= \int_0^1 (2g(w_x)w_{xs} + 2zz_s)dx \\ &= \int_0^1 \{-2g(w_x)_x w_s + 2z[g(w_x)_x - \mu w_s + \dot{\mu} w]\}dx\end{aligned}$$

(by the induced boundary conditions: $w_s(s,0) = w_s(s,1) = 0$)

$$\begin{aligned}&= \int_0^1 \{-2g(w_x)_x[z - \mu w] + 2zg(w_x)_x - 2\mu z w_s + 2\dot{\mu} z w\}dx \\ &= \int_0^1 [2\mu g(w_x)_x w - 2\mu z w_s + 2\dot{\mu} z w]dx \\ &= \int_0^1 [-2\mu g(w_x)w_x - 2\mu z(-\mu w + z) + 2\dot{\mu} z w]dx \\ &= \int_0^1 \{-2\mu[g(w_x)w_x + z^2] + 2\mu^2 z w + 2\dot{\mu} z w\}dx.\end{aligned}$$

Now

$$\int_0^1 w^2 dx \le (1/\pi^2)\int_0^1 w_x^2 dx \le (1/\alpha\pi^2)\int_0^1 g(w_x)w_x dx$$

and since $G(w_x) \le g(w_x)w_x$ we have

$$\begin{aligned}\dot{V} &\le \int_0^1 \{-2\mu[g(w_x)w_x + z^2] + |\mu^2 + \dot{\mu}|(z^2 + w^2)\}dx \\ &\le \int_0^1 \{-2\mu[g(w_x)w_x + z^2] + |\mu^2 + \dot{\mu}|(z^2 + g(w_x)w_x/\alpha\pi^2)\}dx \\ &\le (1/2)\int_0^1 \{-2\mu + (|\mu^2 + \dot{\mu}|/\alpha\pi^2)\}\{2G(w_x) + z^2\}dx\end{aligned}$$

or

$$\dot{V} \le \{-2\mu + (|\mu^2 + \dot{\mu}|/\alpha\pi^2)\}V/2 \tag{11}$$

for t sufficiently large since $2\mu \ge |\mu^2 + \dot{\mu}|/\alpha\pi^2$ for t sufficiently large. The conclusion follows from this.

Note that the integral in the theorem, when changed to the variable t, is

$$\begin{aligned}&\int_0^\infty [-2\{f'(t)/4f^{3/2}(t)\} + |\{(f'(t))^2/16\alpha\pi^2 f^3(t)\} \\ &\qquad + \{2f^{3/2}(t)f''(t) - 3(f'(t))^2\sqrt{f(t)}\}/8\alpha\pi^2 f^3(t)|]\sqrt{f(t)}dt \\ &\quad = \int_0^\infty [\{-f'(t)/2f(t)\} + |\{(f'(t))^2/16\alpha\pi^2 f^{5/2}(t)\} \\ &\qquad + \{2f(t)f''(t) - 3(f'(t))^2\}/8\alpha\pi^2 f^2(t)|]dt.\end{aligned}$$

Example 1. Let $f(t) = e^t$ and $\alpha\pi^2 > 1/4$ so that

$$\begin{aligned}\int_0^\infty &[-2\mu(s) + |\dot{\mu}(s) + \mu^2(s)|/\alpha\pi^2]ds \\ &= \int_0^\infty [-(1/2) + |(1/16\alpha\pi^2 e^{t/2}) - (1/8\alpha\pi^2)|]dt \\ &= -\infty.\end{aligned}$$

Example 2. Let $f(t) = \ln(1+t)$ so that

$$\begin{aligned}\int_1^\infty &[-2\mu(s) + |\dot{\mu}(s) + \mu^2(s)|/\alpha\pi^2]ds \\ &= \int_1^\infty [-(1/2(1+t)\ln(1+t)) + |\{1/16\alpha\pi^2(1+t)^2(\ln(1+t))^{5/2}\} \\ &\quad - [(2\ln(1+t)+3)/8\alpha\pi^2(1+t)^2(\ln(1+t))^2|]dt \\ &= -\infty.\end{aligned}$$

Example 3. Let $f(t) = (1+t)^\beta$, $\beta > 0$. Then

$$\begin{aligned}\int_0^\infty &[-2\mu(s) + |\dot{\mu}(s) + \mu^2(s)|/\alpha\pi^2]ds \\ &= \int_0^\infty [-(\beta/2(1+t)) + |(\beta^2/16\alpha\pi^2(1+t)^2(1+t))^{\beta/2}) \\ &\quad - (\beta(2+\beta)/8\alpha\pi^2(1+t)^2|]dt = -\infty.\end{aligned}$$

From (11) it is very easy to obtain a result on the classical question of limit point-limit circle. If all solutions of (3) are in $L^2[0,\infty)$, then (3) is said to be in the limit circle case, otherwise it is in the limit point case. The terminology is explained, for example, in Coddington and Levinson [7; pp. 225-6]. The literature on the problem is vast and the reader is referred to Devinatz [8].

Definition. Equation (2) is in the limit circle case if every solution $u(t,x)$ defined on $[0,\infty)$ satisfies $\int_0^\infty \int_0^1 u_x g(u_x)dxdt < \infty$.

The next result is an exact counterpart of [5] for (2).

Theorem 7. *Let the conditions of Theorem 6 hold, let $G(r) \geq \beta r g(r)$ for some $\beta > 0$, and let*

$$\begin{aligned}\int_0^\infty &\{(1/\sqrt{f(t)})\exp(1/8\alpha\pi^2)\int_0^t |[f'(x))^2/2f^{5/2}(x)] \\ &+ [2f(x)f''(x) - 3(f'(x))^2]/f^2(x)|dx\}dt < \infty. \qquad (12)\end{aligned}$$

Then (2) is in the limit circle case.

Proof. We have

$$\beta \int_0^1 u_x g(u_x)dx \le \int_0^1 [G(u_x) + z^2]dx \le V(s)$$

and (11). The result now follows by integration of the bound on V obtained from integration of (11).

The next result extends [3] for (1) to (2). One may note that $f(t) = (1+t)^\beta$, $\beta > 0$, satisfies all conditions of this theorem.

Theorem 8. *Suppose that $f'(t) \ge 0$, $f(t) > 0$, $f'(t)/f^{3/2}(t) \le \gamma$ for some $\gamma > 0$, and there is a nonnegative decreasing function $\mu(t)$ such that $f'(t) \ge \mu(t)f(t)$ and $\int_0^\infty \mu(t)dt = \infty$. Let $ug(u) > 0$ if $u \ne 0$, $ug(u) \ge \alpha G(u)$ for some $\alpha > 0$ where $G(u) = \int_0^u g(s)ds$. If $u(t) = u(t,x)$ is a solution of (2) on $[0,\infty)$ with $\int_0^1 u_{xx}^2(t,x)dx \le M$ for some $M > 0$ and all $t \ge 0$, then*

$$\int_0^1 G(u_x(t,x))dx + (1/f(t)) \int_0^1 u_t^2(t,x)dx \to 0 \text{ as } t \to \infty.$$

Proof. Let

$$V(t) = 2\int_0^1 G(u_x(t,x))dx + (1/f(t)) \int_0^1 u_t^2(t,x)dx.$$

Then by using the induced boundary conditions we obtain

$$V'(t) = -[f'(t)/f^2(t)] \int_0^1 u_t^2(t,x)dx.$$

Let

$$y(t) = \left(\int_0^1 u_t^2(t,x)dx\right)^{1/2} / \sqrt{f(t)};$$

then

$$\begin{cases} V(t) = 2\int_0^1 G(u_x(t,x))dx + y^2(t), \\ V'(t) = -[f'(t)/f(t)]y^2(t). \end{cases} \tag{13}$$

Now

$$\liminf_{t\to\infty} y(t) = 0; \tag{14}$$

for if $y^2(t) \ge \delta > 0$ on some interval $[t_1, \infty)$, then

$$V'(t) \le -[f'(t)/f(t)]\delta^2 \text{ on } [t_1, \infty)$$

and a contradiction results from the properties of f. Suppose that

$$\limsup_{t\to\infty} y(t) = \lambda > 0;$$

then $\lim_{t\to\infty} V(t) = c > 0$, c constant. Let $\delta = \min\{1, \lambda/2, 3c\alpha/8(1+\alpha+\gamma\sqrt{M})\}$; then there are sequences $\{t_n\}$, $\{t_n'\}$ having the following properties: $t_n < t_n' \le t_{n+1}$, $y(t_n) = y(t_n') = \delta/2$, $y(t) > \delta/2$ on (t_n, t_n') with $\max_{s\in[t_n,t_n']} y(s) > \delta$, while $y(t) \le \delta$ on $[t_n', t_{n+1}]$. To see that such sequences exist, let t_0 be defined such that $t_0 > 0$, $y(t_0) < \delta/2$ and consider the open set $\{t > t_0,\ y(t) > \delta/2\}$. It follows that $\{t > t_0,\ y(t) > \delta/2\}$ is a union of countable disjoint open intervals H_i $(i = 1, 2, \ldots)$; that is, $\{t > t_0,\ y(t) > \delta/2\} = \bigcup_{j=1}^{\infty} H_j$. Define $H = \{H_j$: there exists $t_j^* \in H_j$ such that $y(t_j^*) > \delta\}$. Since $y'(t)$ is continuous, and consequently bounded on any finite interval, we may assume that $H = \bigcup_{j=1}^{\infty}(a_j, b_j)$, with $a_j < b_j \le a_{j+1}$, $j = 1, 2, \ldots$. Then $y(t) > \delta/2$ on (a_j, b_j), $\max_{a_j\le t\le b_j} y(t) > \delta$, $y(a_j) = y(b_j) = \delta/2$, and $y(t) \le \delta$ on $[b_j, a_{j+1}]$, $j = 1, 2, \ldots$. So $t_n = a_n$, $t_n' = b_n$ are the required sequences.

We shall show that

$$\int_{t_n'}^{t_{n+1}} \sqrt{f(t)}dt \le k \int_{t_n}^{t_n'} \sqrt{f(t)}\,dt, \quad n = 1, 2, \ldots \tag{15}$$

where $k > 0$ is a fixed constant. To that end we first note that

$$\begin{aligned} |(d/dt)y^2(t)| &= \left| -[f'(t)/f^2(t)] \int_0^1 u_t^2 dx + 2\int_0^1 u_t(g(u_x))_x dx \right| \\ &\leqq [f'(t)/f(t)]y^2(t) + \big(2/\sqrt{f(t)}\big)\left(\int_0^1 u_t^2 dx\right)^{1/2} \sqrt{f(t)} \\ &\quad \times \left(\int_0^1 \big(g(u_x)\big)_x^2 dx\right)^{1/2} \end{aligned}$$

and that

$$|(d/dt)y(t)| = [f'(t)/2f^{3/2}(t)]\sqrt{f(t)}y(t) + \sqrt{f(t)}\left(\int_0^1 \big(g(u_x)\big)_x^2 dx\right)^{1/2}$$

if $y(t) \ne 0$.

Now $[f'(t)/f^{3/2}(t)]$ and $y(t)$ are bounded, while $\left(\int_0^1 \big(g(u_x)\big)_x^2 dx\right)^{1/2}$ is bounded since $\int_0^1 u_{xx}^2 dx$ is bounded. Hence, there exists a constant $k_1 > 0$ such that $|y'(t)| \le k_1\sqrt{f(t)}$. This implies that

$$(\delta/2) \le \int_{t_n}^{t_n'} |y'(s)|ds \le k_1 \int_{t_n}^{t_n'} \sqrt{f(s)}ds. \tag{16}$$

On the other hand,

$$
\begin{aligned}
(d/dt)\int_0^1 uu_t dx &= \int_0^1 u_t^2 dx - f(t)\int_0^1 u_x g(u_x)dx \\
&= f(t)y^2(t) - f(t)\int_0^1 u_x g(u_x)dx.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\alpha f(t)\int_0^1 G(u_x(t,x))dx &\le f(t)\int_0^1 u_x g(u_x)dx \\
&\le f(t)y^2(t) - (d/dt)\int_0^1 uu_t dx.
\end{aligned}
$$

As $V(t)\to c>0$ when $t\to\infty$, without loss of generality we may assume that $V(t)\ge 3c/4$ for $t\ge t_1$. Then

$$
\begin{aligned}
2\int_{t_n'}^{t_{n+1}} \sqrt{f(s)}\int_0^1 G(u_x(s,x))dx\,ds &\ge \int_{t_n'}^{t_{n+1}} [(3c/4)-y^2(s)]\sqrt{f(s)}ds \\
&\ge \int_{t_n'}^{t_{n+1}} \sqrt{f(s)}[(3c/4)-\delta^2]ds.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\alpha\int_{t_n'}^{t_{n+1}} \sqrt{f(s)}ds[(3c/8)-(\delta^2/2)] &\le \alpha\int_{t_n'}^{t_{n+1}} \sqrt{f(s)}\int_0^1 G(u_x(s,x))dx \\
\le \int_{t_n'}^{t_{n+1}} \sqrt{f(s)}y^2(s)ds &- \int_{t_n'}^{t_{n+1}} \{[(d/dt)\int_0^1 uu_t dx)]/\sqrt{f(s)}\}ds
\end{aligned}
$$

and so

$$
\begin{aligned}
&[(3c\alpha/8)-(1+(\alpha/2))\delta^2)]\int_{t_n'}^{t_{n+1}} \sqrt{f(s)}ds \\
&\quad\le -\int_{t_n'}^{t_{n+1}} \big(1/\sqrt{f(s)}\big)[(d/dt)\int_0^1 u(s,x)u_t(s,x)dx]ds \\
&\quad= \big(1/\sqrt{f(t_n')}\big)\int_0^1 u(t_n')u_t(t_n')dx - \big(1/\sqrt{f(t_{n+1})}\big)\int_0^1 u(t_{n+1},x)u_t(t_{n+1},x)dx \\
&\qquad - 1/2\int_{t_n'}^{t_{n+1}} [f'(s)/f^{3/2}(s)]\int_0^1 u(s,x)u_t(s,x)dx\,ds \\
&\quad\le \left(\int_0^1 u^2(t_n',x)dx\right)^{1/2} y(t_n') + \left(\int_0^1 u^2(t_{n+1},x)dx\right)^{1/2} y(t_{n+1}) \\
&\qquad + \gamma\sqrt{M}\,\delta\int_{t_n'}^{t_{n+1}} \sqrt{f(s)}ds
\end{aligned}
$$

where $\gamma > 0$ is defined in the theorem. By the definition of δ and the boundedness of u and $y(t)$, we have

$$\int_{t'_n}^{t_{n+1}} \sqrt{f(s)}ds \leq \beta, \text{ for some } \beta > 0, \quad n = 1, 2, \dots . \tag{17}$$

By (16) and (17), (15) follows.

Since $f'(t) \geq 0$, we have

$$t_{n+1} - t'_n \leq k(t'_n - t_n). \tag{18}$$

Let $t > t_n$; then

$$\begin{aligned} V(t) &\leq V(t_1) - \int_{t_1}^{t_n} [f'(s)/f(s)]y^2(s)ds \\ &\leq V(t_1) - (\delta^2/4) \sum_{j=1}^{n} \int_{t_j}^{t'_j} [f'(s)/f(s)]ds \\ &\leq V(t_1) - (\delta^2/4) \sum_{j=1}^{n} \int_{t_j}^{t'_j} \mu(s)ds \\ &\leq V(t_1) - (\delta^2/8) \sum_{j=1}^{n} \left(\int_{t_j}^{t'_j} \mu(s)ds + 1/k \int_{t'_j}^{t_{j+1}} \mu(s)ds \right) \\ &\leq V(t_1) - (\delta^2/8) \min\{1, 1/k\} \int_{t_1}^{t_{n+1}} \mu(s)ds \to \infty \text{ as } n \to \infty, \end{aligned}$$

a contradiction. This implies that $y(t) \to 0$ as $t \to \infty$.

Also,

$$(d/dt) \int_0^1 uu_t dx = \int_0^1 u_t^2 dx - f(t) \int_0^1 u_x g(u_x) dx$$

so that

$$\int_0^1 u_x g(u_x) dx = y^2(t) - (1/f(t))(d/dt) \int_0^1 uu_t dx. \tag{19}$$

Using the facts that $\int_0^1 u^2 dx$ and $[f'(t)/f^{3/2}(t)]$ are bounded, that $f(t) \to \infty$ as $t \to \infty$, and that $y(t) \to 0$ as $t \to \infty$, it follows that

$$\liminf_{t \to \infty} \int_0^1 u_x(t,x) g(u_x(t,x)) dx = 0.$$

In fact, suppose that there exist $c > 0$, $t_1 > 0$ such that

$$\int_0^1 u_x(t,x) g(u_x(t,x)) dx \geq c > 0$$

on $[t_1,\infty)$. We may assume that $|y(t)| \le \min\{\sqrt{c/2}, c/4\gamma M\}$ for $t \ge t_1$. From (19) we have

$$c \le (c/2) - (1/f(t))[(d/dt)\int_0^1 uu_t dx].$$

Thus

$$\begin{aligned}
c/2(t-t_1) &\le -\int_{t_1}^t \{(1/f(s))(d/ds)\int_0^1 u(s,x)u_t(s,x)dx\}ds \\
&= (1/f(t_1))\int_0^1 u(t_1,x)u_t(t_1,x)dx - (1/f(t))\int_0^1 u(t,x)u_t(t,x)dx \\
&\quad - \int_{t_1}^t \{(f'(s)/f^2(s))\int_0^1 u(s,x)u_t(s,x)dx\}ds \\
&\le \{My(t_1)/\sqrt{f(t_1)}\} + \big(My(t)/\sqrt{f(t)}\big) \\
&\quad + \int_{t_1}^t \{(f'(s)/f^{3/2}(s))\Big(\int_0^1 u^2(s,x)dx\Big)^{1/2}\Big(\int_0^1 u_t^2(s,x)dx\Big)^{1/2}/\sqrt{f(s)}\}ds \\
&\le \big(My(t_1)/\sqrt{f(t_1)}\big) + \big(My(t)/\sqrt{f(t)}\big) + \gamma M\int_{t_1}^t y(s)ds \\
&\le \big(My(t_1)/\sqrt{f(t_1)}\big) + \big(My(t)/\sqrt{f(t)}\big) + [c/4(t-t_1)].
\end{aligned}$$

This yields

$$\big(My(t_1)/\sqrt{f(t_1)}\big) + \big(My(t)/\sqrt{f(t)}\big) \ge c/4(t-t_1)$$

which tends to infinity, a contradiction.

As the

$$\liminf_{t\to\infty}\int_0^1 u_x(t,x)g(u_x(t,x))dx = 0$$

we argue that

$$\lim_{t\to\infty}\int_0^1 G(u_x(t,x))dx = 0.$$

Since

$$y^2(t) + 2\int_0^1 G(u_x(t,x))dx \to c$$

as $t\to\infty$, we conclude that $c=0$. This completes the proof.

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Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901-4408

Bolyai Institute, Aradi Vértanúk tere 1, Szeged, Hungary

Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901-4408 (On leave from Northeast Normal University, Changchun, Jilin, PRC)

Current address: Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901-4408

E-mail address: GE0641@SIUCVMB.BITNET