PERIODICITY IN DELAY EQUATIONS BY DIRECT FIXED POINT MAPPING

T. A. Burton
Department of Mathematics
Southern Illinois University
Cabondale, IL 62901-4408

Bo Zhang
Department of Mathematics and Computer Science
Fayetteville State University
Fayetteville, NC 28301-4298

Abstract

Schaefer's fixed point theorem or degree theory is used to study the existence of periodic solutions in functional differential equations $x' = F(t, x_t)$ by constructing a compact homotopy. The construction of such a homotopy is very difficult in practice for nonlinear equations. In this paper we use the direct fixed point mapping technique to link the homotopy to the right-hand side of the equation directly and avoid those difficulties. Applications to linear and nonlinear systems are given. This appeared in Differential Equations and Dynamical Systems, Vol. 6, No. 4, (1998), p. 413-424.

1 Introduction.

We consider the system of functional differential equations

$$x'(t) = F(t, x_t) \tag{1.1}$$

in which $F : R \times BC \to R^n$ is continuous and T-periodic in t. BC is the space of bounded continuous functions $\phi : (-\infty, 0] \to R^n$ with the supremum norm $\|\cdot\|$. For

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each $t \in R$, x_t is defined by $x_t(s) = (t+s)$ for $s \leq 0$.

The existence of periodic solutions of Eq.(1.1) has been the subject of extensive investigations for many years. Our interest here centers on degree-theoretic results, including Schaefer's fixed point theorem. Examples of these may be found in Burton, Eloe, and Islam [2,3], Erbe, Krawcewicz, and Wu [4], Gustafson and Schmitt [5], Hale and Mawhin [6], Krasnoselskii [7], Mawhin [8], Sadovski [9], Serra [10], and Zhang [13]. A common method consists of writing the differential equation as an integral equation which then defines a mapping; if the mapping has a fixed point, then it is a solution of the differential equation. In this paper we use the direct fixed point mapping technique introduced in Burton [1] to construct a homotopy directly from $F(t, \phi)$. This involves writing the solution as an integral equation and it eliminates many of the problems encountered in writing the differential equation as an integral equation. The main difficulty is in selecting the constant of integration. Several different kinds of examples are given which illustrate methods of finding that constant.

Let $R^-,\ R^+,\ R$ denote the intervals $(-\infty, 0], [0, +\infty)$, and $(-\infty, +\infty)$ respectively. $|\cdot|$ denotes the Euclidean norm on R^n. Let $(P_T, \|\cdot\|)$ be the Banach space of continuous T-periodic functions $\phi : R \to R^n$ with the supremum norm and

$$P_T^0 = \{\phi \in P_T | \int_0^T \phi(s)ds = 0\}.$$

2 The main result.

Our result rests on a fixed point theorem of Schaefer [11]. Its relation to Leray-Schauder degree theorem is explained in Smart [12].

Theorem A (Schaefer). Let V be a normed space, H a continuous mapping of V into V which is compact on each bounded subset of V. Then either

(i) the equation $x = \lambda H x$ has a solution for $\lambda = 1$, or

(ii) the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.

Theorem 2.1. Suppose that the following conditions hold.

(i) for each $\phi \in P_T^0$, there is a constant $k_\phi \in R$ such that $\int_0^T F(t, \Phi_t)dt = 0$ where $\Phi(t) = k_\phi + \int_0^t \phi(s)ds$ for each $t \in R$,

(ii) $E : P_T^0 \to P_T$ defined by $E(\phi)(t) = \Phi(t)$ in (i) is continuous and for each $\alpha > 0$, there exists a constant $L_\alpha > 0$ such that $|k_\phi| \le L_\alpha$ whenever $\|\phi\| \le \alpha$.

(iii) there is a constant $B > 0$ such that $\|\phi\| < B$ whenever ϕ is a fixed point of $G_\lambda : P_T^0 \to P_T^0$ defined by $G_\lambda(\phi)(t) = \lambda F(t, \Phi_t)$ for $0 < \lambda < 1$.

Then Eq.(1.1) has a T-periodic solution.

Proof. Let G_λ be defined in (iii). It follows from (i) and the continuity of F that $G_\lambda(\phi) \in P_T^0$. For each $\alpha > 0$, consider $\{E(\phi) : \phi \in P_T^0, \|\phi\| \le \alpha\}$. This set is uniformly bounded by (ii) and is equicontinuous by the definition of Φ. Thus, E is compact by Ascoli-Arzela's theorem. This implies that G_λ is compact since F is continuous. By (iii), $\|\phi\| < B$ whenever ϕ is a fixed point of G_λ. Applying Schaefer's theorem with $\lambda H = G_\lambda$, we conclude that G_λ has a fixed point ϕ for $\lambda = 1$. That is $\phi = G_1(\phi)$ or $\Phi'(t) = F(t, \Phi_t)$. Thus, Φ is a T-periodic solution of Eq.(1.1) and the proof is complete.

Corollary 2.1. Suppose that conditions (i), (ii) of Theorem 2.1 hold and

(iv) $F : R \times P_T \to R^n$ maps bounded sets into bounded sets,

(v) there exists a constant $B > 0$ such that $\|x\| < B$ whenever $x = x(t)$ is a T-periodic solution of

$$x'(t) = \lambda F(t, x_t),\ \lambda \in (0, 1). \tag{2.1}$$

Then Eq.(1.1) has a T-periodic solution.

Proof. We need to show that condition (iii) of Theorem 2.1 holds. Notice that any fixed point ϕ of G_λ corresponds to a T- periodic solution of Eq.(2.1). By (iv), there exists a constant $L = L(B) > 0$ such that $|F(t, \Phi_t)| \le L$ whenever $\|\Phi\| \le B$. If ϕ is a fixed point of G_λ, then $\phi(t) = \Phi'(t) = \lambda F(t, \Phi_t)$. By (v), we have $\|\Phi\| < B$. Thus $\|\phi\| < L$ and (iii) of Theorem 2.1 is satisfied.

Corollary 2.2. Suppose that conditions (i), (ii) of Theorem 2.1 hold and there exist positive constants $M, q,\ 0 < q < 1$, such that

(vi) $|F(t, \Phi_t)| \le q\|\phi\| + M$, for all $\phi \in P_T^0$

where Φ is defined in (i). Then Eq.(1.1) has a T-periodic solution.

Proof. Let ϕ be a fixed point of G_λ. Then

$$|\phi(t)| = \lambda |F(t, \Phi_t)| \le q\|\phi\| + M.$$

Thus, $\|\phi\| \le M/(1-q)$. By Theorem 2.1, Eq.(1.1) has a T-periodic solution.

Finally in this section, we consider the equation

$$x'(t) = L(t, x_t) + p(t) \tag{2.2}$$

where $L : R \times BC \to R^n$ is continuous, linear in ϕ, T-periodic in t, and $p \in P_T$.

Theorem 2.2. Supoose there is an $n \times n$ matrix $L(t, \cdot)$ such that for every $k \in R^n$ there is the relation $L(t, \cdot)k = L(t, k)$. If the linear function $\int_0^T L(t, \cdot)dt$ is invertible and

(v*) there exists a constant $B > 0$ such that $\|x\| < B$ whenever $x = x(t)$ is a T-periodic solution of

$$x'(t) = \lambda[L(t, x_t) + p(t)],\ \lambda \in (0, 1). \tag{2.3}$$

Then Eq.(2.2) has a T-periodic solution.

Proof. Define $F(t, \phi_t) = L(t, \phi_t) + p(t)$. In view of Corollary 2.1, we need to verify conditions (i) and (ii) of Theorem 2.1. Let $\phi \in P_T^0$ and $k \in R^n$. Consider

$$\int_0^T L\Big(t, (k + \int_0^t \phi(s)ds)_t\Big)dt + \int_0^T p(s)ds = 0.$$

Since L is linear with respect to the second argument, we have

$$\int_0^T L(t, k)dt + \int_0^T L\Big(t, (\int_0^t \phi(s)ds)_t\Big)dt + \int_0^T p(s)ds = 0.$$

Thus,

$$k = \Big(\int_0^T L(t, \cdot)dt\Big)^{-1}\Big[-\int_0^T L\Big(t, (\int_0^t \phi(s)ds)_t\Big)dt - \int_0^T p(s)ds\Big].$$

We designate that unique constant as k_ϕ. It is clear that $E : P_T^0 \to P_T$ defined by $E(\phi) = \Phi$ with $\Phi(t) = k_\phi + \int_0^t \phi(s)ds$ is continuous and $\int_0^T F(t, \Phi_t)dt = 0$. Moreover, there exists a constant $\gamma > 0$ such that $|L(t, \psi_t)| \le \gamma\|\psi\|$ for any $\psi \in BC$ since L is continuous and linear in ψ. Thus,

$$\Big|L\Big(t, (\int_0^t \phi(s)ds)_t\Big)\Big| \le \gamma T\|\phi\|$$

and

$$|k_\phi| \le \Big|\Big(\int_0^T L(t,\cdot)dt\Big)^{-1}\Big|(\gamma T\alpha + \|p\|)T =: L_\alpha$$

for $\|\phi\| \le \alpha$. This completes the proof.

Remark 2.1. If $L(t,x_t) = A(t)x(t) + \int_{-\infty}^t B(t,s)x(s)ds + \sum_{k=1}^{+\infty} A_k(t)x(t-h_k)$, where $A(t), B(t,s)$, and $A_k(t)$ are $n \times n$ matrices and $h_k > 0$, then $\int_0^T L(t,\cdot)dt$ is invertible if and only if the matrix

$$\int_0^T \Big(A(t) + \int_{-\infty}^t B(t,s)ds + \sum_{k=1}^{+\infty} A_k(t)\Big)dt$$

has an inverse.

3 Examples.

In this section, we give several examples to illustrate how to apply Theorem 2.1 to some linear and nonlinear delay equations. Our emphasis will be on proving the existence of k_ϕ described in Theorem 2.1 and the use Liapunov functions to derive a priori bounds on periodic solutions. The examples are shown in simple forms for illustrative purposes and they can be easily generalized.

Example 3.1. Consider the scalar equation

$$x'(t) = a(t)x(t) + b(t)x(t-h) + p(t) \tag{3.1}$$

where $h \ge 0$ and a, b, p are continuous and T-periodic. If $a(t)$ is of one sign with $\int_0^T a(t)dt \ne 0$ and there exists a constant $N > 1$ such that $|a(t)| - N|b(t+h)| \ge 0$, then Eq.(3.1) has a T-periodic solution.

Proof. Let $L(t,x_t) = a(t)x(t) + b(t)x(t-h)$. Then $L(t,\cdot) = a(t) + b(t)$. We claim that $\int_0^T L(t,\cdot)dt \ne 0$. Without loss of generality, we assume that $a(t) \ge 0$. Since $b(t+h)$ is also T-periodic, it follows that

$$\int_0^T |b(t)|dt = \int_{-h}^{T-h} |b(s+h)|ds = \int_0^T |b(s+h)|ds.$$

Using the condition $|a(t)| - N|b(t+h)| \ge 0$, we have

$$\int_0^T \Big(a(t) + b(t)\Big)dt \ge \int_0^T \Big(|a(t)| - |b(t)|\Big)dt = \int_0^T \Big(|a(t)| - |b(t+h)|\Big)dt$$

$$= \frac{N-1}{N}\int_0^T a(t)dt + \frac{1}{N}\int_0^T \Big(a(t) - N|b(t+h)|\Big)dt \geq \frac{N-1}{N}\int_0^T a(t)dt > 0.$$

Next, let $V(x) = |x|$ and $x = x(t)$ be a T-periodic solution of

$$x'(t) = \lambda[a(t)x(t) + b(t)x(t-h) + p(t)]. \tag{3.2}$$

Then

$$V'(x(t)) \geq \lambda[a(t)|x(t)| - |b(t)||x(t-h)| - |p(t)|].$$

Integrate the above inequality from 0 to T to obtain

$$\begin{aligned} 0 &= V(x(T)) - V(x(0)) \\ &\geq \lambda\Big[\int_0^T a(t)|x(t)|dt - \int_0^T |b(t)||x(t-h)|dt - \int_0^T |p(t)|dt\Big] \\ &= \lambda\Big[\int_0^T a(t)|x(t)|dt - \int_{-h}^{T-h} |b(t+h)||x(t)|dt - \int_0^T |p(t)|dt\Big] \end{aligned}$$

Notice that $b(t+h)x(t)$ is T-periodic. We have

$$\begin{aligned} 0 &\geq \lambda\Big[\int_0^T a(t)|x(t)|dt - \int_0^T |b(t+h)||x(t)|dt - \int_0^T |p(t)|dt\Big] \\ &= \lambda\Big[\int_0^T (a(t) - |b(t+h)|)|x(t)|dt - \int_0^T |p(t)|dt\Big] \\ &= \lambda\Big[\frac{N-1}{N}\int_0^T a(t)|x(t)|dt + \frac{1}{N}\int_0^T (a(t) - N|b(t+h)|)|x(t)|dt - \int_0^T |p(t)|dt\Big] \\ &\geq \lambda\Big[\frac{N-1}{N}\int_0^T a(t)|x(t)|dt - \|p\|T\Big]. \end{aligned}$$

This implies that $\int_0^T |a(t)||x(t)|dt \leq \|p\|TN/(N-1) =: B_1$. There exists $t^* \in [0,T]$ such that $|x(t^*)|\int_0^T |a(t)|dt \leq \|p\|TN/(N-1)$. Thus,

$$|x(t^*)| \leq \|p\|TN\Big/\Big[(N-1)\int_0^T |a(t)|dt\Big] =: B_2.$$

It follows from Eq.(3.2) that

$$\begin{aligned} \int_0^T |x'(t)|dt &\leq \int_0^T |a(t)||x(t)|dt + \int_0^T |b(t)||x(t-h)|dt + \|p\|T \\ &= \int_0^T |a(t)||x(t)|dt + \int_0^T |b(t+h)||x(t)|dt + \|p\|T \end{aligned}$$

$$\leq \; 2\int_0^T |a(t)||x(t)|dt + \|p\|T \leq 2B_1 + \|p\|T.$$

For $t \in [0,T]$, we have

$$|x(t)| \leq |x(t^*)| + \int_0^T |x'(s)|ds \leq B_2 + 2B_1 + \|p\|T =: B.$$

By Theorem 2.2, Eq.(3.1) has a T-periodic solution.

Example 3.2. Consider the linear Volterra equation

$$x'(t) = A(t)x(t) + \int_{-\infty}^t B(t,s)x(s)ds + p(t) \tag{3.3}$$

where A, B are $n \times n$ matrix functions, $A(t), p(t)$ are continuous on R, and $B(t,s)$ is continuous on $-\infty < s \leq t + \infty$. There exists a constant $T > 0$ such that $A(t+T) = A(t)$, $B(t+T, s+T) = B(t,s)$, $p(t+T) = p(t)$ for all $t \in R$ and $-\infty < s \leq t < +\infty$. Suppose that

(i) $D = \displaystyle\int_0^T \Big(A(t) + \int_{-\infty}^t B(t,s)ds\Big)dt$ is invertible

(ii) $\displaystyle\sup_{t\in[0,T]} \Big[\Big|A(t) + \int_{-\infty}^t B(t,s)ds\Big|KT + \Big(|A(t)| + \int_{-\infty}^t |B(t,s)|ds\Big)T\Big] =: \alpha < 1$

where $K = |D^{-1}| \int_0^T (|A(u)| + \int_{-\infty}^u |B(u,s)|ds)du$. Then Eq.(3.3) has a T-periodic solution.

Proof. Let $F(t, x_t) = A(t)x(t) + \int_{-\infty}^t B(t,s)x(s)ds + p(t)$. It follows from Theorem 2.2 and (i) that for each $\phi \in P_T^0$, there exists a unique $k_\phi \in R$ such that $\int_0^T F(t, \Phi_t)dt = 0$ where $\Phi(t) = k_\phi + \int_0^t \phi(s)ds$. Moreover, the function $E : P_T^0 \to P_T$ defined by $E(\phi) = \Phi$ is continuous on P_T^0 and

$$k_\phi = -D^{-1}\int_0^T \Big(A(t)\int_0^t \phi(s)ds + \int_{-\infty}^t B(t,s)\int_0^s \phi(u)duds + p(t)\Big)dt.$$

Thus, $|k_\phi| \leq KT\|\phi\| + |D^{-1}|\|p\|T$ and

$$|F(t,\Phi_t)| \;\; \leq \;\; \Big|A(t) + \int_{-\infty}^t B(t,s)ds\Big||k_\phi| + \Big(|A(t)| + \int_{-\infty}^t |B(t,s)|ds\Big)T\|\phi\| + \|p\|$$

$$
\begin{aligned}
&\leq \left|A(t)+\int_{-\infty}^{t} B(t,s)ds\right|(KT\|\phi\|+|D^{-1}|\|p\|T) \\
&\quad +(|A(t)|+\int_{-\infty}^{t}|B(t,s)|ds)T\|\phi\|+\|p\| \\
&\leq \alpha\|\phi\|+M
\end{aligned}
$$

where $M=\sup_{t\in[0,T]}|A(t)+\int_{-\infty}^{t}B(t,s)ds||D^{-1}|\|p\|T+\|p\|$. By Corollary 2.2, Eq.(3.3) has a T-periodic solution.

Example 3.3. Consider the two dimensional nonlinear system

$$
x'(t)=Ag(x(t))+\int_{-\infty}^{t}C(t-s)g(x(s))ds+p(t) \tag{3.4}
$$

where $A=\mathrm{diag}(1,-1),\ C(t)=(c_{ij}(t))_{2\times 2},\ g(x)=(x_1^3,x_2^3)^T,\ x=(x_1,x_2)^T,\ p\in P_T^0$. If

$$
\int_{-\infty}^{0}(|c_{1j}(s)|+|c_{2j}(s)|)ds<1,\ j=1,2,
$$

then Eq.(3.4) has a T-periodic solution.

Proof. We verify that all conditions of Corollary 2.1 hold. Let

$$
F(t,x_t)=(F_1(t,x_t),F_2(t,x_t))^T=Ag(x(t))+\int_{-\infty}^{t}C(t-s)g(x(s))ds+p(t).
$$

For $\phi=(\phi_1,\phi_2)^T\in P_T^0$ and $k\in R$, we define $Q(k)=\int_0^T(k+\int_0^t\phi_1(s)ds)^3dt$. Since the quadratic function $Q'(k)=3\int_0^T(k+\int_0^t\phi_1(s)ds)^2dt\geq 0$ and $\lim_{k\to\pm\infty}Q(k)=\pm\infty$, there exists a unique $k_{1\phi}\in R$ such that $Q(k_{1\phi})=0$. Similarly, there exists $k_{2\phi}\in R$ such that $\int_0^T(k_{2\phi}+\int_0^t\phi_2(s)ds)^3dt=0$. Define $k_\phi=(k_{1\phi},k_{2\phi})^T$. We claim that $\int_0^T F(t,\Phi_t)dt=0$, where $\Phi(t)=(\Phi_1(t),\Phi_2(t))^T=k_\phi+\int_0^t\phi(s)ds$. Since $\int_0^t\phi(s)ds$ is T-periodic, for each $s\in R$ we have

$$
\int_0^T\Big(k_{j\phi}+\int_0^{t+s}\phi_j(u)du\Big)^3dt=\int_0^T\Big(k_{j\phi}+\int_0^{t}\phi_j(u)du\Big)^3dt=\int_0^T\Phi_j^3(t)dt=0.
$$

Thus,

$$
\begin{aligned}
F_1(t,\Phi_t) &= \Big(k_{1\phi}+\int_0^t\phi_1(s)ds\Big)^3+\int_{-\infty}^{0}C_{11}(-s)\Big(k_{1\phi}+\int_0^{t+s}\phi_1(u)du\Big)^3ds \\
&\quad +\int_{-\infty}^{0}C_{12}(-s)\Big(k_{2\phi}+\int_0^{t+s}\phi_2(u)du\Big)^3ds+p_1(t)
\end{aligned}
$$

and

$$\int_0^T F_1(t,\Phi_t)dt = \int_0^T \Phi_1^3(t)dt + \int_{-\infty}^0 C_{11}(-s)ds\int_0^T \Phi_1^3(t)dt + \int_{-\infty}^0 C_{12}(-s)ds\int_0^T \Phi_2^3(t)dt + \int_0^T p_1(t)dt = 0. \tag{3.5}$$

Similarly, we have

$$\int_0^T F_2(t,\Phi_t)dt = -\int_0^T \Phi_2^3(t)dt + \int_{-\infty}^0 C_{21}(-s)ds\int_0^T \Phi_1^3(t)dt + \int_{-\infty}^0 C_{22}(-s)ds\int_0^T \Phi_2^3(t)dt + \int_0^T p_2(t)dt = 0. \tag{3.6}$$

Thus, $\int_0^T F(t,\Phi_t)dt = 0$. It is clear that $|k_{j\phi}| \le T\|\phi_j\|$ for $j=1,2$. Thus, $|k_\phi| \le 2T\|\phi\|$. We now show that $E: P_T^0 \to P_T$ defined by $E(\phi)(t) = \Phi(t) = k_\phi + \int_0^t \phi(s)ds$ is continuous. Let $\{\phi_n\}$ be a sequence in P_T^0 and $\phi_n \to \phi \in P_T^0$ as $n \to +\infty$. We show that $k_{\phi_n} \to k_\phi$ as $n \to +\infty$. By way of contradiction, if $k_{\phi_n} \not\to k_\phi$, then there exists a subsequence, say $\{k_{\phi_n}\}$ again, and $\mu > 0$ with $|k_{\phi_n} - k_\phi| \ge \mu$. We may assume $|k_{1\phi_n} - k_{1\phi}| \ge \mu$. Let $\phi_n = (\phi_{1n},\phi_{2n})^T$ and $\phi = (\phi_1,\phi_2)^T$. By the definitions of $k_{\phi_n} = (k_{1\phi_n}, k_{2\phi_n})^T$ and $k_\phi = (k_{1\phi}, k_{2\phi})^T$, we have

$$\begin{aligned} 0 &= \Big|\int_0^T \Big(k_{1\phi_n} + \int_0^t \phi_{1n}(s)ds\Big)^3 dt - \int_0^T \Big(k_{1\phi} + \int_0^t \phi_1(s)ds\Big)^3 dt\Big| \\ &= \Big|\int_0^T \Big(k_{1\phi_n} - k_{1\phi} + \int_0^t \phi_{1n}(s)ds - \int_0^t \phi_1(s)ds\Big)\Big(\Phi_{1n}^2(t) + \Phi_{1n}(t)\Phi_1(t) + \Phi_1^2(t)\Big)dt\Big| \end{aligned}$$

where $\Phi_{1n}(t) = k_{1\phi_n} + \int_0^t \phi_{1n}(s)ds$. Since $\phi_n \to \phi$, there exists a constant $Q_1 > 0$ such that $\|\phi_n\| \le Q_1$ for all $n = 1,2,\cdots$. Thus, $|k_{\phi_n}| \le 2TQ_1$ and there exists a subsequence $\{k_{\phi_{n_j}}\}$ of $\{k_{\phi_n}\}$ such that $k_{\phi_{n_j}} \to k_* = (k_{1*}, k_{2*})^T$ as $j \to +\infty$. Applying Lebesgue's convergence theorem and letting $n_j \to +\infty$ in the above equality, we have

$$0 = \Big|\int_0^T (k_{1*} - k_{1\phi})\Big(\Phi_{1*}^2(t) + \Phi_{1*}(t)\Phi_1(t) + \Phi_1^2(t)\Big)dt\Big|$$

where $\Phi_{1*}(t) = k_{1*} + \int_0^t \phi_1(s)ds$. Thus,

$$\begin{aligned} 0 &\ge \frac{|k_{1*} - k_{1\phi}|}{2}\int_0^T \Big[\Phi_{1*}^2(t) + \Phi_1^2(t)\Big]dt \\ &= \frac{|k_{1*} - k_{1\phi}|}{2}\int_0^T \Big[\Big(k_{1*} + \int_0^t \phi_1(s)ds\Big)^2 + \Big(k_{1\phi} + \int_0^t \phi_1(s)ds\Big)^2\Big]dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{|k_{1*}-k_{1\phi}|}{2}\int_0^T \Big[\Big(k_{1*}-k_{1\phi}+k_{1\phi}+\int_0^t \phi_1(s)ds\Big)^2+\Big(k_{1\phi}+\int_0^t \phi_1(s)ds\Big)^2\Big]dt \\
&\geq \frac{|k_{1*}-k_{1\phi}|}{2}\int_0^T \frac{1}{2}(k_{1*}-k_{1\phi})^2 dt \\
&= \frac{1}{4}T|k_{1*}-k_{1\phi}|^3 \geq \frac{1}{4}T\mu^3 > 0,
\end{aligned}
$$

a contradiction. Similarly, we can show that $k_{2\phi_n} \to k_{2\phi}$. Thus, $k_{\phi_n} \to k_\phi$ as $n \to +\infty$. It is clear that $\Big|\int_0^t \phi_n(s)ds - \int_0^t \phi(s)ds\Big| \leq T\|\phi_n - \phi\| \to 0$ as $n \to +\infty$. This shows that $E : P_T^0 \to P_T$ is continuous.

It is clear that $F : R \times P_T \to R^2$ maps bounded sets into bounded sets. We now show there exists a constant $B > 0$ such that $\|x\| < B$ whenever $x = x(t)$ is a T-periodic solution of

$$
x'(t) = \lambda\Big[Ag(x(t)) + \int_{-\infty}^t C(t-s)g(x(s))ds + p(t)\Big] \tag{3.7}
$$

where $0 < \lambda < 1$. Let $x = (x_1, x_2)^T$ and define $V(x) = -|x_1| + |x_2|$. If $x = x(t)$ is a T-periodic solution of (3.7), then

$$
\begin{aligned}
V'(x(t)) \;&\leq\; -\lambda(|x_1(t)|^3 + |x_2(t)|^3) + \lambda\int_{-\infty}^0 (|c_{11}(-s)| + |c_{21}(-s)|)|x_1(t+s)|^3 ds \\
&\quad + \lambda\int_{-\infty}^0 (|c_{12}(-s)| + |c_{22}(-s)|)|x_2(t+s)|^3 ds + \lambda\|p_1\| + \lambda\|p_2\|. \qquad (3.8)
\end{aligned}
$$

Integrating (3.8) from 0 to T, we find constants α_{ij} so that

$$
\begin{aligned}
0 &= V(x(T)) - V(x(0)) \\
&\leq -\lambda\int_0^T (|x_1(t)|^3 + |x_2(t)|^3)dt + \lambda\int_{-\infty}^0 (|c_{11}(-s)| + |c_{21}(-s)|)ds\int_0^T |x_1(t)|^3 dt \\
&\quad + \lambda\int_{-\infty}^0 (|c_{12}(-s)| + |c_{22}(-s)|)ds\int_0^T |x_2(t)|^3 dt + \lambda T(\|p_1\| + \|p_2\|) \\
&= -\lambda(1-\alpha_{11}-\alpha_{21})\int_0^T |x_1(t)|^3 dt - \lambda(1-\alpha_{12}-\alpha_{22})\int_0^T |x_2(t)|^3 dt + \lambda T(\|p_1\| + \|p_2\|).
\end{aligned}
$$

Thus, there exists a constant $B_1 > 0$ such that $\int_0^T |x(t)|^3 dt < B_1$. It follows from Eq.(3.7) that

$$
\int_0^T |x'(t)|dt \;\leq\; \int_0^T |A||g(x(t))|dt + \int_0^T\int_{-\infty}^0 |C(-s)||g(x(t+s))|dsdt + \|p\|T
$$

$$
\begin{aligned}
&\leq \ (|A| + \int_{-\infty}^{0} |C(-s)|ds) \int_0^T |x(t)|^3 dt + \|p\|T \\
&\leq \ (|A| + \int_{-\infty}^{0} |C(-s)|ds) B_1 + \|p\|T.
\end{aligned}
$$

By Sobolev's inequality, there exists a constant $B > 0$ such that $\|x\| < B$. Since all conditions of Corollary 2.1 are satisfied, we conclude that Eq.(3.4) has a T-periodic solution.

Remark 3.1. It is clear from (3.5) and (3.6) that the same conclusion of Example 3.3 follows if $p \in P_T^0$ is replaced by $p \in P_T$ and

$$
\det D \neq 0, \quad D = A + \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \quad \alpha_{ij} = \int_{-\infty}^{0} C_{ij}(-s)ds.
$$

Remark 3.2. For nonlinear vector equations there is the problem of deciding the order in which we should find the appropriate constants k_ϕ to satisfy condition (i) of Theorem 2.1. We now illustrate the order (but not the details) for finding those constants for the well-known Liénard system

$$
\begin{aligned}
x_1'(t) &= x_2(t) - N(x_1(t)), \\
x_2'(t) &= -g(x_1(t-r)) + p(t, x_1(t), x_2(t))
\end{aligned}
\tag{3.9}
$$

where $p(t,u,v)$ is T-periodic in t. Let $F_1(t,x_t) = x_2(t) - N(x_1(t))$, $F_2(t,x_t) = -g(x_1(t-r)) + p(t, x_1(t), x_2(t))$, and $F(t,x_t) = (F_1(t,x_t), F_2(t,x_t))^T$. For $\phi = (\phi_1,\phi_2)^T \in P_T^0$ and $k = (k_1,k_2)^T$, define $\Phi(t) = k + \int_0^t \phi(s)ds$. Set

$$
\int_0^T F_1(t,\Phi_t)dt = \int_0^T \Big(k_2 + \int_0^t \phi_2(s)ds\Big)dt - \int_0^T N\Big(k_1 + \int_0^t \phi_1(s)ds\Big)dt = 0.
$$

This yields,

$$
k_2 = \frac{1}{T}\Big[- \int_0^T \int_0^t \phi_2(s)dsdt + \int_0^T N\Big(k_1 + \int_0^t \phi_1(s)ds\Big)dt\Big]. \tag{3.10}
$$

Substitute (3.10) into $\int_0^T F_2(t,x_t)dt = 0$ to solve k_1 and then solve for k_2 by (3.10). Obviously, conditions on the functions must now be given to ensure the requirements of Theorem 2.1. If $p(t,u,v) \equiv p(t)$, we may solve k_1 directly from

$$
-\int_0^T g\Big(k_1 + \int_0^t \phi_1(s)ds\Big)dt + \int_0^T p(s)ds = 0.
$$

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